Note that a heat transfer problem may involve different kinds of boundary conditions on different surfaces. For example, a plate may be subject to *heat flux* on one surface while losing or gaining heat by *convection* from the other surface. Also, the two boundary conditions in a direction may be specified *at the same boundary,* while no condition is imposed on the other boundary. For example, specifying the temperature and heat flux at $x = 0$ of a plate of thickness L will result in a unique solution for the one-dimensional steady temperature distribution in the plate, including the value of temperature at the surface $x = L$. Although not necessary, there is nothing wrong with specifying more than two boundary conditions in a specified direction, provided that there is no contradiction. The extra conditions in this case can be used to verify the results.

2–5 ▪ SOLUTION OF STEADY ONE-DIMENSIONAL HEAT CONDUCTION PROBLEMS

So far we have derived the differential equations for heat conduction in various coordinate systems and discussed the possible boundary conditions. A heat conduction problem can be formulated by specifying the applicable differential equation and a set of proper boundary conditions.

In this section we will solve a wide range of heat conduction problems in rectangular, cylindrical, and spherical geometries. We will limit our attention to problems that result in *ordinary differential equations* such as the *steady one-dimensional* heat conduction problems. We will also assume *constant thermal conductivity,* but will consider variable conductivity later in this chapter. If you feel rusty on differential equations or haven't taken differential equations yet, no need to panic. *Simple integration* is all you need to solve the steady one-dimensional heat conduction problems.

The solution procedure for solving heat conduction problems can be summarized as (1) *formulate* the problem by obtaining the applicable differential equation in its simplest form and specifying the boundary conditions, (2) obtain the *general solution* of the differential equation, and (3) apply the *boundary conditions* and determine the arbitrary constants in the general solution (Fig. 2–40). This is demonstrated below with examples.

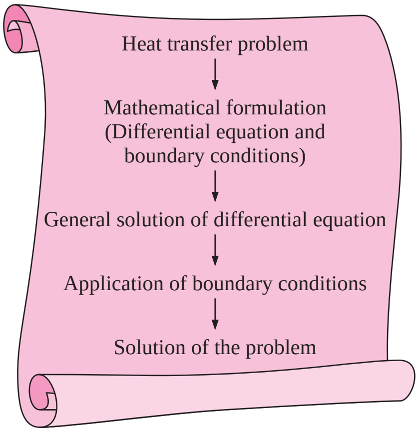


FIGURE 2–40
Basic steps involved in the solution of heat transfer problems.

EXAMPLE 2–11 Heat Conduction in a Plane Wall

Consider a large plane wall of thickness $L = 0.2$ m, thermal conductivity $k = 1.2$ W/m · °C, and surface area $A = 15$ m^2. The two sides of the wall are maintained at constant temperatures of $T_1 = 120$°C and $T_2 = 50$°C, respectively, as shown in Figure 2–41. Determine (*a*) the variation of temperature within the wall and the value of temperature at $x = 0.1$ m and (*b*) the rate of heat conduction through the wall under steady conditions.

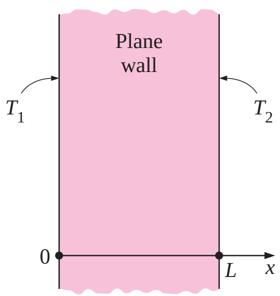


FIGURE 2–41
Schematic for Example 2–11.

SOLUTION A plane wall with specified surface temperatures is given. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat conduction is steady. **2** Heat conduction is one-dimensional since the wall is large relative to its thickness and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 1.2$ W/m · °C.

Analysis (*a*) Taking the direction normal to the surface of the wall to be the *x*-direction, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 120°\text{C}$$
$$T(L) = T_2 = 50°\text{C}$$

The differential equation is linear and second order, and a quick inspection of it reveals that it has a single term involving derivatives and no terms involving the unknown function T as a factor. Thus, it can be solved by direct integration. Noting that an integration reduces the order of a derivative by one, the general solution of the differential equation above can be obtained by two simple successive integrations, each of which introduces an integration constant.

Integrating the differential equation once with respect to x yields

$$\frac{dT}{dx} = C_1$$

where C_1 is an arbitrary constant. Notice that the order of the derivative went down by one as a result of integration. As a check, if we take the derivative of this equation, we will obtain the original differential equation. This equation is not the solution yet since it involves a derivative.

Integrating one more time, we obtain

$$T(x) = C_1x + C_2$$

which is the general solution of the differential equation (Fig. 2–42). The general solution in this case resembles the general formula of a straight line whose slope is C_1 and whose value at $x = 0$ is C_2. This is not surprising since the second derivative represents the change in the slope of a function, and a zero second derivative indicates that the slope of the function remains constant. Therefore, *any straight line* is a solution of this differential equation.

The general solution contains two unknown constants C_1 and C_2, and thus we need two equations to determine them uniquely and obtain the specific solution. These equations are obtained by forcing the general solution to satisfy the specified boundary conditions. The application of each condition yields one equation, and thus we need to specify two conditions to determine the constants C_1 and C_2.

When applying a boundary condition to an equation, *all occurrences of the dependent and independent variables and any derivatives are replaced by the specified values.* Thus the only unknowns in the resulting equations are the arbitrary constants.

The first boundary condition can be interpreted as *in the general solution, replace all the x's by zero and* $T(x)$ *by* T_1. That is (Fig. 2–43),

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

Differential equation:

$$\frac{d^2T}{dx^2} = 0$$

Integrate:

$$\frac{dT}{dx} = C_1$$

Integrate again:

$$T(x) = C_1x + C_2$$

General solution ↗ $T(x)$; Arbitrary constants ↗ C_1, C_2

FIGURE 2–42
Obtaining the general solution of a simple second order differential equation by integration.

Boundary condition:

$$T(0) = T_1$$

General solution:

$$T(x) = C_1x + C_2$$

Applying the boundary condition:

$$\underbrace{T(x)}_{\substack{\uparrow\\0\\T_1}} = C_1\underbrace{x}_{\substack{\uparrow\\0}} + C_2$$

Substituting:

$$T_1 = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

It cannot involve x or $T(x)$ after the boundary condition is applied.

FIGURE 2–43
When applying a boundary condition to the general solution at a specified point, all occurrences of the dependent and independent variables should be replaced by their specified values at that point.

The second boundary condition can be interpreted as *in the general solution, replace all the x's by L and T(x) by T_2*. That is,

$$T(L) = C_1L + C_2 \quad \rightarrow \quad T_2 = C_1L + T_1 \quad \rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

Substituting the C_1 and C_2 expressions into the general solution, we obtain

$$T(x) = \frac{T_2 - T_1}{L}x + T_1 \tag{2-56}$$

which is the desired solution since it satisfies not only the differential equation but also the two specified boundary conditions. That is, differentiating Eq. 2–56 with respect to x twice will give d^2T/dx^2, which is the given differential equation, and substituting $x = 0$ and $x = L$ into Eq. 2–56 gives $T(0) = T_1$ and $T(L) = T_2$, respectively, which are the specified conditions at the boundaries.

Substituting the given information, the value of the temperature at $x = 0.1$ m is determined to be

$$T(0.1 \text{ m}) = \frac{(50 - 120)°\text{C}}{0.2 \text{ m}}(0.1 \text{ m}) + 120°\text{C} = \mathbf{85°C}$$

(*b*) The rate of heat conduction anywhere in the wall is determined from Fourier's law to be

$$\dot{Q}_{\text{wall}} = -kA\frac{dT}{dx} = -kAC_1 = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} \tag{2-57}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (1.2 \text{ W/m} \cdot °\text{C})(15 \text{ m}^2)\frac{(120 - 50)°\text{C}}{0.2 \text{ m}} = \mathbf{6300 \text{ W}}$$

Discussion Note that under steady conditions, the rate of heat conduction through a plane wall is constant.

EXAMPLE 2–12 A Wall with Various Sets of Boundary Conditions

Consider steady one-dimensional heat conduction in a large plane wall of thickness L and constant thermal conductivity k with no heat generation. Obtain expressions for the variation of temperature within the wall for the following pairs of boundary conditions (Fig. 2–44):

(*a*) $-k\dfrac{dT(0)}{dx} = \dot{q}_0 = 40 \text{ W/cm}^2$ and $T(0) = T_0 = 15°\text{C}$

(*b*) $-k\dfrac{dT(0)}{dx} = \dot{q}_0 = 40 \text{ W/cm}^2$ and $-k\dfrac{dT(L)}{dx} = \dot{q}_L = -25 \text{ W/cm}^2$

(*c*) $-k\dfrac{dT(0)}{dx} = \dot{q}_0 = 40 \text{ W/cm}^2$ and $-k\dfrac{dT(L)}{dx} = \dot{q}_0 = 40 \text{ W/cm}^2$

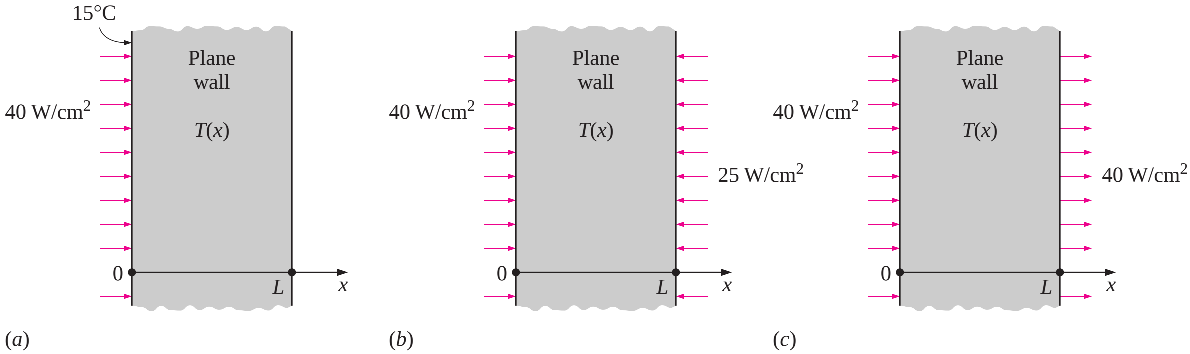


FIGURE 2–44
Schematic for Example 2–12.

SOLUTION This is a steady one-dimensional heat conduction problem with constant thermal conductivity and no heat generation in the medium, and the heat conduction equation in this case can be expressed as (Eq. 2–17)

$$\frac{d^2T}{dx^2} = 0$$

whose general solution was determined in the previous example by direct integration to be

$$T(x) = C_1x + C_2$$

where C_1 and C_2 are two arbitrary integration constants. The specific solutions corresponding to each specified pair of boundary conditions are determined as follows.

(*a*) In this case, both boundary conditions are specified at the same boundary at $x = 0$, and no boundary condition is specified at the other boundary at $x = L$. Noting that

$$\frac{dT}{dx} = C_1$$

the application of the boundary conditions gives

$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad \rightarrow \quad -kC_1 = \dot{q}_0 \quad \rightarrow \quad C_1 = -\frac{\dot{q}_0}{k}$$

and

$$T(0) = T_0 \quad \rightarrow \quad T_0 = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_0$$

Substituting, the specific solution in this case is determined to be

$$T(x) = -\frac{\dot{q}_0}{k} + T_0$$

Therefore, the two boundary conditions can be specified at the same boundary, and it is not necessary to specify them at different locations. In fact, the fundamental theorem of linear ordinary differential equations guarantees that a

Differential equation:

$$T''(x) = 0$$

General solution:

$$T(x) = C_1x + C_2$$

(*a*) *Unique solution:*

$$\left.\begin{aligned}-kT'(0) &= \dot{q}_0\\ T(0) &= T_0\end{aligned}\right\} T(x) = -\frac{\dot{q}_0}{k}x + T_0$$

(*b*) *No solution:*

$$\left.\begin{aligned}-kT'(0) &= \dot{q}_0\\ -kT'(L) &= \dot{q}_L\end{aligned}\right\} T(x) = \text{None}$$

(*c*) *Multiple solutions:*

$$\left.\begin{aligned}-kT'(0) &= \dot{q}_0\\ -kT'(L) &= \dot{q}_0\end{aligned}\right\} T(x) = -\frac{\dot{q}_0}{k}x + \underset{\text{Arbitrary}}{\underset{\uparrow}{C_2}}$$

FIGURE 2–45
A boundary-value problem may have a unique solution, infinitely many solutions, or no solutions at all.

unique solution exists when both conditions are specified at the same location. But no such guarantee exists when the two conditions are specified at different boundaries, as you will see below.

(*b*) In this case different heat fluxes are specified at the two boundaries. The application of the boundary conditions gives

$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad\rightarrow\quad -kC_1 = \dot{q}_0 \quad\rightarrow\quad C_1 = -\frac{\dot{q}_0}{k}$$

and

$$-k\frac{dT(L)}{dx} = \dot{q}_L \quad\rightarrow\quad -kC_1 = \dot{q}_L \quad\rightarrow\quad C_1 = -\frac{\dot{q}_L}{k}$$

Since $\dot{q}_0 \neq \dot{q}_L$ and the constant C_1 cannot be equal to two different things at the same time, there is no solution in this case. This is not surprising since this case corresponds to supplying heat to the plane wall from both sides and expecting the temperature of the wall to remain steady (not to change with time). This is impossible.

(*c*) In this case, the same values for heat flux are specified at the two boundaries. The application of the boundary conditions gives

$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad\rightarrow\quad -kC_1 = \dot{q}_0 \quad\rightarrow\quad C_1 = -\frac{\dot{q}_0}{k}$$

and

$$-k\frac{dT(L)}{dx} = \dot{q}_0 \quad\rightarrow\quad -kC_1 = \dot{q}_0 \quad\rightarrow\quad C_1 = -\frac{\dot{q}_0}{k}$$

Thus, both conditions result in the same value for the constant C_1, but no value for C_2. Substituting, the specific solution in this case is determined to be

$$T(x) = -\frac{\dot{q}_0}{k}x + C_2$$

which is not a unique solution since C_2 is arbitrary. This solution represents a family of straight lines whose slope is $-\dot{q}_0/k$. Physically, this problem corresponds to requiring the rate of heat supplied to the wall at $x = 0$ be equal to the rate of heat removal from the other side of the wall at $x = L$. But this is a consequence of the heat conduction through the wall being steady, and thus the second boundary condition does not provide any new information. So it is not surprising that the solution of this problem is not unique. The three cases discussed above are summarized in Figure 2–45.

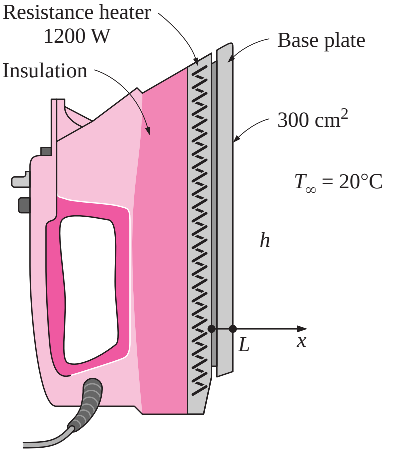


FIGURE 2–46
Schematic for Example 2–13.

EXAMPLE 2–13 Heat Conduction in the Base Plate of an Iron

Consider the base plate of a 1200-W household iron that has a thickness of L = 0.5 cm, base area of A = 300 cm^2, and thermal conductivity of k = 15 W/m · °C. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside, and the outer surface loses heat to the surroundings at T_∞ = 20°C by convection, as shown in Figure 2–46.

Taking the convection heat transfer coefficient to be $h = 80\ \text{W/m}^2 \cdot °\text{C}$ and disregarding heat loss by radiation, obtain an expression for the variation of temperature in the base plate, and evaluate the temperatures at the inner and the outer surfaces.

SOLUTION The base plate of an iron is considered. The variation of temperature in the plate and the surface temperatures are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since the surface area of the base plate is large relative to its thickness, and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation in the medium. **5** Heat transfer by radiation is negligible. **6** The upper part of the iron is well insulated so that the entire heat generated in the resistance wires is transferred to the base plate through its inner surface.

Properties The thermal conductivity is given to be $k = 15\ \text{W/m} \cdot °\text{C}$.

Analysis The inner surface of the base plate is subjected to uniform heat flux at a rate of

$$\dot{q}_0 = \frac{\dot{Q}_0}{A_{\text{base}}} = \frac{1200\ \text{W}}{0.03\ \text{m}^2} = 40{,}000\ \text{W/m}^2$$

The outer side of the plate is subjected to the convection condition. Taking the direction normal to the surface of the wall as the x-direction with its origin on the inner surface, the differential equation for this problem can be expressed as (Fig. 2–47)

$$\frac{d^2T}{dx^2} = 0$$

with the boundary conditions

$$-k\frac{dT(0)}{dx} = \dot{q}_0 = 40{,}000\ \text{W/m}^2$$

$$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty]$$

The general solution of the differential equation is again obtained by two successive integrations to be

$$\frac{dT}{dx} = C_1$$

and

$$T(x) = C_1x + C_2 \qquad \textbf{(a)}$$

where C_1 and C_2 are arbitrary constants. Applying the first boundary condition,

$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad \rightarrow \quad -kC_1 = \dot{q}_0 \quad \rightarrow \quad C_1 = -\frac{\dot{q}_0}{k}$$

Noting that $dT/dx = C_1$ and $T(L) = C_1L + C_2$, the application of the second boundary condition gives

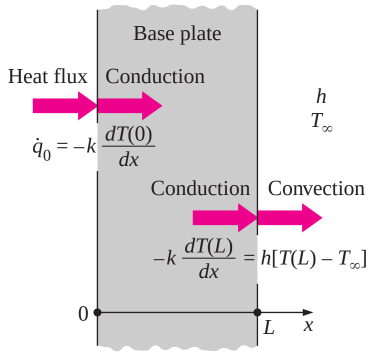


FIGURE 2–47
The boundary conditions on the base plate of the iron discussed in Example 2–13.

$$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty] \quad \rightarrow \quad -kC_1 = h[(C_1L + C_2) - T_\infty]$$

Substituting $C_1 = -\dot{q}_0/k$ and solving for C_2, we obtain

$$C_2 = T_\infty + \frac{\dot{q}_0}{h} + \frac{\dot{q}_0}{k}L$$

Now substituting C_1 and C_2 into the general solution (*a*) gives

$$T(x) = T_\infty + \dot{q}_0\left(\frac{L - x}{k} + \frac{1}{h}\right) \qquad \textbf{(b)}$$

which is the solution for the variation of the temperature in the plate. The temperatures at the inner and outer surfaces of the plate are determined by substituting $x = 0$ and $x = L$, respectively, into the relation (*b*):

$$\begin{aligned} T(0) &= T_\infty + \dot{q}_0\left(\frac{L}{k} + \frac{1}{h}\right) \\ &= 20°\text{C} + (40{,}000\ \text{W/m}^2)\left(\frac{0.005\ \text{m}}{15\ \text{W/m}\cdot°\text{C}} + \frac{1}{80\ \text{W/m}^2\cdot°\text{C}}\right) = \mathbf{533°C} \end{aligned}$$

and

$$T(L) = T_\infty + \dot{q}_0\left(0 + \frac{1}{h}\right) = 20°\text{C} + \frac{40{,}000\ \text{W/m}^2}{80\ \text{W/m}^2\cdot°\text{C}} = \mathbf{520°C}$$

Discussion Note that the temperature of the inner surface of the base plate will be 13°C higher than the temperature of the outer surface when steady operating conditions are reached. Also note that this heat transfer analysis enables us to calculate the temperatures of surfaces that we cannot even reach. This example demonstrates how the heat flux and convection boundary conditions are applied to heat transfer problems.

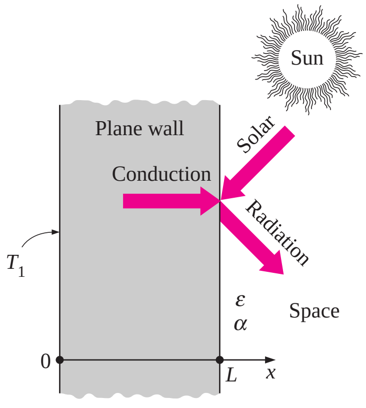


FIGURE 2–48
Schematic for Example 2–14.

EXAMPLE 2–14 Heat Conduction in a Solar Heated Wall

Consider a large plane wall of thickness $L = 0.06$ m and thermal conductivity $k = 1.2$ W/m · °C in space. The wall is covered with white porcelain tiles that have an emissivity of $\varepsilon = 0.85$ and a solar absorptivity of $\alpha = 0.26$, as shown in Figure 2–48. The inner surface of the wall is maintained at $T_1 = 300$ K at all times, while the outer surface is exposed to solar radiation that is incident at a rate of $\dot{q}_{solar} = 800$ W/m^2. The outer surface is also losing heat by radiation to deep space at 0 K. Determine the temperature of the outer surface of the wall and the rate of heat transfer through the wall when steady operating conditions are reached. What would your response be if no solar radiation was incident on the surface?

SOLUTION A plane wall in space is subjected to specified temperature on one side and solar radiation on the other side. The outer surface temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since the wall is large relative to its thickness, and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 1.2$ W/m · °C.

Analysis Taking the direction normal to the surface of the wall as the *x*-direction with its origin on the inner surface, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 300 \text{ K}$$
$$-k\frac{dT(L)}{dx} = \varepsilon\sigma[T(L)^4 - T_{\text{space}}^4] - \alpha\dot{q}_{\text{solar}}$$

where $T_{\text{space}} = 0$. The general solution of the differential equation is again obtained by two successive integrations to be

$$T(x) = C_1x + C_2 \qquad (a)$$

where C_1 and C_2 are arbitrary constants. Applying the first boundary condition yields

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

Noting that $dT/dx = C_1$ and $T(L) = C_1L + C_2 = C_1L + T_1$, the application of the second boundary conditions gives

$$-k\frac{dT(L)}{dx} = \varepsilon\sigma T(L)^4 - \alpha\dot{q}_{\text{solar}} \quad \rightarrow \quad -kC_1 = \varepsilon\sigma(C_1L + T_1)^4 - \alpha\dot{q}_{\text{solar}}$$

Although C_1 is the only unknown in this equation, we cannot get an explicit expression for it because the equation is nonlinear, and thus we cannot get a closed-form expression for the temperature distribution. This should explain why we do our best to avoid nonlinearities in the analysis, such as those associated with radiation.

Let us back up a little and denote the outer surface temperature by $T(L) = T_L$ instead of $T(L) = C_1L + T_1$. The application of the second boundary condition in this case gives

$$-k\frac{dT(L)}{dx} = \varepsilon\sigma T(L)^4 - \alpha\dot{q}_{\text{solar}} \quad \rightarrow \quad -kC_1 = \varepsilon\sigma T_L^4 - \alpha\dot{q}_{\text{solar}}$$

Solving for C_1 gives

$$C_1 = \frac{\alpha\dot{q}_{\text{solar}} - \varepsilon\sigma T_L^4}{k} \qquad (b)$$

Now substituting C_1 and C_2 into the general solution (*a*), we obtain

$$T(x) = \frac{\alpha\dot{q}_{\text{solar}} - \varepsilon\sigma T_L^4}{k}x + T_1 \qquad (c)$$

(1) *Rearrange the equation to be solved:*

$$T_L = 310.4 - 0.240975\left(\frac{T_L}{100}\right)^4$$

The equation is in the proper form since the left side consists of T_L only.

(2) *Guess the value of T_L, say 300 K, and substitute into the right side of the equation. It gives*

$$T_L = 290.2 \text{ K}$$

(3) *Now substitute this value of T_L into the right side of the equation and get*

$$T_L = 293.1 \text{ K}$$

(4) *Repeat step (3) until convergence to desired accuracy is achieved. The subsequent iterations give*

$$T_L = 292.6 \text{ K}$$
$$T_L = 292.7 \text{ K}$$
$$T_L = 292.7 \text{ K}$$

Therefore, the solution is $T_L = 292.7$ K. The result is independent of the initial guess.

FIGURE 2–49
A simple method of solving a nonlinear equation is to arrange the equation such that the unknown is alone on the left side while everything else is on the right side, and to iterate after an initial guess until convergence.

which is the solution for the variation of the temperature in the wall in terms of the unknown outer surface temperature T_L. At $x = L$ it becomes

$$T_L = \frac{\alpha \dot{q}_{\text{solar}} - \varepsilon\sigma T_L^4}{k} L + T_1 \qquad (d)$$

which is an implicit relation for the outer surface temperature T_L. Substituting the given values, we get

$$T_L = \frac{0.26 \times (800 \text{ W/m}^2) - 0.85 \times (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)\, T_L^4}{1.2 \text{ W/m} \cdot \text{K}} (0.06 \text{ m}) + 300 \text{ K}$$

which simplifies to

$$T_L = 310.4 - 0.240975\left(\frac{T_L}{100}\right)^4$$

This equation can be solved by one of the several nonlinear equation solvers available (or by the old fashioned trial-and-error method) to give (Fig. 2–49)

$$T_L = \mathbf{292.7 \text{ K}}$$

Knowing the outer surface temperature and knowing that it must remain constant under steady conditions, the temperature distribution in the wall can be determined by substituting the T_L value above into Eq. (*c*):

$$T(x) = \frac{0.26 \times (800 \text{ W/m}^2) - 0.85 \times (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(292.7 \text{ K})^4}{1.2 \text{ W/m} \cdot \text{K}} x + 300 \text{ K}$$

which simplifies to

$$T(x) = (-121.5 \text{ K/m})x + 300 \text{ K}$$

Note that the outer surface temperature turned out to be lower than the inner surface temperature. Therefore, the heat transfer through the wall will be toward the outside despite the absorption of solar radiation by the outer surface. Knowing both the inner and outer surface temperatures of the wall, the steady rate of heat conduction through the wall can be determined from

$$\dot{q} = k\frac{T_0 - T_L}{L} = (1.2 \text{ W/m} \cdot \text{K})\frac{(300 - 292.7) \text{ K}}{0.06 \text{ m}} = \mathbf{146 \text{ W/m}^2}$$

Discussion In the case of no incident solar radiation, the outer surface temperature, determined from Eq. (*d*) by setting $\dot{q}_{\text{solar}} = 0$, will be $T_L =$ **284.3 K.** It is interesting to note that the solar energy incident on the surface causes the surface temperature to increase by about 8 K only when the inner surface temperature of the wall is maintained at 300 K.

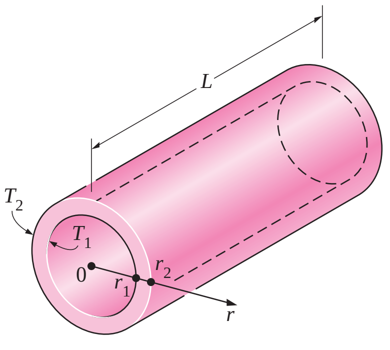


FIGURE 2–50
Schematic for Example 2–15.

EXAMPLE 2–15 Heat Loss through a Steam Pipe

Consider a steam pipe of length $L = 20$ m, inner radius $r_1 = 6$ cm, outer radius $r_2 = 8$ cm, and thermal conductivity $k = 20$ W/m · °C, as shown in Figure 2–50. The inner and outer surfaces of the pipe are maintained at average temperatures of $T_1 = 150$°C and $T_2 = 60$°C, respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.

SOLUTION A steam pipe is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction, and thus $T = T(r)$. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 20$ W/m · °C.

Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 150°\text{C}$$
$$T(r_2) = T_2 = 60°\text{C}$$

Integrating the differential equation once with respect to r gives

$$r\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Again integrating with respect to r gives (Fig. 2–51)

$$T(r) = C_1 \ln r + C_2 \qquad \textbf{(a)}$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in Eq. (*a*) with the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad C_1 \ln r_1 + C_2 = T_1$$
$$T(r_2) = T_2 \quad \rightarrow \quad C_1 \ln r_2 + C_2 = T_2$$

which are two equations in two unknowns, C_1 and C_2. Solving them simultaneously gives

$$C_1 = \frac{T_2 - T_1}{\ln(r_2/r_1)} \quad \text{and} \quad C_2 = T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)}\ln r_1$$

Substituting them into Eq. (*a*) and rearranging, the variation of temperature within the pipe is determined to be

$$T(r) = \left(\frac{\ln(r/r_1)}{\ln(r_2/r_1)}\right)(T_2 - T_1) + T_1 \qquad \textbf{(2-58)}$$

The rate of heat loss from the steam is simply the total rate of heat conduction through the pipe, and is determined from Fourier's law to be

Differential equation:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Integrate:

$$r\frac{dT}{dr} = C_1$$

Divide by r ($r \neq 0$):

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Integrate again:

$$T(r) = C_1 \ln r + C_2$$

which is the general solution.

FIGURE 2–51
Basic steps involved in the solution of the steady one-dimensional heat conduction equation in cylindrical coordinates.

$$\dot{Q}_{\text{cylinder}} = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{C_1}{r} = -2\pi kLC_1 = 2\pi kL\frac{T_1 - T_2}{\ln(r_2/r_1)} \quad \textbf{(2-59)}$$

The numerical value of the rate of heat conduction through the pipe is determined by substituting the given values

$$\dot{Q} = 2\pi(20 \text{ W/m}\cdot{}^\circ\text{C})(20 \text{ m})\frac{(150 - 60)^\circ\text{C}}{\ln(0.08/0.06)} = \mathbf{786\ kW}$$

DISCUSSION Note that the total rate of heat transfer through a pipe is constant, but the heat flux is not since it decreases in the direction of heat transfer with increasing radius since $\dot{q} = \dot{Q}/(2\pi rL)$.

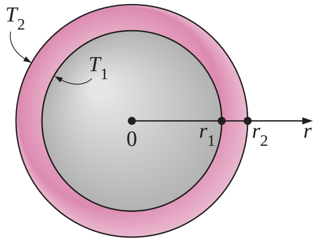


FIGURE 2–52
Schematic for Example 2–16.

EXAMPLE 2–16 Heat Conduction through a Spherical Shell

Consider a spherical container of inner radius r_1 = 8 cm, outer radius r_2 = 10 cm, and thermal conductivity k = 45 W/m · °C, as shown in Figure 2–52. The inner and outer surfaces of the container are maintained at constant temperatures of T_1 = 200°C and T_2 = 80°C, respectively, as a result of some chemical reactions occurring inside. Obtain a general relation for the temperature distribution inside the shell under steady conditions, and determine the rate of heat loss from the container.

SOLUTION A spherical container is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the midpoint, and thus $T = T(r)$. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be k = 45 W/m · °C.

Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 200^\circ\text{C}$$
$$T(r_2) = T_2 = 80^\circ\text{C}$$

Integrating the differential equation once with respect to r yields

$$r^2\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r^2 to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r^2}$$

Again integrating with respect to r gives

$$T(r) = -\frac{C_1}{r} + C_2 \qquad (a)$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in the relation above by the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad -\frac{C_1}{r_1} + C_2 = T_1$$

$$T(r_2) = T_2 \quad \rightarrow \quad -\frac{C_1}{r_2} + C_2 = T_2$$

which are two equations in two unknowns, C_1 and C_2 . Solving them simultaneously gives

$$C_1 = -\frac{r_1 r_2}{r_2 - r_1}(T_1 - T_2) \qquad \text{and} \qquad C_2 = \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1}$$

Substituting into Eq. (a), the variation of temperature within the spherical shell is determined to be

$$T(r) = \frac{r_1 r_2}{r(r_2 - r_1)}(T_1 - T_2) + \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1} \qquad \textbf{(2-60)}$$

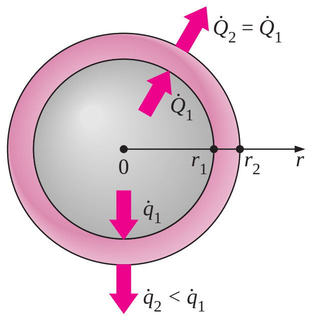
The rate of heat loss from the container is simply the total rate of heat conduction through the container wall and is determined from Fourier's law

$$\dot{Q}_{\text{sphere}} = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{C_1}{r^2} = -4\pi k C_1 = 4\pi k r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1} \qquad \textbf{(2-61)}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = 4\pi(45 \text{ W/m}\cdot{}^\circ\text{C})(0.08 \text{ m})(0.10 \text{ m})\frac{(200 - 80)^\circ\text{C}}{(0.10 - 0.08)\text{ m}} = \mathbf{27{,}140 \text{ W}}$$

Discussion Note that the total rate of heat transfer through a spherical shell is constant, but the heat flux, $\dot{q} = \dot{Q}/4\pi r^2$, is not since it decreases in the direction of heat transfer with increasing radius as shown in Figure 2–53.



$$\dot{q}_1 = \frac{\dot{Q}_1}{A_1} = \frac{27.14 \text{ kW}}{4\pi(0.08 \text{ m})^2} = 337.5 \text{ kW/m}^2$$

$$\dot{q}_2 = \frac{\dot{Q}_2}{A_2} = \frac{27.14 \text{ kW}}{4\pi(0.10 \text{ m})^2} = 216.0 \text{ kW/m}^2$$

FIGURE 2–53
During steady one-dimensional heat conduction in a spherical (or cylindrical) container, the total rate of heat transfer remains constant, but the heat flux decreases with increasing radius.

2–6 HEAT GENERATION IN A SOLID

Many practical heat transfer applications involve the conversion of some form of energy into *thermal* energy in the medium. Such mediums are said to involve internal *heat generation*, which manifests itself as a rise in temperature throughout the medium. Some examples of heat generation are *resistance heating* in wires, exothermic *chemical reactions* in a solid, and *nuclear reactions* in nuclear fuel rods where electrical, chemical, and nuclear energies are converted to heat, respectively (Fig. 2–54). The absorption of radiation throughout the volume of a semitransparent medium such as water can also be considered as heat generation within the medium, as explained earlier.

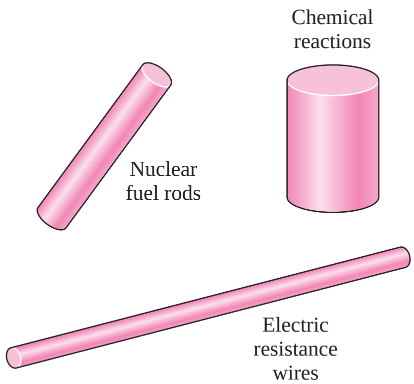


FIGURE 2–54
Heat generation in solids is commonly encountered in practice.