Therefore, lumped system analysis is *not* applicable. However, we can still use it to get a "rough" estimate of the time of death. The exponent b in this case is

$$b = \frac{hA_s}{\rho C_p V} = \frac{h}{\rho C_p L_c} = \frac{8 \text{ W/m}^2 \cdot {}^\circ\text{C}}{(996 \text{ kg/m}^3)(4178 \text{ J/kg} \cdot {}^\circ\text{C})(0.0689 \text{ m})}$$
$$= 2.79 \times 10^{-5} \text{ s}^{-1}$$

We now substitute these values into Eq. 4–4,

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \longrightarrow \frac{25 - 20}{37 - 20} = e^{-(2.79 \times 10^{-5} \text{ s}^{-1})t}$$

which yields

$$t = 43{,}860 \text{ s} = \mathbf{12.2 \text{ h}}$$

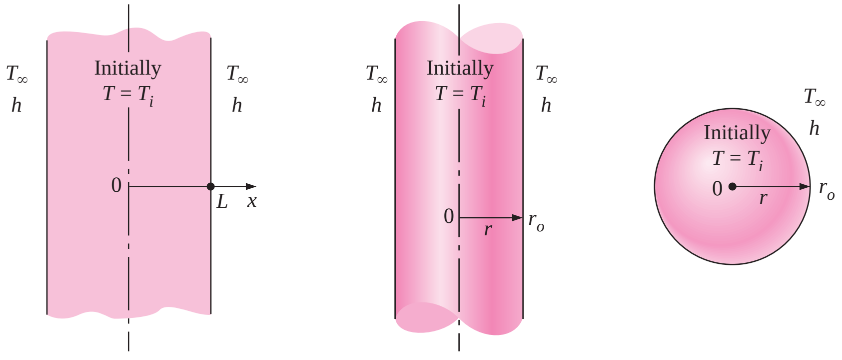
Therefore, as a rough estimate, the person died about 12 h before the body was found, and thus the time of death is 5 AM. This example demonstrates how to obtain "ball park" values using a simple analysis.

4–2 ▪ TRANSIENT HEAT CONDUCTION IN LARGE PLANE WALLS, LONG CYLINDERS, AND SPHERES WITH SPATIAL EFFECTS

In Section, 4–1, we considered bodies in which the variation of temperature within the body was negligible; that is, bodies that remain nearly *isothermal* during a process. Relatively *small* bodies of *highly conductive* materials approximate this behavior. In general, however, the temperature within a body will change from point to point as well as with time. In this section, we consider the variation of temperature with *time* and *position* in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.

Consider a plane wall of thickness $2L$, a long cylinder of radius r_o, and a sphere of radius r_o initially at a *uniform temperature* T_i, as shown in Fig. 4–11. At time $t = 0$, each geometry is placed in a large medium that is at a constant temperature T_∞ and kept in that medium for $t > 0$. Heat transfer takes place between these bodies and their environments by convection with a *uniform* and *constant* heat transfer coefficient h. Note that all three cases possess geometric and thermal symmetry: the plane wall is symmetric about its *center plane* ($x = 0$), the cylinder is symmetric about its *centerline* ($r = 0$), and the sphere is symmetric about its *center point* ($r = 0$). We neglect *radiation* heat transfer between these bodies and their surrounding surfaces, or incorporate the radiation effect into the convection heat transfer coefficient h.

The variation of the temperature profile with *time* in the plane wall is illustrated in Fig. 4–12. When the wall is first exposed to the surrounding medium at $T_\infty < T_i$ at $t = 0$, the entire wall is at its initial temperature T_i. But the wall temperature at and near the surfaces starts to drop as a result of heat transfer from the wall to the surrounding medium. This creates a *temperature*



(*a*) A large plane wall (*b*) A long cylinder (*c*) A sphere

FIGURE 4–11
Schematic of the simple geometries in which heat transfer is one-dimensional.

gradient in the wall and initiates heat conduction from the inner parts of the wall toward its outer surfaces. Note that the temperature at the center of the wall remains at T_i until $t = t_2$, and that the temperature profile within the wall remains symmetric at all times about the center plane. The temperature profile gets flatter and flatter as time passes as a result of heat transfer, and eventually becomes uniform at $T = T_\infty$. That is, the wall reaches *thermal equilibrium* with its surroundings. At that point, the heat transfer stops since there is no longer a temperature difference. Similar discussions can be given for the long cylinder or sphere.

The formulation of the problems for the determination of the one-dimensional transient temperature distribution $T(x, t)$ in a wall results in a partial differential equation, which can be solved using advanced mathematical techniques. The solution, however, normally involves infinite series, which are inconvenient and time-consuming to evaluate. Therefore, there is clear motivation to present the solution in *tabular* or *graphical* form. However, the solution involves the parameters x, L, t, k, α, h, T_i, and T_∞, which are too many to make any graphical presentation of the results practical. In order to reduce the number of parameters, we nondimensionalize the problem by defining the following dimensionless quantities:

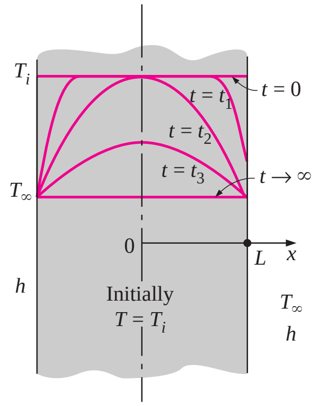


FIGURE 4–12
Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$.

Dimensionless temperature: $$\theta(x, t) = \frac{T(x, t) - T_\infty}{T_i - T_\infty}$$

Dimensionless distance from the center: $$X = \frac{x}{L}$$

Dimensionless heat transfer coefficient: $$\text{Bi} = \frac{hL}{k} \qquad \textbf{(Biot number)}$$

Dimensionless time: $$\tau = \frac{\alpha t}{L^2} \qquad \textbf{(Fourier number)}$$

The nondimensionalization enables us to present the temperature in terms of three parameters only: X, Bi, and τ. This makes it practical to present the solution in graphical form. The dimensionless quantities defined above for a plane wall can also be used for a *cylinder* or *sphere* by replacing the space variable x by r and the half-thickness L by the outer radius r_o. Note that the characteristic length in the definition of the Biot number is taken to be the *half-thickness L* for the plane wall, and the *radius* r_o for the long cylinder and sphere instead of V/A used in lumped system analysis.

The one-dimensional transient heat conduction problem just described can be solved exactly for any of the three geometries, but the solution involves infinite series, which are difficult to deal with. However, the terms in the solutions converge rapidly with increasing time, and for $\tau > 0.2$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2 percent. We are usually interested in the solution for times with $\tau > 0.2$, and thus it is very convenient to express the solution using this **one-term approximation,** given as

Plane wall:
$$\theta(x, t)_{\text{wall}} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2 \tag{4-10}$$

Cylinder:
$$\theta(r, t)_{\text{cyl}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2 \tag{4-11}$$

Sphere:
$$\theta(r, t)_{\text{sph}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2 \tag{4-12}$$

where the constants A_1 and λ_1 are functions of the Bi number only, and their values are listed in Table 4–1 against the Bi number for all three geometries. The function J_0 is the zeroth-order Bessel function of the first kind, whose value can be determined from Table 4–2. Noting that $\cos(0) = J_0(0) = 1$ and the limit of $(\sin x)/x$ is also 1, these relations simplify to the next ones at the center of a plane wall, cylinder, or sphere:

Center of plane wall ($x = 0$):
$$\theta_{0,\,\text{wall}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \tag{4-13}$$

Center of cylinder ($r = 0$):
$$\theta_{0,\,\text{cyl}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \tag{4-14}$$

Center of sphere ($r = 0$):
$$\theta_{0,\,\text{sph}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \tag{4-15}$$

Once the Bi number is known, the above relations can be used to determine the temperature anywhere in the medium. The determination of the constants A_1 and λ_1 usually requires interpolation. For those who prefer reading charts to interpolating, the relations above are plotted and the one-term approximation solutions are presented in graphical form, known as the *transient temperature charts.* Note that the charts are sometimes difficult to read, and they are subject to reading errors. Therefore, the relations above should be preferred to the charts.

The transient temperature charts in Figs. 4–13, 4–14, and 4–15 for a large plane wall, long cylinder, and sphere were presented by M. P. Heisler in 1947 and are called **Heisler charts.** They were supplemented in 1961 with transient heat transfer charts by H. Gröber. There are *three* charts associated with each geometry: the first chart is to determine the temperature T_o at the *center* of the geometry at a given time t. The second chart is to determine the temperature at *other locations* at the same time in terms of T_o. The third chart is to determine the total amount of *heat transfer* up to the time t. These plots are valid for $\tau > 0.2$.

TABLE 4–1

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane Wall		*Cylinder*		*Sphere*	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–2

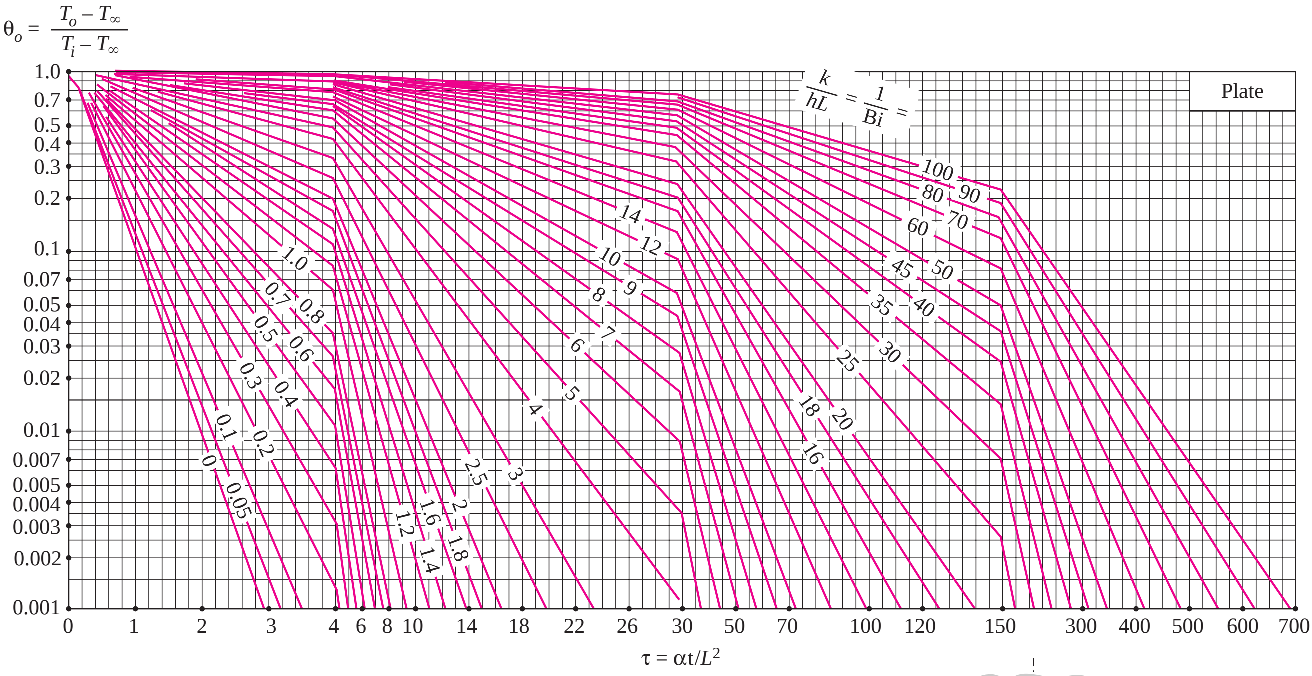
The zeroth- and first-order Bessel functions of the first kind

ξ	$J_o(\xi)$	$J_1(\xi)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

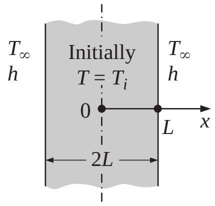
Note that the case $1/\text{Bi} = k/hL = 0$ corresponds to $h \to \infty$, which corresponds to the case of *specified surface temperature* T_∞. That is, the case in which the surfaces of the body are suddenly brought to the temperature T_∞ at $t = 0$ and kept at T_∞ at all times can be handled by setting h to infinity (Fig. 4–16).

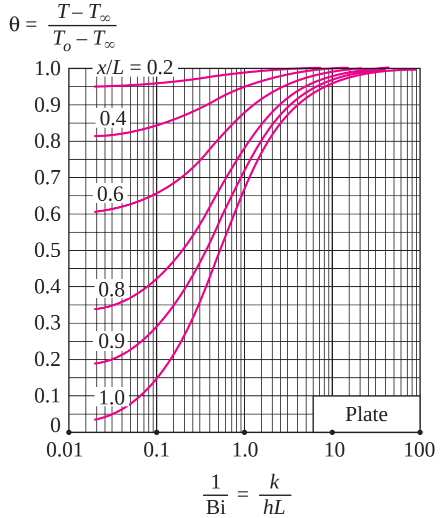
The temperature of the body changes from the initial temperature T_i to the temperature of the surroundings T_∞ at the end of the transient heat conduction process. Thus, the *maximum* amount of heat that a body can gain (or lose if $T_i > T_\infty$) is simply the *change* in the *energy content* of the body. That is,

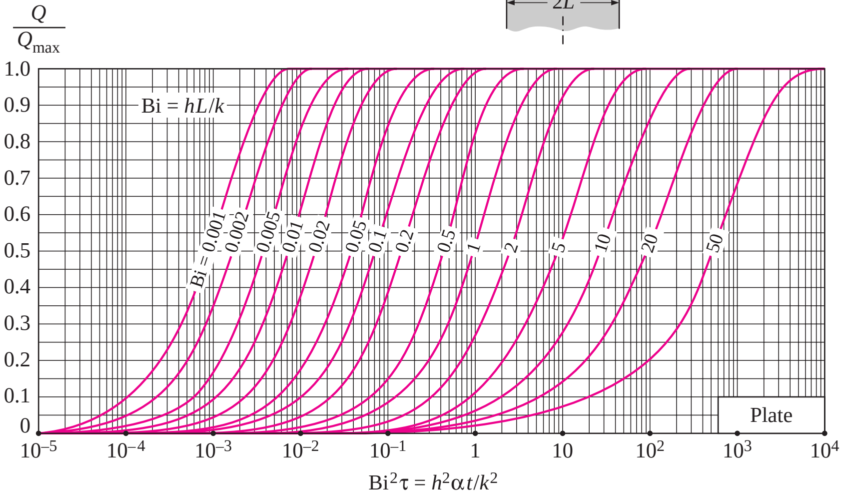
$$Q_{\max} = mC_p(T_\infty - T_i) = \rho VC_p(T_\infty - T_i) \qquad \text{(kJ)} \qquad \textbf{(4-16)}$$



(*a*) Midplane temperature (from M. P. Heisler)







(*b*) Temperature distribution (from M. P. Heisler) (*c*) Heat transfer (from H. Gröber et al.)

FIGURE 4–13

Transient temperature and heat transfer charts for a plane wall of thickness 2L initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h.

where m is the mass, V is the volume, ρ is the density, and C_p is the specific heat of the body. Thus, Q_{max} represents the amount of heat transfer for $t \to \infty$. The amount of heat transfer Q at a finite time t will obviously be less than this

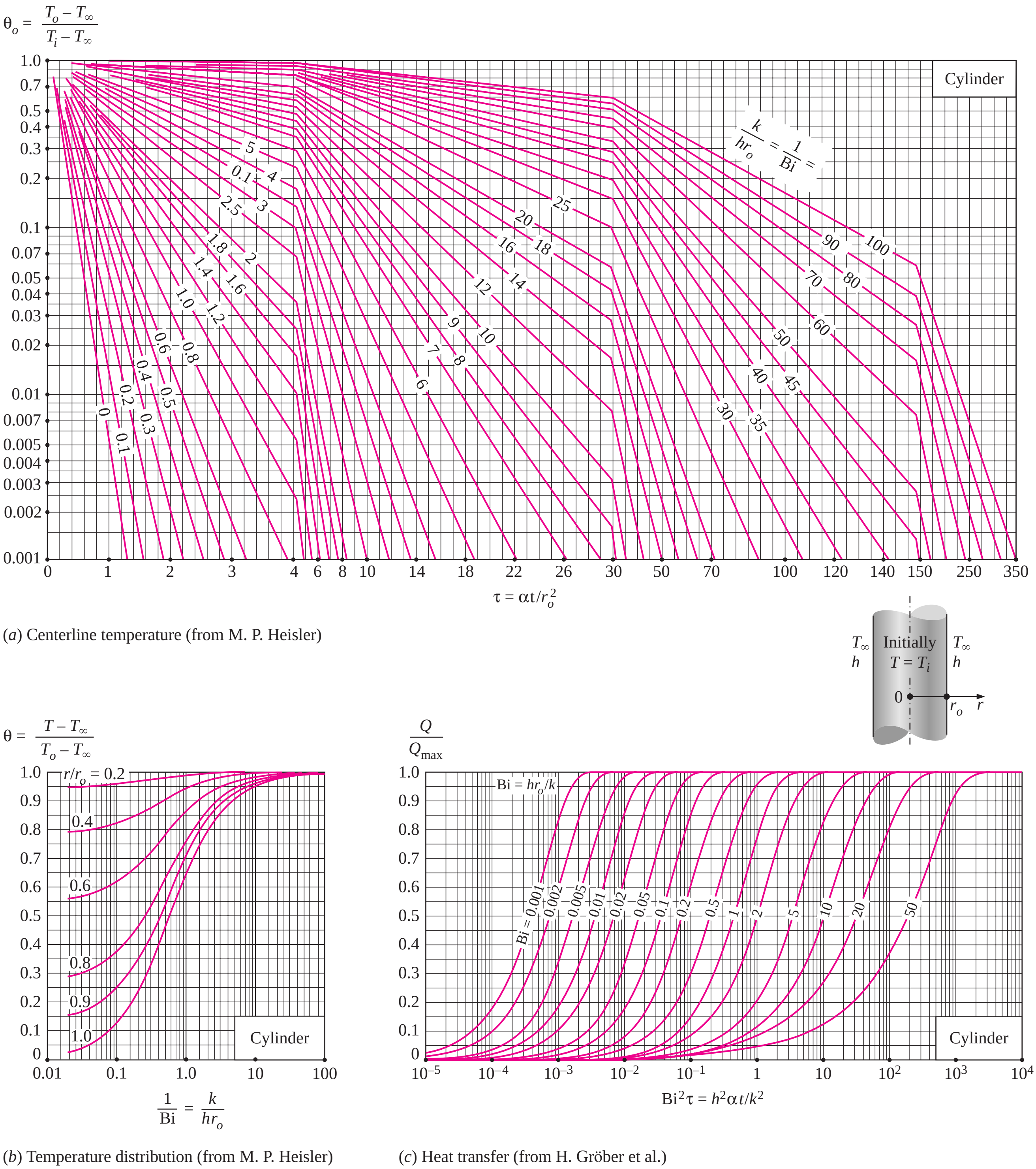
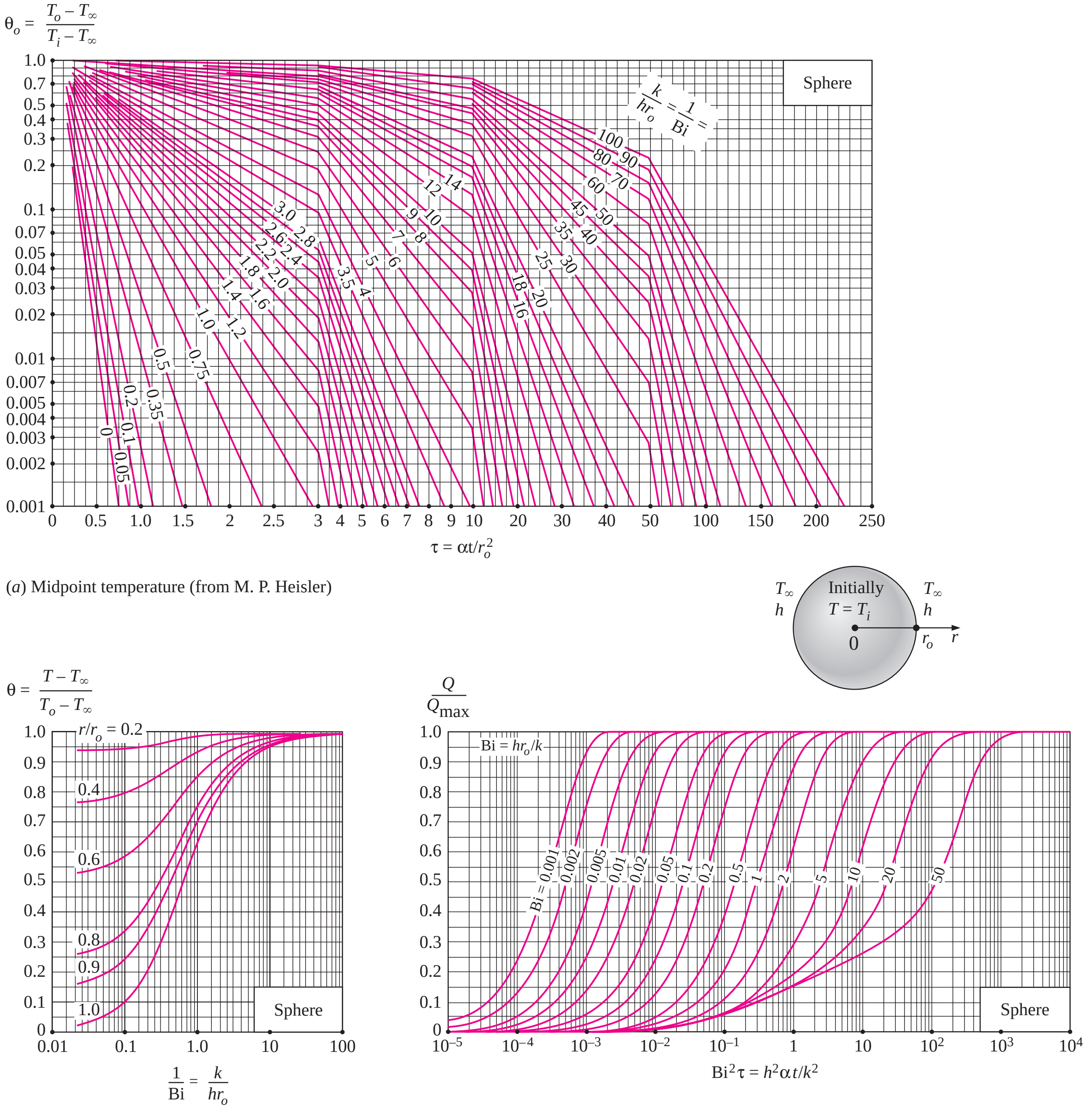


FIGURE 4–14

Transient temperature and heat transfer charts for a long cylinder of radius r_o initially at a uniform temperature T_i subjected to convection from all sides to an environment at temperature T_∞ with a convection coefficient of h.



(*a*) Midpoint temperature (from M. P. Heisler)

(*b*) Temperature distribution (from M. P. Heisler)

(*c*) Heat transfer (from H. Gröber et al.)

FIGURE 4–15

Transient temperature and heat transfer charts for a sphere of radius r_o initially at a uniform temperature T_i subjected to convection from all sides to an environment at temperature T_∞ with a convection coefficient of h.

maximum. The ratio Q/Q_{max} is plotted in Figures 4–13*c*, 4–14*c*, and 4–15*c* against the variables Bi and $h^2\alpha t/k^2$ for the large plane wall, long cylinder, and sphere, respectively. Note that once the *fraction* of heat transfer Q/Q_{max} has been determined from these charts for the given t, the actual amount of heat transfer by that time can be evaluated by multiplying this fraction by Q_{max}. A *negative* sign for Q_{max} indicates that heat is *leaving* the body (Fig. 4–17).

The fraction of heat transfer can also be determined from these relations, which are based on the one-term approximations already discussed:

Plane wall:
$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{0,\,wall}\frac{\sin\lambda_1}{\lambda_1} \quad \textbf{(4-17)}$$

Cylinder:
$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - 2\theta_{0,\,cyl}\frac{J_1(\lambda_1)}{\lambda_1} \quad \textbf{(4-18)}$$

Sphere:
$$\left(\frac{Q}{Q_{max}}\right)_{sph} = 1 - 3\theta_{0,\,sph}\frac{\sin\lambda_1 - \lambda_1\cos\lambda_1}{\lambda_1^3} \quad \textbf{(4-19)}$$

The use of the Heisler/Gröber charts and the one-term solutions already discussed is limited to the conditions specified at the beginning of this section: the body is initially at a *uniform* temperature, the temperature of the medium surrounding the body and the convection heat transfer coefficient are *constant* and *uniform*, and there is no *energy generation* in the body.

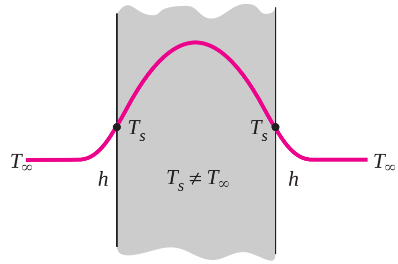
We discussed the physical significance of the *Biot number* earlier and indicated that it is a measure of the relative magnitudes of the two heat transfer mechanisms: *convection* at the surface and *conduction* through the solid. A *small* value of Bi indicates that the inner resistance of the body to heat conduction is *small* relative to the resistance to convection between the surface and the fluid. As a result, the temperature distribution within the solid becomes fairly uniform, and lumped system analysis becomes applicable. Recall that when $\text{Bi} < 0.1$, the error in assuming the temperature within the body to be *uniform* is negligible.

To understand the physical significance of the *Fourier number* τ, we express it as (Fig. 4–18)

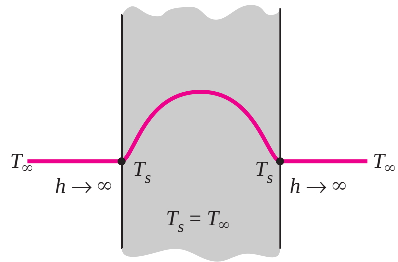
$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2\,(1/L)}{\rho C_p L^3/t}\frac{\Delta T}{\Delta T} = \frac{\text{The rate at which heat is \textit{conducted} across } L \text{ of a body of volume } L^3}{\text{The rate at which heat is \textit{stored} in a body of volume } L^3} \quad (4\text{-}20)$$

Therefore, the Fourier number is a measure of *heat conducted* through a body relative to *heat stored*. Thus, a large value of the Fourier number indicates faster propagation of heat through a body.

Perhaps you are wondering about what constitutes an infinitely large plate or an infinitely long cylinder. After all, nothing in this world is infinite. A plate whose thickness is small relative to the other dimensions can be modeled as an infinitely large plate, except very near the outer edges. But the edge effects on large bodies are usually negligible, and thus a large plane wall such as the wall of a house can be modeled as an infinitely large wall for heat transfer purposes. Similarly, a long cylinder whose diameter is small relative to its length can be analyzed as an infinitely long cylinder. The use of the transient temperature charts and the one-term solutions is illustrated in the following examples.



(*a*) Finite convection coefficient



(*b*) Infinite convection coefficient

FIGURE 4–16
The specified surface temperature corresponds to the case of convection to an environment at T_∞ with a convection coefficient h that is *infinite*.

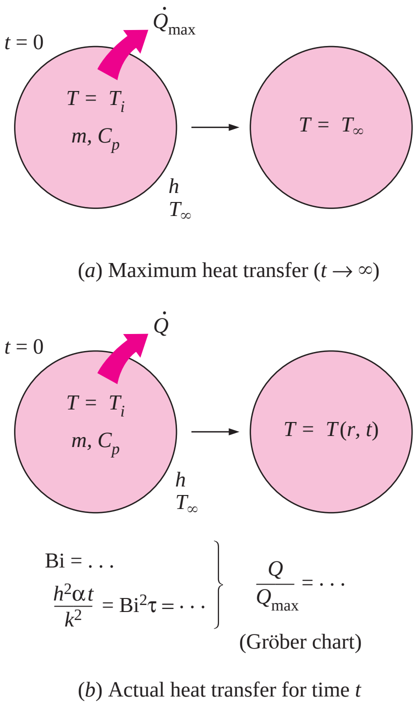


FIGURE 4–17
The fraction of total heat transfer Q/Q_{max} up to a specified time t is determined using the Gröber charts.

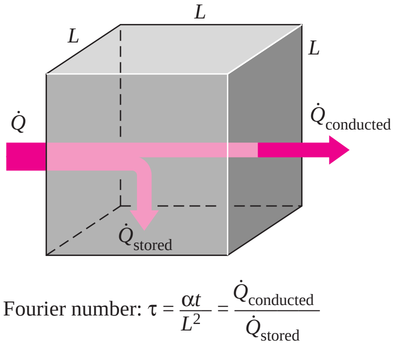


FIGURE 4–18
Fourier number at time t can be viewed as the ratio of the rate of heat conducted to the rate of heat stored at that time.

EXAMPLE 4–3 Boiling Eggs

An ordinary egg can be approximated as a 5-cm-diameter sphere (Fig. 4–19). The egg is initially at a uniform temperature of 5°C and is dropped into boiling water at 95°C. Taking the convection heat transfer coefficient to be $h = 1200$ W/m² · °C, determine how long it will take for the center of the egg to reach 70°C.

SOLUTION An egg is cooked in boiling water. The cooking time of the egg is to be determined.

Assumptions **1** The egg is spherical in shape with a radius of $r_0 = 2.5$ cm. **2** Heat conduction in the egg is one-dimensional because of thermal symmetry about the midpoint. **3** The thermal properties of the egg and the heat transfer coefficient are constant. **4** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The water content of eggs is about 74 percent, and thus the thermal conductivity and diffusivity of eggs can be approximated by those of water at the average temperature of $(5 + 70)/2 = 37.5$°C; $k = 0.627$ W/m · °C and $\alpha = k/\rho C_p = 0.151 \times 10^{-6}$ m²/s (Table A-9).

Analysis The temperature within the egg varies with radial distance as well as time, and the temperature at a specified location at a given time can be determined from the Heisler charts or the one-term solutions. Here we will use the latter to demonstrate their use. The Biot number for this problem is

$$\text{Bi} = \frac{hr_0}{k} = \frac{(1200\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.025\ \text{m})}{0.627\ \text{W/m} \cdot {}^\circ\text{C}} = 47.8$$

which is much greater than 0.1, and thus the lumped system analysis is not applicable. The coefficients λ_1 and A_1 for a sphere corresponding to this Bi are, from Table 4–1,

$$\lambda_1 = 3.0753, \qquad A_1 = 1.9958$$

Substituting these and other values into Eq. 4–15 and solving for τ gives

$$\frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \longrightarrow \frac{70 - 95}{5 - 95} = 1.9958 e^{-(3.0753)^2 \tau} \longrightarrow \tau = 0.209$$

which is greater than 0.2, and thus the one-term solution is applicable with an error of less than 2 percent. Then the cooking time is determined from the definition of the Fourier number to be

$$t = \frac{\tau r_o^2}{\alpha} = \frac{(0.209)(0.025\ \text{m})^2}{0.151 \times 10^{-6}\ \text{m}^2/\text{s}} = 865\ \text{s} \approx \mathbf{14.4\ min}$$

Therefore, it will take about 15 min for the center of the egg to be heated from 5°C to 70°C.

Discussion Note that the Biot number in lumped system analysis was defined differently as $\text{Bi} = hL_c/k = h(r/3)/k$. However, either definition can be used in determining the applicability of the lumped system analysis unless $\text{Bi} \approx 0.1$.

EXAMPLE 4–4 Heating of Large Brass Plates in an Oven

In a production facility, large brass plates of 4 cm thickness that are initially at a uniform temperature of 20°C are heated by passing them through an oven that is maintained at 500°C (Fig. 4–20). The plates remain in the oven for a period of 7 min. Taking the combined convection and radiation heat transfer coefficient to be $h = 120$ W/m$^2 \cdot$ °C, determine the surface temperature of the plates when they come out of the oven.

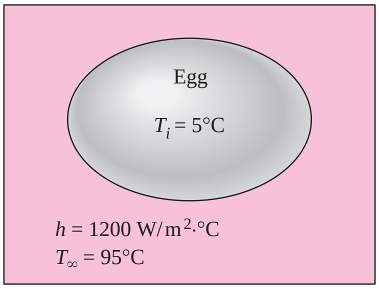


FIGURE 4–19
Schematic for Example 4–3.

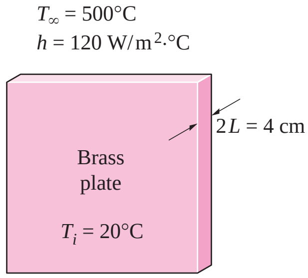


FIGURE 4–20
Schematic for Example 4–4.

SOLUTION Large brass plates are heated in an oven. The surface temperature of the plates leaving the oven is to be determined.

Assumptions **1** Heat conduction in the plate is one-dimensional since the plate is large relative to its thickness and there is thermal symmetry about the center plane. **2** The thermal properties of the plate and the heat transfer coefficient are constant. **3** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The properties of brass at room temperature are $k = 110$ W/m · °C, $\rho = 8530$ kg/m^3, $C_p = 380$ J/kg · °C, and $\alpha = 33.9 \times 10^{-6}$ m^2/s (Table A-3). More accurate results are obtained by using properties at average temperature.

Analysis The temperature at a specified location at a given time can be determined from the Heisler charts or one-term solutions. Here we will use the charts to demonstrate their use. Noting that the half-thickness of the plate is $L = 0.02$ m, from Fig. 4–13 we have

$$\left.\begin{aligned} \frac{1}{\text{Bi}} &= \frac{k}{hL} = \frac{100 \text{ W/m} \cdot {}^\circ\text{C}}{(120 \text{ W/m}^2 \cdot {}^\circ\text{C})(0.02 \text{ m})} = 45.8 \\ \tau &= \frac{\alpha t}{L^2} = \frac{(33.9 \times 10^{-6} \text{ m}^2\text{/s})(7 \times 60 \text{ s})}{(0.02 \text{ m})^2} = 35.6 \end{aligned}\right\} \frac{T_o - T_\infty}{T_i - T_\infty} = 0.46$$

Also,

$$\left.\begin{aligned} \frac{1}{\text{Bi}} &= \frac{k}{hL} = 45.8 \\ \frac{x}{L} &= \frac{L}{L} = 1 \end{aligned}\right\} \frac{T - T_\infty}{T_o - T_\infty} = 0.99$$

Therefore,

$$\frac{T - T_\infty}{T_i - T_\infty} = \frac{T - T_\infty}{T_o - T_\infty}\frac{T_o - T_\infty}{T_i - T_\infty} = 0.46 \times 0.99 = 0.455$$

and

$$T = T_\infty + 0.455(T_i - T_\infty) = 500 + 0.455(20 - 500) = \mathbf{282°C}$$

Therefore, the surface temperature of the plates will be 282°C when they leave the oven.

Discussion We notice that the Biot number in this case is Bi = 1/45.8 = 0.022, which is much less than 0.1. Therefore, we expect the lumped system analysis to be applicable. This is also evident from $(T - T_\infty)/(T_o - T_\infty) = 0.99$, which indicates that the temperatures at the center and the surface of the plate relative to the surrounding temperature are within 1 percent of each other.

Noting that the error involved in reading the Heisler charts is typically at least a few percent, the lumped system analysis in this case may yield just as accurate results with less effort.

The heat transfer surface area of the plate is $2A$, where A is the face area of the plate (the plate transfers heat through both of its surfaces), and the volume of the plate is $V = (2L)A$, where L is the half-thickness of the plate. The exponent b used in the lumped system analysis is determined to be

$$b = \frac{hA_s}{\rho C_p V} = \frac{h(2A)}{\rho C_p (2LA)} = \frac{h}{\rho C_p L}$$

$$= \frac{120 \text{ W/m}^2 \cdot °\text{C}}{(8530 \text{ kg/m}^3)(380 \text{ J/kg} \cdot °\text{C})(0.02 \text{ m})} = 0.00185 \text{ s}^{-1}$$

Then the temperature of the plate at $t = 7$ min $= 420$ s is determined from

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \longrightarrow \frac{T(t) - 500}{20 - 500} = e^{-(0.00185 \text{ s}^{-1})(420 \text{ s})}$$

It yields

$$T(t) = 279°\text{C}$$

which is practically identical to the result obtained above using the Heisler charts. Therefore, we can use lumped system analysis with confidence when the Biot number is sufficiently small.

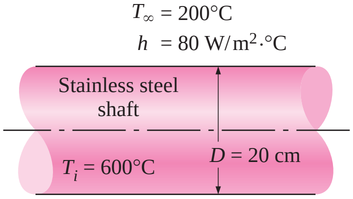


FIGURE 4–21
Schematic for Example 4–5.

EXAMPLE 4–5 **Cooling of a Long Stainless Steel Cylindrical Shaft**

A long 20-cm-diameter cylindrical shaft made of stainless steel 304 comes out of an oven at a uniform temperature of 600°C (Fig. 4–21). The shaft is then allowed to cool slowly in an environment chamber at 200°C with an average heat transfer coefficient of $h = 80$ W/m$^2 \cdot$ °C. Determine the temperature at the center of the shaft 45 min after the start of the cooling process. Also, determine the heat transfer per unit length of the shaft during this time period.

SOLUTION A long cylindrical shaft at 600°C is allowed to cool slowly. The center temperature and the heat transfer per unit length are to be determined.

Assumptions **1** Heat conduction in the shaft is one-dimensional since it is long and it has thermal symmetry about the centerline. **2** The thermal properties of the shaft and the heat transfer coefficient are constant. **3** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The properties of stainless steel 304 at room temperature are $k = 14.9$ W/m $\cdot$ °C, $\rho = 7900$ kg/m^3, $C_p = 477$ J/kg $\cdot$ °C, and $\alpha = 3.95 \times 10^{-6}$ m^2/s (Table A-3). More accurate results can be obtained by using properties at average temperature.

Analysis The temperature within the shaft may vary with the radial distance r as well as time, and the temperature at a specified location at a given time can be determined from the Heisler charts. Noting that the radius of the shaft is $r_o = 0.1$ m, from Fig. 4–14 we have

$$\left.\begin{aligned} \frac{1}{\text{Bi}} &= \frac{k}{hr_o} = \frac{14.9 \text{ W/m} \cdot °\text{C}}{(80 \text{ W/m}^2 \cdot °\text{C})(0.1 \text{ m})} = 1.86 \\ \tau &= \frac{\alpha t}{r_o^2} = \frac{(3.95 \times 10^{-6} \text{ m}^2/\text{s})(45 \times 60 \text{ s})}{(0.1 \text{ m})^2} = 1.07 \end{aligned}\right\} \frac{T_o - T_\infty}{T_i - T_\infty} = 0.40$$

and

$$T_o = T_\infty + 0.4(T_i - T_\infty) = 200 + 0.4(600 - 200) = \mathbf{360°C}$$

Therefore, the center temperature of the shaft will drop from 600°C to 360°C in 45 min.

To determine the actual heat transfer, we first need to calculate the maximum heat that can be transferred from the cylinder, which is the sensible energy of the cylinder relative to its environment. Taking $L = 1$ m,

$$\begin{aligned} m &= \rho V = \rho \pi r_o^2 L = (7900 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(1 \text{ m}) = 248.2 \text{ kg} \\ Q_{\text{max}} &= mC_p(T_\infty - T_i) = (248.2 \text{ kg})(0.477 \text{ kJ/kg} \cdot °\text{C})(600 - 200)°\text{C} \\ &= 47{,}354 \text{ kJ} \end{aligned}$$

The dimensionless heat transfer ratio is determined from Fig. 4–14*c* for a long cylinder to be

$$\left.\begin{aligned} \text{Bi} &= \frac{1}{1/\text{Bi}} = \frac{1}{1.86} = 0.537 \\ \frac{h^2 \alpha t}{k^2} &= \text{Bi}^2\tau = (0.537)^2(1.07) = 0.309 \end{aligned}\right\} \frac{Q}{Q_{\text{max}}} = 0.62$$

Therefore,

$$Q = 0.62 Q_{\text{max}} = 0.62 \times (47{,}354 \text{ kJ}) = \mathbf{29{,}360 \text{ kJ}}$$

which is the total heat transfer from the shaft during the first 45 min of the cooling.

ALTERNATIVE SOLUTION We could also solve this problem using the one-term solution relation instead of the transient charts. First we find the Biot number

$$\text{Bi} = \frac{hr_o}{k} = \frac{(80 \text{ W/m}^2 \cdot °\text{C})(0.1 \text{ m})}{14.9 \text{ W/m} \cdot °\text{C}} = 0.537$$

The coefficients λ_1 and A_1 for a cylinder corresponding to this Bi are determined from Table 4–1 to be

$$\lambda_1 = 0.970, \qquad A_1 = 1.122$$

Substituting these values into Eq. 4–14 gives

$$\theta_0 = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} = 1.122 e^{-(0.970)^2(1.07)} = 0.41$$

and thus

$$T_o = T_\infty + 0.41(T_i - T_\infty) = 200 + 0.41(600 - 200) = \mathbf{364°C}$$

The value of $J_1(\lambda_1)$ for $\lambda_1 = 0.970$ is determined from Table 4–2 to be 0.430. Then the fractional heat transfer is determined from Eq. 4–18 to be

$$\frac{Q}{Q_{\max}} = 1 - 2\theta_0 \frac{J_1(\lambda_1)}{\lambda_1} = 1 - 2 \times 0.41 \frac{0.430}{0.970} = 0.636$$

and thus

$$Q = 0.636 Q_{\max} = 0.636 \times (47{,}354 \text{ kJ}) = \mathbf{30{,}120 \text{ kJ}}$$

Discussion The slight difference between the two results is due to the reading error of the charts.

4–3 ▪ TRANSIENT HEAT CONDUCTION IN SEMI-INFINITE SOLIDS

A semi-infinite solid is an idealized body that has a *single plane surface* and extends to infinity in all directions, as shown in Fig. 4–22. This idealized body is used to indicate that the temperature change in the part of the body in which we are interested (the region close to the surface) is due to the thermal conditions on a single surface. The earth, for example, can be considered to be a semi-infinite medium in determining the variation of temperature near its surface. Also, a thick wall can be modeled as a semi-infinite medium if all we are interested in is the variation of temperature in the region near one of the surfaces, and the other surface is too far to have any impact on the region of interest during the time of observation.

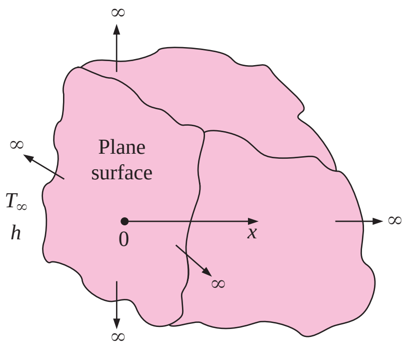


FIGURE 4–22
Schematic of a semi-infinite body.

Consider a semi-infinite solid that is at a uniform temperature T_i. At time $t = 0$, the surface of the solid at $x = 0$ is exposed to convection by a fluid at a constant temperature T_∞, with a heat transfer coefficient h. This problem can be formulated as a partial differential equation, which can be solved analytically for the transient temperature distribution $T(x, t)$. The solution obtained is presented in Fig. 4–23 graphically for the *nondimensionalized temperature* defined as

$$1 - \theta(x, t) = 1 - \frac{T(x, t) - T_\infty}{T_i - T_\infty} = \frac{T(x, t) - T_i}{T_\infty - T_i} \qquad \textbf{(4-21)}$$

against the dimensionless variable $x/(2\sqrt{\alpha t})$ for various values of the parameter $h\sqrt{\alpha t}/k$.

Note that the values on the vertical axis correspond to $x = 0$, and thus represent the surface temperature. The curve $h\sqrt{\alpha t}/k = \infty$ corresponds to $h \to \infty$, which corresponds to the case of *specified temperature* T_∞ at the surface at $x = 0$. That is, the case in which the surface of the semi-infinite body is suddenly brought to temperature T_∞ at $t = 0$ and kept at T_∞ at all times can be handled by setting h to infinity. The specified surface temperature case is closely

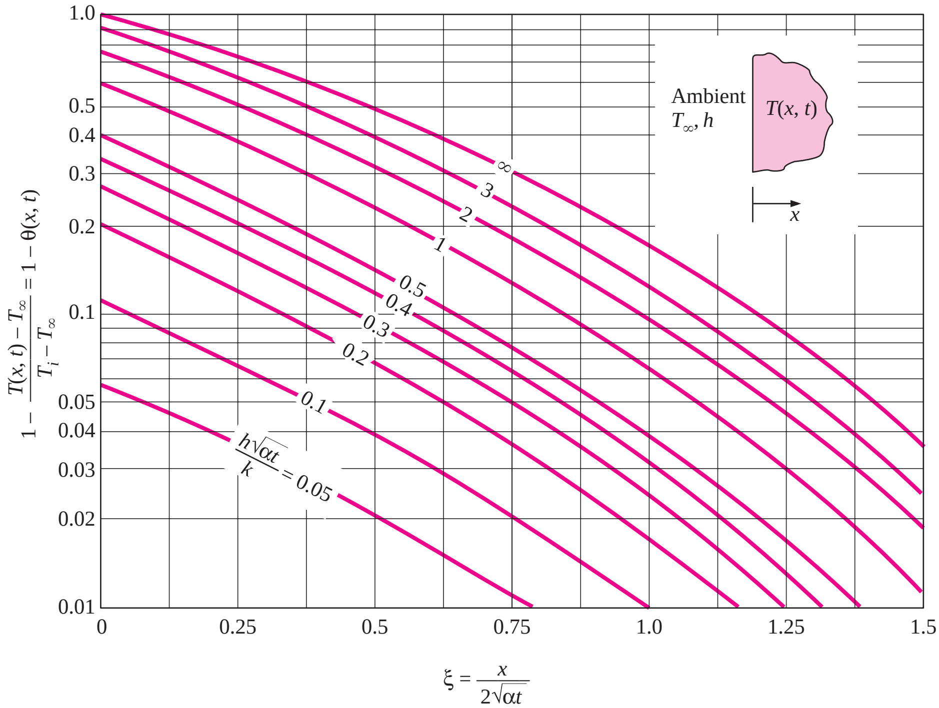


FIGURE 4–23

Variation of temperature with position and time in a semi-infinite solid initially at T_i subjected to convection to an environment at T_∞ with a convection heat transfer coefficient of h (from P. J. Schneider, Ref. 10).

approximated in practice when condensation or boiling takes place on the surface. For a *finite* heat transfer coefficient h, the surface temperature approaches the fluid temperature T_∞ as the time t approaches infinity.

The exact solution of the transient one-dimensional heat conduction problem in a semi-infinite medium that is initially at a uniform temperature of T_i and is suddenly subjected to convection at time $t = 0$ has been obtained, and is expressed as

$$\frac{T(x, t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\left[\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right] \quad \textbf{(4-22)}$$

where the quantity erfc (ξ) is the **complementary error function,** defined as

$$\text{erfc}\,(\xi) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\xi e^{-u^2}\,du \quad \textbf{(4-23)}$$

Despite its simple appearance, the integral that appears in the above relation cannot be performed analytically. Therefore, it is evaluated numerically for different values of ξ, and the results are listed in Table 4–3. For the special case of $h \to \infty$, the surface temperature T_s becomes equal to the fluid temperature T_∞, and Eq. 4–22 reduces to

$$\frac{T(x, t) - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad (4\text{-}24)$$

TABLE 4–3

The complementary error function

ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)
0.00	1.00000	0.38	0.5910	0.76	0.2825	1.14	0.1069	1.52	0.03159	1.90	0.00721
0.02	0.9774	0.40	0.5716	0.78	0.2700	1.16	0.10090	1.54	0.02941	1.92	0.00662
0.04	0.9549	0.42	0.5525	0.80	0.2579	1.18	0.09516	1.56	0.02737	1.94	0.00608
0.06	0.9324	0.44	0.5338	0.82	0.2462	1.20	0.08969	1.58	0.02545	1.96	0.00557
0.08	0.9099	0.46	0.5153	0.84	0.2349	1.22	0.08447	1.60	0.02365	1.98	0.00511
0.10	0.8875	0.48	0.4973	0.86	0.2239	1.24	0.07950	1.62	0.02196	2.00	0.00468
0.12	0.8652	0.50	0.4795	0.88	0.2133	1.26	0.07476	1.64	0.02038	2.10	0.00298
0.14	0.8431	0.52	0.4621	0.90	0.2031	1.28	0.07027	1.66	0.01890	2.20	0.00186
0.16	0.8210	0.54	0.4451	0.92	0.1932	1.30	0.06599	1.68	0.01751	2.30	0.00114
0.18	0.7991	0.56	0.4284	0.94	0.1837	1.32	0.06194	1.70	0.01612	2.40	0.00069
0.20	0.7773	0.58	0.4121	0.96	0.1746	1.34	0.05809	1.72	0.01500	2.50	0.00041
0.22	0.7557	0.60	0.3961	0.98	0.1658	1.36	0.05444	1.74	0.01387	2.60	0.00024
0.24	0.7343	0.62	0.3806	1.00	0.1573	1.38	0.05098	1.76	0.01281	2.70	0.00013
0.26	0.7131	0.64	0.3654	1.02	0.1492	1.40	0.04772	1.78	0.01183	2.80	0.00008
0.28	0.6921	0.66	0.3506	1.04	0.1413	1.42	0.04462	1.80	0.01091	2.90	0.00004
0.30	0.6714	0.68	0.3362	1.06	0.1339	1.44	0.04170	1.82	0.01006	3.00	0.00002
0.32	0.6509	0.70	0.3222	1.08	0.1267	1.46	0.03895	1.84	0.00926	3.20	0.00001
0.34	0.6306	0.72	0.3086	1.10	0.1198	1.48	0.03635	1.86	0.00853	3.40	0.00000
0.36	0.6107	0.74	0.2953	1.12	0.1132	1.50	0.03390	1.88	0.00784	3.60	0.00000

This solution corresponds to the case when the temperature of the exposed surface of the medium is suddenly raised (or lowered) to T_s at $t = 0$ and is maintained at that value at all times. Although the graphical solution given in Fig. 4–23 is a plot of the exact analytical solution given by Eq. 4–23, it is subject to reading errors, and thus is of limited accuracy.

EXAMPLE 4–6 **Minimum Burial Depth of Water Pipes to Avoid Freezing**

In areas where the air temperature remains below 0°C for prolonged periods of time, the freezing of water in underground pipes is a major concern. Fortunately, the soil remains relatively warm during those periods, and it takes weeks for the subfreezing temperatures to reach the water mains in the ground. Thus, the soil effectively serves as an insulation to protect the water from subfreezing temperatures in winter.

The ground at a particular location is covered with snow pack at −10°C for a continuous period of three months, and the average soil properties at that location are $k = 0.4$ W/m · °C and $\alpha = 0.15 \times 10^{-6}$ m²/s (Fig. 4–24). Assuming an initial uniform temperature of 15°C for the ground, determine the minimum burial depth to prevent the water pipes from freezing.

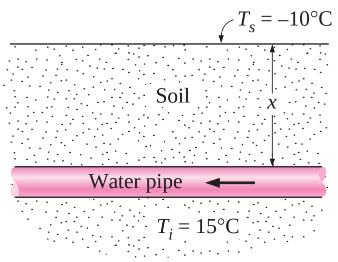


FIGURE 4–24
Schematic for Example 4–6.

SOLUTION The water pipes are buried in the ground to prevent freezing. The minimum burial depth at a particular location is to be determined.

Assumptions **1** The temperature in the soil is affected by the thermal conditions at one surface only, and thus the soil can be considered to be a semi-infinite medium with a specified surface temperature of −10°C. **2** The thermal properties of the soil are constant.

Properties The properties of the soil are as given in the problem statement.

Analysis The temperature of the soil surrounding the pipes will be 0°C after three months in the case of minimum burial depth. Therefore, from Fig. 4–23, we have

$$\left.\begin{aligned} &\frac{h\sqrt{\alpha t}}{k} = \infty \qquad \text{(since } h \to \infty) \\ &1 - \frac{T(x, t) - T_\infty}{T_i - T_\infty} = 1 - \frac{0 - (-10)}{15 - (-10)} = 0.6 \end{aligned}\right\} \xi = \frac{x}{2\sqrt{\alpha t}} = 0.36$$

We note that

$$t = (90 \text{ days})(24 \text{ h/day})(3600 \text{ s/h}) = 7.78 \times 10^6 \text{ s}$$

and thus

$$x = 2\xi\sqrt{\alpha t} = 2 \times 0.36\sqrt{(0.15 \times 10^{-6} \text{ m}^2\text{/s})(7.78 \times 10^6 \text{ s})} = \mathbf{0.77\ m}$$

Therefore, the water pipes must be buried to a depth of at least 77 cm to avoid freezing under the specified harsh winter conditions.

ALTERNATIVE SOLUTION The solution of this problem could also be determined from Eq. 4–24:

$$\frac{T(x, t) - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \longrightarrow \frac{0 - 15}{-10 - 15} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) = 0.60$$

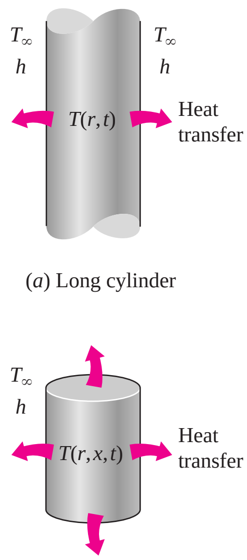
The argument that corresponds to this value of the complementary error function is determined from Table 4–3 to be $\xi = 0.37$. Therefore,

$$x = 2\xi\sqrt{\alpha t} = 2 \times 0.37\sqrt{(0.15 \times 10^{-6} \text{ m}^2\text{/s})(7.78 \times 10^6 \text{ s})} = \mathbf{0.80\ m}$$

Again, the slight difference is due to the reading error of the chart.

4–4 ▪ TRANSIENT HEAT CONDUCTION IN MULTIDIMENSIONAL SYSTEMS

The transient temperature charts presented earlier can be used to determine the temperature distribution and heat transfer in *one-dimensional* heat conduction problems associated with a large plane wall, a long cylinder, a sphere, and a semi-infinite medium. Using a superposition approach called the **product solution,** these charts can also be used to construct solutions for the *two-dimensional* transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, or a semi-infinite cylinder or plate, and even *three-dimensional* problems associated with geometries such as a rectangular prism or a semi-infinite rectangular bar, provided that *all* surfaces of the solid are subjected to convection to the *same* fluid at temperature



(*a*) Long cylinder

(*b*) Short cylinder (two-dimensional)

FIGURE 4–25
The temperature in a short cylinder exposed to convection from all surfaces varies in both the radial and axial directions, and thus heat is transferred in both directions.

T_∞, with the *same* heat transfer coefficient h, and the body involves no heat generation (Fig. 4–25). The solution in such multidimensional geometries can be expressed as the *product* of the solutions for the one-dimensional geometries whose intersection is the multidimensional geometry.

Consider a *short cylinder* of height a and radius r_o initially at a uniform temperature T_i. There is no heat generation in the cylinder. At time $t = 0$, the cylinder is subjected to convection from all surfaces to a medium at temperature T_∞ with a heat transfer coefficient h. The temperature within the cylinder will change with x as well as r and time t since heat transfer will occur from the top and bottom of the cylinder as well as its side surfaces. That is, $T = T(r, x, t)$ and thus this is a two-dimensional transient heat conduction problem. When the properties are assumed to be constant, it can be shown that the solution of this two-dimensional problem can be expressed as

$$\left(\frac{T(r, x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{short}\\ \text{cylinder}}} = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{plane}\\ \text{wall}}} \left(\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{infinite}\\ \text{cylinder}}} \tag{4-25}$$

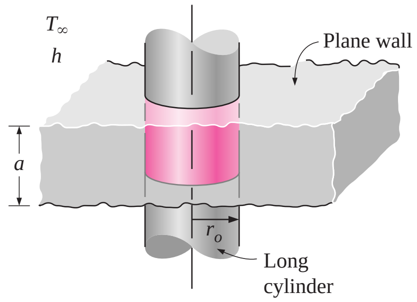


FIGURE 4–26
A short cylinder of radius r_o and height a is the *intersection* of a long cylinder of radius r_o and a plane wall of thickness a.

That is, the solution for the two-dimensional short cylinder of height a and radius r_o is equal to the *product* of the nondimensionalized solutions for the one-dimensional plane wall of thickness a and the long cylinder of radius r_o, which are the two geometries whose intersection is the short cylinder, as shown in Fig. 4–26. We generalize this as follows: *the solution for a multidimensional geometry is the product of the solutions of the one-dimensional geometries whose intersection is the multidimensional body.*

For convenience, the one-dimensional solutions are denoted by

$$\theta_{\text{wall}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{plane}\\ \text{wall}}}$$

$$\theta_{\text{cyl}}(r, t) = \left(\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{infinite}\\ \text{cylinder}}}$$

$$\theta_{\text{semi-inf}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{semi-infinite}\\ \text{solid}}} \tag{4-26}$$

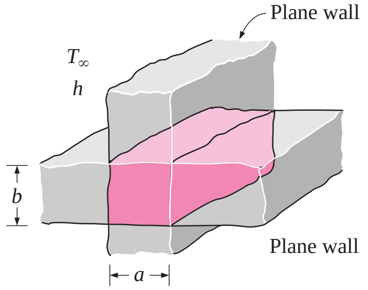


FIGURE 4–27
A long solid bar of rectangular profile $a \times b$ is the *intersection* of two plane walls of thicknesses a and b.

For example, the solution for a long solid bar whose cross section is an $a \times b$ rectangle is the intersection of the two infinite plane walls of thicknesses a and b, as shown in Fig. 4–27, and thus the transient temperature distribution for this rectangular bar can be expressed as

$$\left(\frac{T(x, y, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{rectangular}\\ \text{bar}}} = \theta_{\text{wall}}(x, t)\theta_{\text{wall}}(y, t) \tag{4-27}$$

The proper forms of the product solutions for some other geometries are given in Table 4–4. It is important to note that the x-coordinate is measured from the *surface* in a semi-infinite solid, and from the *midplane* in a plane wall. The radial distance r is always measured from the centerline.

Note that the solution of a *two-dimensional* problem involves the product of *two* one-dimensional solutions, whereas the solution of a *three-dimensional* problem involves the product of *three* one-dimensional solutions.

A modified form of the product solution can also be used to determine the total transient heat transfer to or from a multidimensional geometry by using the one-dimensional values, as shown by L. S. Langston in 1982. The