double-valued function $Z^{1/2}$. Each branch of $Z^{1/2}$ yields a branch of $(z - z_0)^{1/2}$. More precisely, when $Z = Re^{i\theta}$, branches of $Z^{1/2}$ are

$$Z^{1/2} = \sqrt{R}\exp\frac{i\theta}{2} \qquad (R > 0, \alpha < \theta < \alpha + 2\pi),$$

according to equation (8) in Sec. 97. Hence if we write

$$R = |z - z_0|, \quad \Theta = \text{Arg}\,(z - z_0), \quad \text{and} \quad \theta = \arg(z - z_0),$$

two branches of $(z - z_0)^{1/2}$ are

$$G_0(z) = \sqrt{R}\exp\frac{i\Theta}{2} \qquad (R > 0, -\pi < \Theta < \pi) \tag{1}$$

and

$$g_0(z) = \sqrt{R}\exp\frac{i\theta}{2} \qquad (R > 0, 0 < \theta < 2\pi). \tag{2}$$

The branch of $Z^{1/2}$ that was used in writing $G_0(z)$ is defined at all points in the Z plane except for the origin and points on the ray Arg $Z = \pi$. The transformation $w = G_0(z)$ is, therefore, a one to one mapping of the domain

$$|z - z_0| > 0, \quad -\pi < \text{Arg}\,(z - z_0) < \pi$$

onto the right half Re $w > 0$ of the w plane (Fig. 126). The transformation $w = g_0(z)$ maps the domain

$$|z - z_0| > 0, \quad 0 < \arg(z - z_0) < 2\pi$$

in a one to one manner onto the upper half plane Im $w > 0$.

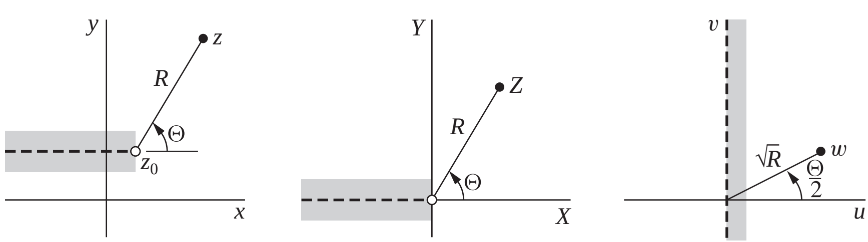

FIGURE 126
$w = G_0(z)$.

EXAMPLE 2. For an instructive but less elementary example, we now consider the double-valued function $(z^2 - 1)^{1/2}$. Using established properties of logarithms, we can write

$$(z^2 - 1)^{1/2} = \exp\left[\frac{1}{2}\log(z^2 - 1)\right] = \exp\left[\frac{1}{2}\log(z - 1) + \frac{1}{2}\log(z + 1)\right],$$

or

$$(z^2 - 1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2} \qquad (z \neq \pm 1). \tag{3}$$

Consequently, if $f_1(z)$ is a branch of $(z-1)^{1/2}$ defined on a domain D_1 and $f_2(z)$ is a branch of $(z+1)^{1/2}$ defined on a domain D_2, the product $f(z) = f_1(z) f_2(z)$ is a branch of $(z^2-1)^{1/2}$ defined at all points lying in both D_1 and D_2.

In order to obtain a specific branch of $(z^2-1)^{1/2}$, we use the branch of $(z-1)^{1/2}$ and the branch of $(z+1)^{1/2}$ given by equation (2). If we write

$$r_1 = |z-1| \quad \text{and} \quad \theta_1 = \arg(z-1),$$

that branch of $(z-1)^{1/2}$ is

$$f_1(z) = \sqrt{r_1} \exp \frac{i\theta_1}{2} \qquad (r_1 > 0, 0 < \theta_1 < 2\pi).$$

The branch of $(z+1)^{1/2}$ given by equation (2) is

$$f_2(z) = \sqrt{r_2} \exp \frac{i\theta_2}{2} \qquad (r_2 > 0, 0 < \theta_2 < 2\pi),$$

where

$$r_2 = |z+1| \quad \text{and} \quad \theta_2 = \arg(z+1).$$

The product of these two branches is, therefore, the branch f of $(z^2-1)^{1/2}$ defined by means of the equation

$$f(z) = \sqrt{r_1 r_2} \exp \frac{i(\theta_1 + \theta_2)}{2}, \tag{4}$$

where

$$r_k > 0, \quad 0 < \theta_k < 2\pi \qquad (k = 1, 2).$$

As illustrated in Fig. 127, the branch f is defined everywhere in the z plane except on the ray $r_2 \geq 0, \theta_2 = 0$, which is the portion $x \geq -1$ of the x axis.

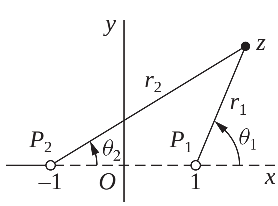

FIGURE 127

The branch f of $(z^2-1)^{1/2}$ given in equation (4) can be extended to a function

$$F(z) = \sqrt{r_1 r_2} \exp \frac{i(\theta_1 + \theta_2)}{2}, \tag{5}$$

where

$$r_k > 0, \quad 0 \le \theta_k < 2\pi \quad (k = 1, 2) \qquad \text{and} \qquad r_1 + r_2 > 2.$$

As we shall now see, this function is analytic everywhere in its domain of definition, which is the entire z plane except for the segment $-1 \le x \le 1$ of the x axis.

Since $F(z) = f(z)$ for all z in the domain of definition of F except on the ray $r_1 > 0, \theta_1 = 0$, we need only show that F is analytic on that ray. To do this, we form the product of the branches of $(z-1)^{1/2}$ and $(z+1)^{1/2}$ which are given by equation (1). That is, we consider the function

$$G(z) = \sqrt{r_1 r_2}\exp\frac{i(\Theta_1 + \Theta_2)}{2},$$

where

$$r_1 = |z-1|, \quad r_2 = |z+1|, \quad \Theta_1 = \text{Arg}\,(z-1), \quad \Theta_2 = \text{Arg}\,(z+1)$$

and where

$$r_k > 0, \quad -\pi < \Theta_k < \pi \qquad (k = 1, 2).$$

Observe that G is analytic in the entire z plane except for the ray $r_1 \ge 0, \Theta_1 = \pi$. Now $F(z) = G(z)$ when the point z lies above or on the ray $r_1 > 0, \Theta_1 = 0$; for then $\theta_k = \Theta_k$ $(k = 1, 2)$. When z lies below that ray, $\theta_k = \Theta_k + 2\pi$ $(k = 1, 2)$. Consequently, $\exp(i\theta_k/2) = -\exp(i\Theta_k/2)$; and this means that

$$\exp\frac{i(\theta_1+\theta_2)}{2} = \left(\exp\frac{i\theta_1}{2}\right)\left(\exp\frac{i\theta_2}{2}\right) = \exp\frac{i(\Theta_1+\Theta_2)}{2}.$$

So again, $F(z) = G(z)$. Since $F(z)$ and $G(z)$ are the same in a domain containing the ray $r_1 > 0, \Theta_1 = 0$ and since G is analytic in that domain, F is analytic there. Hence F *is analytic everywhere except on the line segment* P_2P_1 *in* Fig. 127.

The function F defined by equation (5) cannot itself be extended to a function which is analytic at points on the line segment P_2P_1. This is because the value on the right in equation (5) jumps from $i\sqrt{r_1 r_2}$ to numbers near $-i\sqrt{r_1 r_2}$ as the point z moves downward across that line segment, and the extension would not even be continuous there.

The transformation $w = F(z)$ is, as we shall see, a one to one mapping of the domain D_z consisting of all points in the z plane except those on the line segment P_2P_1 onto the domain D_w consisting of the entire w plane with the exception of the segment $-1 \le v \le 1$ of the v axis (Fig. 128).

Before verifying this, we note that if $z = iy$ $(y > 0)$, then

$$r_1 = r_2 > 1 \quad \text{and} \quad \theta_1 + \theta_2 = \pi;$$

hence the positive y axis is mapped by $w = F(z)$ onto that part of the v axis for which $v > 1$. The negative y axis is, moreover, mapped onto that part of the v axis for which $v < -1$. Each point in the upper half $y > 0$ of the domain D_z is mapped into the upper half $v > 0$ of the w plane, and each point in the lower half $y < 0$

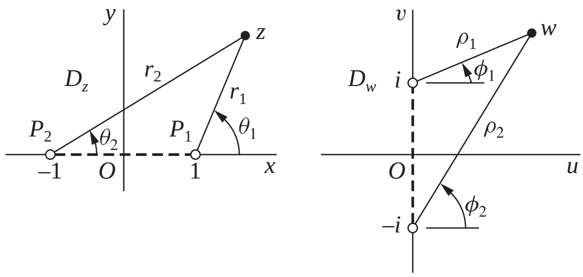

FIGURE 128
$w = F(z)$.

of the domain D_z is mapped into the lower half $v < 0$ of the w plane. Also, the ray $r_1 > 0, \theta_1 = 0$ is mapped onto the positive real axis in the w plane, and the ray $r_2 > 0, \theta_2 = \pi$ is mapped onto the negative real axis there.

To show that the transformation $w = F(z)$ is *one to one*, we observe that if $F(z_1) = F(z_2)$, then $z_1^2 - 1 = z_2^2 - 1$. From this, it follows that $z_1 = z_2$ or $z_1 = -z_2$. However, because of the manner in which F maps the upper and lower halves of the domain D_z, as well as the portions of the real axis lying in D_z, the case $z_1 = -z_2$ is impossible. Thus, if $F(z_1) = F(z_2)$, then $z_1 = z_2$; and F is one to one.

We can show that F maps the domain D_z *onto* the domain D_w by finding a function H mapping D_w into D_z with the property that if $z = H(w)$, then $w = F(z)$. This will show that for any point w in D_w, there exists a point z in D_z such that $F(z) = w$; that is, the mapping F is onto. The mapping H will be the inverse of F.

To find H, we first note that if w is a value of $(z^2 - 1)^{1/2}$ for a specific z, then $w^2 = z^2 - 1$; and z is, therefore, a value of $(w^2 + 1)^{1/2}$ for that w. The function H will be a branch of the double-valued function

$$(w^2 + 1)^{1/2} = (w - i)^{1/2}(w + i)^{1/2} \qquad (w \neq \pm i).$$

Following our procedure for obtaining the function $F(z)$, we write $w - i = \rho_1 \exp(i\phi_1)$ and $w + i = \rho_2 \exp(i\phi_2)$. (See Fig. 128.) With the restrictions

$$\rho_k > 0, \ -\frac{\pi}{2} \leq \phi_k < \frac{3\pi}{2} \quad (k = 1, 2) \qquad \text{and} \qquad \rho_1 + \rho_2 > 2,$$

we then write

$$H(w) = \sqrt{\rho_1 \rho_2} \exp \frac{i(\phi_1 + \phi_2)}{2}, \tag{6}$$

the domain of definition being D_w. The transformation $z = H(w)$ maps points of D_w lying above or below the u axis onto points above or below the x axis, respectively. It maps the positive u axis into that part of the x axis where $x > 1$ and the negative u axis into that part of the negative x axis where $x < -1$. If $z = H(w)$, then $z^2 = w^2 + 1$; and so $w^2 = z^2 - 1$. Since z is in D_z and since $F(z)$ and $-F(z)$ are the two values of $(z^2 - 1)^{1/2}$ for a point in D_z, we see that $w = F(z)$ or $w = -F(z)$. But it is evident from the manner in which F and H map the upper and lower halves of their domains of definition, including the portions of the real axes lying in those domains, that $w = F(z)$.

Mappings by branches of double-valued functions

$$w = (z^2 + Az + B)^{1/2} = [(z - z_0)^2 - z_1^2]^{1/2} \qquad (z_1 \neq 0), \tag{7}$$

where $A = -2z_0$ and $B = z_0^2 = z_1^2$, can be treated with the aid of the results found for the function F in Example 2 just above and the successive transformations

$$Z = \frac{z - z_0}{z_1}, \quad W = (Z^2 - 1)^{1/2}, \quad w = z_1 W. \tag{8}$$

EXERCISES

1. The branch F of $(z^2 - 1)^{1/2}$ in Example 2, Sec. 98, was defined in terms of the coordinates $r_1, r_2, \theta_1, \theta_2$. Explain geometrically why the conditions $r_1 > 0, 0 < \theta_1 + \theta_2 < \pi$ describe the first quadrant $x > 0, y > 0$ of the z plane. Then show that $w = F(z)$ maps that quadrant onto the first quadrant $u > 0, v > 0$ of the w plane.

Suggestion: To show that the quadrant $x > 0, y > 0$ in the z plane is described, note that $\theta_1 + \theta_2 = \pi$ at each point on the positive y axis and that $\theta_1 + \theta_2$ decreases as a point z moves to the right along a ray $\theta_2 = c$ $(0 < c < \pi/2)$.

2. For the mapping $w = F(z)$ of the first quadrant in the z plane onto the first quadrant in the w plane in Exercise 1, show that

$$u = \frac{1}{\sqrt{2}}\sqrt{r_1 r_2 + x^2 - y^2 - 1} \quad \text{and} \quad v = \frac{1}{\sqrt{2}}\sqrt{r_1 r_2 - x^2 + y^2 + 1},$$

where

$$(r_1 r_2)^2 = (x^2 + y^2 + 1)^2 - 4x^2,$$

and that the image of the portion of the hyperbola $x^2 - y^2 = 1$ in the first quadrant is the ray $v = u$ $(u > 0)$.

3. Show that in Exercise 2 the domain D that lies under the hyperbola and in the first quadrant of the z plane is described by the conditions $r_1 > 0, 0 < \theta_1 + \theta_2 < \pi/2$. Then show that the image of D is the octant $0 < v < u$. Sketch the domain D and its image.

4. Let F be the branch of $(z^2 - 1)^{1/2}$ that was defined in Example 2, Sec. 98, and let $z_0 = r_0 \exp(i\theta_0)$ be a fixed complex number, where $r_0 > 0$ and $0 \le \theta_0 < 2\pi$. Show that a branch F_0 of $(z^2 - z_0^2)^{1/2}$ whose branch cut is the line segment between the points z_0 and $-z_0$ can be written $F_0(z) = z_0 F(Z)$, where $Z = z/z_0$.

5. Write $z - 1 = r_1 \exp(i\theta_1)$ and $z + 1 = r_2 \exp(i\Theta_2)$, where

$$0 < \theta_1 < 2\pi \quad \text{and} \quad -\pi < \Theta_2 < \pi,$$

to define a branch of the function

(a) $(z^2 - 1)^{1/2}$; (b) $\left(\dfrac{z-1}{z+1}\right)^{1/2}$.

In each case, the branch cut should consist of the two rays $\theta_1 = 0$ and $\Theta_2 = \pi$.

6. Using the notation in Sec. 98, show that the function

$$w = \left(\frac{z-1}{z+1}\right)^{1/2} = \sqrt{\frac{r_1}{r_2}} \exp \frac{i(\theta_1 - \theta_2)}{2}$$

is a branch with the same domain of definition D_z and the same branch cut as the function $w = F(z)$ in that section. Show that this transformation maps D_z onto the right half plane $\rho > 0$, $-\pi/2 < \phi < \pi/2$, where the point $w = 1$ is the image of the point $z = \infty$. Also, show that the inverse transformation is

$$z = \frac{1 + w^2}{1 - w^2} \qquad (\text{Re}\, w > 0).$$

(Compare with Exercise 7, Sec. 97.)

7. Show that the transformation in Exercise 6 maps the region outside the unit circle $|z| = 1$ in the upper half of the z plane onto the region in the first quadrant of the w plane between the line $v = u$ and the u axis. Sketch the two regions.

8. Write $z = r\exp(i\Theta)$, $z - 1 = r_1 \exp(i\Theta_1)$, and $z + 1 = r_2 \exp(i\Theta_2)$, where the values of all three arguments lie between $-\pi$ and π. Then define a branch of the function $[z(z^2 - 1)]^{1/2}$ whose branch cut consists of the two segments $x \le -1$ and $0 \le x \le 1$ of the x axis.

99. RIEMANN SURFACES

The remaining two sections of this chapter constitute a brief introduction to the concept of a mapping defined on a *Riemann surface*, which is a generalization of the complex plane consisting of more than one sheet. The theory rests on the fact that at each point on such a surface only one value of a given multiple-valued function is assigned. The material in these two sections will not be used in the chapters to follow, and the reader may skip to Chap. 9 without disruption.

Once a Riemann surface is devised for a given function, the function is single-valued on the surface and the theory of single-valued functions applies there. Complexities arising because the function is multiple-valued are thus relieved by a geometric device. However, the description of those surfaces and the arrangement of proper connections between the sheets can become quite involved. We limit our attention to fairly simple examples and begin with a surface for $\log z$.

EXAMPLE 1. Corresponding to each nonzero number z, the multiple-valued function

$$\log z = \ln r + i\theta \tag{1}$$

has infinitely many values. To describe $\log z$ as a single-valued function, we replace the z plane, with the origin deleted, by a surface on which a new point is located whenever the argument of the number z is increased or decreased by 2π, or an integral multiple of 2π.

We treat the z plane, with the origin deleted, as a thin sheet R_0 which is cut along the positive half of the real axis. On that sheet, let θ range from 0 to 2π. Let a second sheet R_1 be cut in the same way and placed in front of the sheet R_0. The lower edge of the slit in R_0 is then joined to the upper edge of the slit in R_1. On R_1, the angle θ ranges from 2π to 4π; so, when z is represented by a point on R_1, the imaginary component of $\log z$ ranges from 2π to 4π.

A sheet R_2 is then cut in the same way and placed in front of R_1. The lower edge of the slit in R_1 is joined to the upper edge of the slit in this new sheet, and similarly for sheets $R_3, R_4, \ldots$. A sheet R_{-1} on which θ varies from 0 to -2π is cut and placed behind R_0, with the lower edge of its slit connected to the upper edge of the slit in R_0; the sheets $R_{-2}, R_{-3}, \ldots$ are constructed in like manner. The coordinates r and θ of a point on any sheet can be considered as polar coordinates of the projection of the point onto the original z plane, the angular coordinate θ being restricted to a definite range of 2π radians on each sheet.

Consider any continuous curve on this connected surface of infinitely many sheets. As a point z describes that curve, the values of $\log z$ vary continuously since θ, in addition to r, varies continuously; and $\log z$ now assumes just one value corresponding to each point on the curve. For example, as the point makes a complete cycle around the origin on the sheet R_0 over the path indicated in Fig. 129, the angle changes from 0 to 2π. As it moves across the ray $\theta = 2\pi$, the point passes to the sheet R_1 of the surface. As the point completes a cycle in R_1, the angle θ varies from 2π to 4π; and as it crosses the ray $\theta = 4\pi$, the point passes to the sheet R_2.

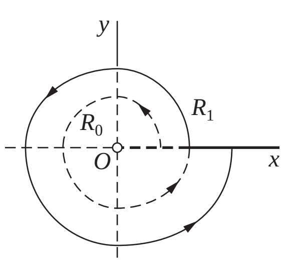

FIGURE 129

The surface described here is a Riemann surface for $\log z$. It is a connected surface of infinitely many sheets, arranged so that $\log z$ is a single-valued function of points on it.

The transformation $w = \log z$ maps the whole Riemann surface in a one to one manner onto the entire w plane. The image of the sheet R_0 is the strip $0 \le v \le 2\pi$ (see Example 3, Sec. 95). As a point z moves onto the sheet R_1 over the arc shown

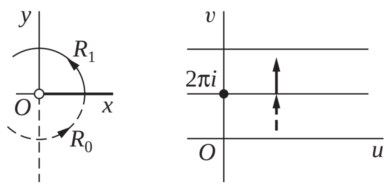

FIGURE 130

in Fig. 130, its image w moves upward across the line $v = 2\pi$, as indicated in that figure.

Note that $\log z$, defined on the sheet R_1, represents the analytic continuation (Sec. 27) of the single-valued analytic function

$$f(z) = \ln r + i\theta \qquad (0 < \theta < 2\pi)$$

upward across the positive real axis. In this sense, $\log z$ is not only a single-valued function of all points z on the Riemann surface but also an *analytic* function at all points there.

The sheets could, of course, be cut along the negative real axis or along any other ray from the origin, and properly joined along the slits, to form other Riemann surfaces for $\log z$.

EXAMPLE 2. Corresponding to each point in the z plane other than the origin, the square root function

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \tag{2}$$

has two values. A Riemann surface for $z^{1/2}$ is obtained by replacing the z plane with a surface made up of two sheets R_0 and R_1, each cut along the positive real axis and with R_1 placed in front of R_0. The lower edge of the slit in R_0 is joined to the upper edge of the slit in R_1, and the lower edge of the slit in R_1 is joined to the upper edge of the slit in R_0.

As a point z starts from the upper edge of the slit in R_0 and describes a continuous circuit around the origin in the counterclockwise direction (Fig. 131),

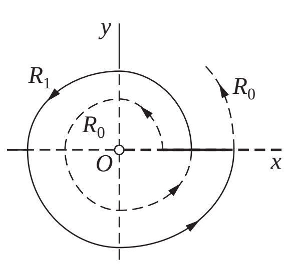

FIGURE 131

the angle θ increases from 0 to 2π. The point then passes from the sheet R_0 to the sheet R_1, where θ increases from 2π to 4π. As the point moves still further, it passes back to the sheet R_0, where the values of θ can vary from 4π to 6π or from 0 to 2π, a choice that does not affect the value of $z^{1/2}$, etc. Note that the value of $z^{1/2}$ at a point where the circuit passes from the sheet R_0 to the sheet R_1 is different from the value of $z^{1/2}$ at a point where the circuit passes from the sheet R_1 to the sheet R_0.

We have thus constructed a Riemann surface on which $z^{1/2}$ is single-valued for each nonzero z. In that construction, the edges of the sheets R_0 and R_1 are joined in pairs in such a way that the resulting surface is closed and connected. The points where two of the edges are joined are distinct from the points where the other two edges are joined. Thus it is physically impossible to build a model of that Riemann surface. In visualizing a Riemann surface, it is important to understand how we are to proceed when we arrive at an edge of a slit.

The origin is a special point on this Riemann surface. It is common to both sheets, and a curve around the origin on the surface must wind around it twice in order to be a closed curve. A point of this kind on a Riemann surface is called a *branch point.*

The image of the sheet R_0 under the transformation $w = z^{1/2}$ is the upper half of the w plane since the argument of w is $\theta/2$ on R_0, where $0 \le \theta/2 \le \pi$. Likewise, the image of the sheet R_1 is the lower half of the w plane. As defined on either sheet, the function is the analytic continuation, across the cut, of the function defined on the other sheet. In this respect, the single-valued function $z^{1/2}$ of points on the Riemann surface is analytic at all points except the origin.

EXERCISES

1. Describe the Riemann surface for $\log z$ obtained by cutting the z plane along the negative real axis. Compare this Riemann surface with the one obtained in Example 1, Sec. 99.

2. Determine the image under the transformation $w = \log z$ of the sheet R_n, where n is an arbitrary integer, of the Riemann surface for $\log z$ given in Example 1, Sec. 99.

3. Verify that under the transformation $w = z^{1/2}$, the sheet R_1 of the Riemann surface for $z^{1/2}$ given in Example 2, Sec. 99, is mapped onto the lower half of the w plane.

4. Describe the curve, on a Riemann surface for $z^{1/2}$, whose image is the entire circle $|w| = 1$ under the transformation $w = z^{1/2}$.

5. Let C denote the positively oriented circle $|z - 2| = 1$ on the Riemann surface described in Example 2, Sec. 99, for $z^{1/2}$, where the upper half of that circle lies on the sheet R_0 and the lower half on R_1. Note that for each point z on C, one can write

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \quad \text{where} \quad 4\pi - \frac{\pi}{2} < \theta < 4\pi + \frac{\pi}{2}.$$

State why it follows that

$$\int_C z^{1/2}\,dz = 0.$$

Generalize this result to fit the case of the other simple closed curves that cross from one sheet to another without enclosing the branch points. Generalize to other functions, thus extending the Cauchy–Goursat theorem to integrals of multiple-valued functions.

100. SURFACES FOR RELATED FUNCTIONS

We consider here Riemann surfaces for two composite functions involving simple polynomials and the square root function.

EXAMPLE 1. Let us describe a Riemann surface for the double-valued function

$$f(z) = (z^2 - 1)^{1/2} = \sqrt{r_1 r_2}\exp\frac{i(\theta_1 + \theta_2)}{2}, \tag{1}$$

where $z - 1 = r_1 \exp(i\theta_1)$ and $z + 1 = r_2 \exp(i\theta_2)$. A branch of this function, with the line segment P_2P_1 between the branch points $z = \pm 1$ serving as a branch cut (Fig. 132), was described in Example 2, Sec. 98. That branch is as written above, with the restrictions $r_k > 0, 0 \le \theta_k < 2\pi$ $(k = 1, 2)$ and $r_1 + r_2 > 2$. The branch is not defined on the segment P_2P_1.

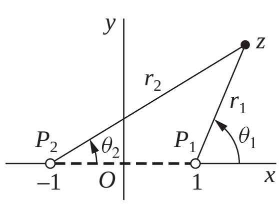

FIGURE 132

A Riemann surface for the double-valued function (1) must consist of two sheets R_0 and R_1. Let both sheets be cut along the segment P_2P_1. The lower edge of the slit in R_0 is then joined to the upper edge of the slit in R_1, and the lower edge in R_1 is joined to the upper edge in R_0.

On the sheet R_0, let the angles θ_1 and θ_2 range from 0 to 2π. If a point on the sheet R_0 describes a simple closed curve that encloses the segment P_2P_1 once in the counterclockwise direction, then both θ_1 and θ_2 change by the amount 2π upon the return of the point to its original position. The change in $(\theta_1 + \theta_2)/2$ is also 2π, and the value of f is unchanged. If a point starting on the sheet R_0 describes a path that passes twice around just the branch point $z = 1$, it crosses from the sheet

R_0 onto the sheet R_1 and then back onto the sheet R_0 before it returns to its original position. In this case, the value of θ_1 changes by the amount 4π, while the value of θ_2 does not change at all. Similarly, for a circuit twice around the point $z = -1$, the value of θ_2 changes by 4π, while the value of θ_1 remains unchanged. Again, the change in $(\theta_1 + \theta_2)/2$ is 2π; and the value of f is unchanged. Thus, on the sheet R_0, the range of the angles θ_1 and θ_2 may be extended by changing both θ_1 and θ_2 by the same integral multiple of 2π or by changing just one of the angles by a multiple of 4π. In either case, the total change in both angles is an even integral multiple of 2π.

To obtain the range of values for θ_1 and θ_2 on the sheet R_1, we note that if a point starts on the sheet R_0 and describes a path around just one of the branch points once, it crosses onto the sheet R_1 and does not return to the sheet R_0. In this case, the value of one of the angles is changed by 2π, while the value of the other remains unchanged. Hence, on the sheet R_1, one angle can range from 2π to 4π, while the other ranges from 0 to 2π. Their sum then ranges from 2π to 4π, and the value of $(\theta_1 + \theta_2)/2$, which is the argument of $f(z)$, ranges from π to 2π. Again, the range of the angles is extended by changing the value of just one of the angles by an integral multiple of 4π or by changing the value of both angles by the same integral multiple of 2π.

The double-valued function (1) may now be considered as a single-valued function of the points on the Riemann surface just constructed. The transformation $w = f(z)$ maps each of the sheets used in the construction of that surface onto the entire w plane.

EXAMPLE 2. Consider the double-valued function

$$f(z) = [z(z^2 - 1)]^{1/2} = \sqrt{rr_1r_2}\,\exp\frac{i(\theta + \theta_1 + \theta_2)}{2} \tag{2}$$

(Fig. 133). The points $z = 0, \pm 1$ are branch points of this function. We note that if the point z describes a circuit that includes all three of those points, the argument of $f(z)$ changes by the angle 3π and the value of the function thus changes. Consequently, a branch cut must run from one of those branch points to the point at infinity in order to describe a single-valued branch of f. Hence the point at infinity is also a branch point, as one can show by noting that the function $f(1/z)$ has a branch point at $z = 0$.

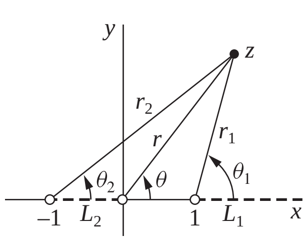

FIGURE 133

Let two sheets be cut along the line segment L_2 from $z = -1$ to $z = 0$ and along the part L_1 of the real axis to the right of the point $z = 1$. We specify that each of the three angles θ, θ_1, and θ_2 may range from 0 to 2π on the sheet R_0 and from 2π to 4π on the sheet R_1. We also specify that the angles corresponding to a point on either sheet may be changed by integral multiples of 2π in such a way that the sum of the three angles changes by an integral multiple of 4π. The value of the function f is, therefore, unaltered.

A Riemann surface for the double-valued function (2) is obtained by joining the lower edges in R_0 of the slits along L_1 and L_2 to the upper edges in R_1 of the slits along L_1 and L_2, respectively. The lower edges in R_1 of the slits along L_1 and L_2 are then joined to the upper edges in R_0 of the slits along L_1 and L_2, respectively. It is readily verified with the aid of Fig. 133 that one branch of the function is represented by its values at points on R_0 and the other branch at points on R_1.

EXERCISES

1. Describe a Riemann surface for the triple-valued function $w = (z-1)^{1/3}$, and point out which third of the w plane represents the image of each sheet of that surface.

2. Corresponding to each point on the Riemann surface described in Example 2, Sec. 100, for the function $w = f(z)$ in that example, there is just one value of w. Show that corresponding to each value of w, there are, in general, three points on the surface.

3. Describe a Riemann surface for the multiple-valued function

$$f(z) = \left(\frac{z-1}{z}\right)^{1/2}.$$

4. Note that the Riemann surface described in Example 1, Sec. 100, for $(z^2-1)^{1/2}$ is also a Riemann surface for the function

$$g(z) = z + (z^2-1)^{1/2}.$$

Let f_0 denote the branch of $(z^2-1)^{1/2}$ defined on the sheet R_0, and show that the branches g_0 and g_1 of g on the two sheets are given by the equations

$$g_0(z) = \frac{1}{g_1(z)} = z + f_0(z).$$

5. In Exercise 4, the branch f_0 of $(z^2-1)^{1/2}$ can be described by means of the equation

$$f_0(z) = \sqrt{r_1 r_2}\left(\exp\frac{i\theta_1}{2}\right)\left(\exp\frac{i\theta_2}{2}\right),$$

where θ_1 and θ_2 range from 0 to 2π and

$$z-1 = r_1 \exp(i\theta_1), \quad z+1 = r_2 \exp(i\theta_2).$$

Note that

$$2z = r_1 \exp(i\theta_1) + r_2 \exp(i\theta_2),$$

and show that the branch g_0 of the function $g(z) = z + (z^2 - 1)^{1/2}$ can be written in the form

$$g_0(z) = \frac{1}{2}\left(\sqrt{r_1}\exp\frac{i\theta_1}{2} + \sqrt{r_2}\exp\frac{i\theta_2}{2}\right)^2.$$

Find $g_0(z)\overline{g_0(z)}$ and note that $r_1 + r_2 \geq 2$ and $\cos[(\theta_1 - \theta_2)/2] \geq 0$ for all z, to prove that $|g_0(z)| \geq 1$. Then show that the transformation $w = z + (z^2 - 1)^{1/2}$ maps the sheet R_0 of the Riemann surface onto the region $|w| \geq 1$, the sheet R_1 onto the region $|w| \leq 1$, and the branch cut between the points $z = \pm 1$ onto the circle $|w| = 1$. Note that the transformation used here is an inverse of the transformation

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right).$$

CHAPTER

9

CONFORMAL MAPPING

In this chapter, we introduce and develop the concept of a conformal mapping, with emphasis on connections between such mappings and harmonic functions (Sec. 26). Applications to physical problems will follow in Chap. 10.

101. PRESERVATION OF ANGLES

Let C be a smooth arc (Sec. 39), represented by the equation

$$z = z(t) \qquad (a \le t \le b),$$

and let $f(z)$ be a function defined at all points z on C. The equation

$$w = f[z(t)] \qquad (a \le t \le b)$$

is a parametric representation of the image Γ of C under the transformation $w = f(z)$.

Suppose that C passes through a point $z_0 = z(t_0)$ $(a < t_0 < b)$ at which f is analytic and that $f'(z_0) \neq 0$. According to the chain rule verified in Exercise 5, Sec. 39, if $w(t) = f[z(t)]$, then

$$w'(t_0) = f'[z(t_0)]z'(t_0); \tag{1}$$

and this means that (see Sec. 8)

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0). \tag{2}$$

Statement (2) is useful in relating the directions of C and Γ at the points z_0 and $w_0 = f(z_0)$, respectively.

To be specific, let θ_0 denote a value of $\arg z'(t_0)$ and let ϕ_0 be a value of $\arg w'(t_0)$. According to the discussion of unit tangent vectors **T** near the end of Sec. 39, the number θ_0 is the angle of inclination of a directed line tangent to C at z_0 and ϕ_0 is the angle of inclination of a directed line tangent to Γ at the point $w_0 = f(z_0)$. (See Fig. 134.) In view of statement (2), there is a value ψ_0 of $\arg f'[z(t_0)]$ such that

$$\phi_0 = \psi_0 + \theta_0. \tag{3}$$

Thus $\phi_0 - \theta_0 = \psi_0$, and we find that the angles ϕ_0 and θ_0 differ by the *angle of rotation*

$$\psi_0 = \arg f'(z_0). \tag{4}$$

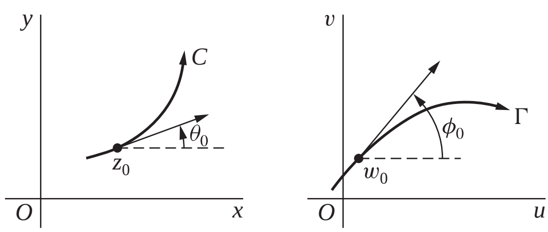

FIGURE 134
$\phi_0 = \psi_0 + \theta_0$.

Now let C_1 and C_2 be two smooth arcs passing through z_0, and let θ_1 and θ_2 be angles of inclination of directed lines tangent to C_1 and C_2, respectively, at z_0. We know from the preceding paragraph that the quantities

$$\phi_1 = \psi_0 + \theta_1 \quad \text{and} \quad \phi_2 = \psi_0 + \theta_2$$

are angles of inclination of directed lines tangent to the image curves Γ_1 and Γ_2, respectively, at the point $w_0 = f(z_0)$. Thus $\phi_2 - \phi_1 = \theta_2 - \theta_1$; that is, the angle $\phi_2 - \phi_1$ from Γ_1 to Γ_2 is the same in *magnitude* and *sense* as the angle $\theta_2 - \theta_1$ from C_1 to C_2. Those angles are denoted by α in Fig. 135.

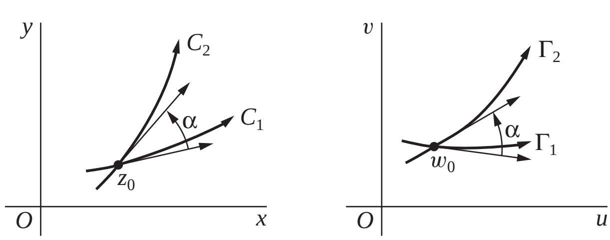

FIGURE 135

Because of this angle-preserving property, a transformation $w = f(z)$ is said to be *conformal* at a point z_0 if f is analytic there and $f'(z_0) \neq 0$. Such a transformation is actually conformal at each point in some neighborhood of z_0. For it must be analytic in a neighborhood of z_0 (Sec. 24); and since its derivative f' is continuous in that neighborhood (Sec. 52), Theorem 2 in Sec. 18 tells us that there is also a neighborhood of z_0 throughout which $f'(z) \neq 0$.

A transformation $w = f(z)$, defined on a domain D, is referred to as a conformal transformation, or *conformal mapping,* when it is conformal at each point in D. That is, the mapping is conformal in D if f is analytic in D and its derivative f' has no zeros there. Each of the elementary functions studied in Chap. 3 can be used to define a transformation that is conformal in some domain.

EXAMPLE 1. The mapping $w = e^z$ is conformal throughout the entire z plane since $(e^z)' = e^z \neq 0$ for each z. Consider any two lines $x = c_1$ and $y = c_2$ in the z plane, the first directed upward and the second directed to the right. According to Example 1 in Sec. 14, their images under the mapping $w = e^z$ are a positively oriented circle centered at the origin and a ray from the origin, respectively. As illustrated in Fig. 20 (Sec. 14), the angle between the lines at their point of intersection is a right angle in the negative direction, and the same is true of the angle between the circle and the ray at the corresponding point in the w plane. The conformality of the mapping $w = e^z$ is also illustrated in Figs. 7 and 8 of Appendix 2.

EXAMPLE 2. Consider two smooth arcs which are level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the real and imaginary components, respectively, of a function

$$f(z) = u(x, y) + iv(x, y),$$

and suppose that they intersect at a point z_0 where f is analytic and $f'(z_0) \neq 0$. The transformation $w = f(z)$ is conformal at z_0 and maps these arcs into the lines $u = c_1$ and $v = c_2$, which are orthogonal at the point $w_0 = f(z_0)$. According to our theory, then, the arcs must be orthogonal at z_0. This has already been verified and illustrated in Exercises 7 through 11 of Sec. 26.

A mapping that preserves the magnitude of the angle between two smooth arcs but not necessarily the sense is called an *isogonal mapping.*

EXAMPLE 3. The transformation $w = \overline{z}$, which is a reflection in the real axis, is isogonal but not conformal. If it is followed by a conformal transformation, the resulting transformation $w = f(\overline{z})$ is also isogonal but not conformal.

Suppose that f is not a constant function and is analytic at a point z_0. If, in addition, $f'(z_0) = 0$, then z_0 is called a *critical point* of the transformation $w = f(z)$.

EXAMPLE 4. The point $z_0 = 0$ is a critical point of the transformation

$$w = 1 + z^2,$$

which is a composition of the mappings

$$Z = z^2 \quad \text{and} \quad w = 1 + Z.$$

A ray $\theta = \alpha$ from the point $z_0 = 0$ is evidently mapped onto the ray from the point $w_0 = 1$ whose angle of inclination is 2α, and the angle between any two rays drawn from $z_0 = 0$ is doubled by the transformation.

More generally, it can be shown that if z_0 is a critical point of a transformation $w = f(z)$, there is an integer m ($m \geq 2$) such that the angle between any two smooth arcs passing through z_0 is multiplied by m under that transformation. The integer m is the smallest positive integer such that $f^{(m)}(z_0) \neq 0$. Verification of these facts is left to the exercises.

102. SCALE FACTORS

Another property of a transformation $w = f(z)$ that is conformal at a point z_0 is obtained by considering the modulus of $f'(z_0)$. From the definition of derivative and a property of limits involving moduli that was derived in Exercise 7, Sec. 18, we know that

$$(1) \qquad |f'(z_0)| = \left| \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

Now $|z - z_0|$ is the length of a line segment joining z_0 and z, and $|f(z) - f(z_0)|$ is the length of the line segment joining the points $f(z_0)$ and $f(z)$ in the w plane. Evidently, then, if z is near the point z_0, the ratio

$$\frac{|f(z) - f(z_0)|}{|z - z_0|}$$

of the two lengths is approximately the number $|f'(z_0)|$. Note that $|f'(z_0)|$ represents an expansion if it is greater than unity and a contraction if it is less than unity.

Although the angle of rotation $\arg f'(z)$ (Sec. 101) and the *scale factor* $|f'(z)|$ vary, in general, from point to point, it follows from the continuity of f' (see Sec. 52) that their values are approximately $\arg f'(z_0)$ and $|f'(z_0)|$ at points z near z_0. Hence the image of a small region in a neighborhood of z_0 *conforms* to the original region in the sense that it has approximately the same shape. A large region may, however, be transformed into a region that bears no resemblance to the original one.

EXAMPLE. When $f(z) = z^2$, the transformation

$$w = f(z) = x^2 - y^2 + i2xy$$

is conformal at the point $z = 1 + i$, where the half lines

$$y = x \ (x \geq 0) \quad \text{and} \quad x = 1 \ (y \geq 0)$$

intersect. We denote those half lines by C_1 and C_2 (Fig. 136), with positive sense upward. Observe that the angle from C_1 to C_2 is $\pi/4$ at their point of intersection. Since the image of a point $z = (x, y)$ is a point in the w plane whose rectangular coordinates are

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy,$$

the half line C_1 is transformed into the curve Γ_1 with parametric representation

$$u = 0, \quad v = 2x^2 \qquad (0 \leq x < \infty). \tag{2}$$

Thus Γ_1 is the upper half $v \geq 0$ of the v axis. The half line C_2 is transformed into the curve Γ_2 represented by the equations

$$u = 1 - y^2, \quad v = 2y \qquad (0 \leq y < \infty). \tag{3}$$

Hence Γ_2 is the upper half of the parabola $v^2 = -4(u - 1)$. Note that in each case, the positive sense of the image curve is upward.

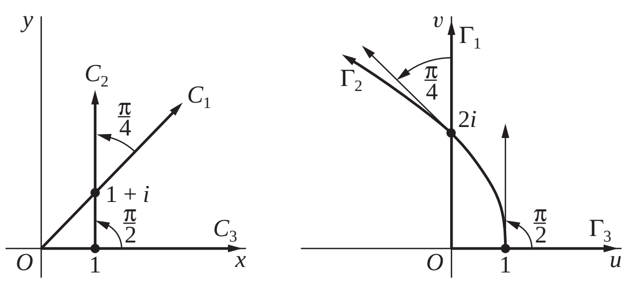

FIGURE 136
$w = z^2$.

If u and v are the variables in representation (3) for the image curve Γ_2, then

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = \frac{2}{-2y} = -\frac{2}{v}.$$

In particular, $dv/du = -1$ when $v = 2$. Consequently, the angle from the image curve Γ_1 to the image curve Γ_2 at the point $w = f(1 + i) = 2i$ is $\pi/4$, as required by the conformality of the mapping at $z = 1 + i$. The angle of rotation $\pi/4$ at the point $z = 1 + i$ is, of course, a value of

$$\arg[f'(1 + i)] = \arg[2(1 + i)] = \frac{\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The scale factor at that point is the number

$$|f'(1 + i)| = |2(1 + i)| = 2\sqrt{2}.$$

To illustrate how the angle of rotation and the scale factor can change from point to point, we note that they are 0 and 2, respectively, at the point $z = 1$ since $f'(1) = 2$. See Fig. 136, where the curves C_2 and Γ_2 are the ones just discussed and where the nonnegative x axis C_3 is transformed into the nonnegative u axis Γ_3.

103. LOCAL INVERSES

A transformation $w = f(z)$ that is conformal at a point z_0 has a *local inverse* there. That is, if $w_0 = f(z_0)$, then there exists a unique transformation $z = g(w)$, which is defined and analytic in a neighborhood N of w_0, such that $g(w_0) = z_0$ and $f[g(w)] = w$ for all points w in N. The derivative of $g(w)$ is, moreover,

$$g'(w) = \frac{1}{f'(z)}. \tag{1}$$

We note from expression (1) that the transformation $z = g(w)$ is itself conformal at w_0.

Assuming that $w = f(z)$ is, in fact, conformal at z_0, let us verify the existence of such an inverse, which is a direct consequence of results in advanced calculus.* As noted in Sec. 101, the conformality of the transformation $w = f(z)$ at z_0 implies that there is some neighborhood of z_0 throughout which f is analytic. Hence if we write

$$z = x + iy, \quad z_0 = x_0 + iy_0, \quad \text{and} \quad f(z) = u(x, y) + iv(x, y),$$

we know that there is a neighborhood of the point (x_0, y_0) throughout which the functions $u(x, y)$ and $v(x, y)$, along with their partial derivatives of all orders, are continuous (see Sec. 52).

Now the pair of equations

$$u = u(x, y), \quad v = v(x, y) \tag{2}$$

represents a transformation from the neighborhood just mentioned into the uv plane. Moreover, the determinant

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y,$$

which is known as the *Jacobian* of the transformation, is nonzero at the point (x_0, y_0). For, in view of the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$, one can write J as

$$J = (u_x)^2 + (v_x)^2 = |f'(z)|^2;$$

and $f'(z_0) \neq 0$ since the transformation $w = f(z)$ is conformal at z_0. The above continuity conditions on the functions $u(x, y)$ and $v(x, y)$ and their derivatives,

*The results from advanced calculus to be used here appear in, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 241–247, 1983.

together with this condition on the Jacobian, are sufficient to ensure the existence of a local inverse of transformation (2) at (x_0, y_0). That is, if

$$u_0 = u(x_0, y_0) \quad \text{and} \quad v_0 = v(x_0, y_0), \tag{3}$$

then there is a unique continuous transformation

$$x = x(u, v), \quad y = y(u, v), \tag{4}$$

defined on a neighborhood N of the point (u_0, v_0) and mapping that point onto (x_0, y_0), such that equations (2) hold when equations (4) hold. Also, in addition to being continuous, the functions (4) have continuous first-order partial derivatives satisfying the equations

$$x_u = \frac{1}{J}v_y, \quad x_v = -\frac{1}{J}u_y, \quad y_u = -\frac{1}{J}v_x, \quad y_v = \frac{1}{J}u_x \tag{5}$$

throughout N.

If we write $w = u + iv$ and $w_0 = u_0 + iv_0$, as well as

$$g(w) = x(u, v) + iy(u, v), \tag{6}$$

the transformation $z = g(w)$ is evidently the local inverse of the original transformation $w = f(z)$ at z_0. Transformations (2) and (4) can be written

$$u + iv = u(x, y) + iv(x, y) \quad \text{and} \quad x + iy = x(u, v) + iy(u, v);$$

and these last two equations are the same as

$$w = f(z) \quad \text{and} \quad z = g(w),$$

where g has the desired properties. Equations (5) can be used to show that g is analytic in N. Details are left to the exercises, where expression (1) for $g'(w)$ is also derived.

EXAMPLE. We know from Example 1, Sec. 101, that if $f(z) = e^z$, the transformation $w = f(z)$ is conformal everywhere in the z plane and, in particular, at the point $z_0 = 2\pi i$. The image of this choice of z_0 is the point $w_0 = 1$. When points in the w plane are expressed in the form $w = \rho \exp(i\phi)$, the local inverse at z_0 can be obtained by writing $g(w) = \log w$, where $\log w$ denotes the branch

$$\log w = \ln \rho + i\phi \qquad (\rho > 0, \pi < \theta < 3\pi)$$

of the logarithmic function, restricted to any neighborhood of w_0 that does not contain the origin. Observe that

$$g(1) = \ln 1 + i2\pi = 2\pi i$$

and that when w is in the neighborhood,

$$f[g(w)] = \exp(\log w) = w.$$

Also

$$g'(w) = \frac{d}{dw} \log w = \frac{1}{w} = \frac{1}{\exp z},$$

in accordance with equation (1).

Note that if the point $z_0 = 0$ is chosen, one can use the principal branch

$$\text{Log } w = \ln \rho + i\phi \qquad (\rho > 0, -\pi < \phi < \pi)$$

of the logarithmic function to define g. In this case, $g(1) = 0$.

EXERCISES

1. Determine the angle of rotation at the point $z_0 = 2 + i$ when $w = z^2$, and illustrate it for some particular curve. Show that the scale factor at that point is $2\sqrt{5}$.

2. What angle of rotation is produced by the transformation $w = 1/z$ at the point

(a) $z_0 = 1$; *(b)* $z_0 = i$?

Ans. *(a)* π; *(b)* 0.

3. Show that under the transformation $w = 1/z$, the images of the lines $y = x - 1$ and $y = 0$ are the circle $u^2 + v^2 - u - v = 0$ and the line $v = 0$, respectively. Sketch all four curves, determine corresponding directions along them, and verify the conformality of the mapping at the point $z_0 = 1$.

4. Show that the angle of rotation at a nonzero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n$ $(n = 1, 2, \ldots)$ is $(n - 1)\theta_0$. Determine the scale factor of the transformation at that point.

Ans. nr_0^{n-1}.

5. Show that the transformation $w = \sin z$ is conformal at all points except

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Note that this is in agreement with the mapping of directed line segments shown in Figs. 9, 10, and 11 of Appendix 2.

6. Find the local inverse of the transformation $w = z^2$ at the point

(a) $z_0 = 2$; *(b)* $z_0 = -2$; *(c)* $z_0 = -i$.

Ans. *(a)* $w^{1/2} = \sqrt{\rho}\, e^{i\phi/2}$ $(\rho > 0, -\pi < \phi < \pi)$;
(c) $w^{1/2} = \sqrt{\rho}\, e^{i\phi/2}$ $(\rho > 0, 2\pi < \phi < 4\pi)$.

7. In Sec. 103, it was pointed out that the components $x(u, v)$ and $y(u, v)$ of the inverse function $g(w)$ defined by equation (6) there are continuous and have continuous first-order partial derivatives in a neighborhood N. Use equations (5), Sec. 103, to show that the Cauchy–Riemann equations $x_u = y_v, x_v = -y_u$ hold in N. Then conclude that $g(w)$ is analytic in that neighborhood.

8. Show that if $z = g(w)$ is the local inverse of a conformal transformation $w = f(z)$ at a point z_0, then

$$g'(w) = \frac{1}{f'(z)}$$

at points w in a neighborhood N where g is analytic (Exercise 7).

Suggestion: Start with the fact that $f[g(w)] = w$, and apply the chain rule for differentiating composite functions.

9. Let C be a smooth arc lying in a domain D throughout which a transformation $w = f(z)$ is conformal, and let Γ denote the image of C under that transformation. Show that Γ is also a smooth arc.

10. Suppose that a function f is analytic at z_0 and that

$$f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, \quad f^{(m)}(z_0) \neq 0$$

for some positive integer m ($m \geq 1$). Also, write $w_0 = f(z_0)$.

(a) Use the Taylor series for f about the point z_0 to show that there is a neighborhood of z_0 in which the difference $f(z) - w_0$ can be written

$$f(z) - w_0 = (z - z_0)^m \frac{f^{(m)}(z_0)}{m!}[1 + g(z)],$$

where $g(z)$ is continuous at z_0 and $g(z_0) = 0$.

(b) Let Γ be the image of a smooth arc C under the transformation $w = f(z)$, as shown in Fig. 134 (Sec. 101), and note that the angles of inclination θ_0 and ϕ_0 in that figure are limits of $\arg(z - z_0)$ and $\arg[f(z) - w_0]$, respectively, as z approaches z_0 along the arc C. Then use the result in part *(a)* to show that θ_0 and ϕ_0 are related by the equation

$$\phi_0 = m\theta_0 + \arg f^{(m)}(z_0).$$

(c) Let α denote the angle between two smooth arcs C_1 and C_2 passing through z_0, as shown on the left in Fig. 135 (Sec. 101). Show how it follows from the relation obtained in part *(b)* that the corresponding angle between the image curves Γ_1 and Γ_2 at the point $w_0 = f(z_0)$ is $m\alpha$. (Note that the transformation is conformal at z_0 when $m = 1$ and that z_0 is a critical point when $m \geq 2$.)

104. HARMONIC CONJUGATES

We saw in Sec. 26 that if a function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic in a domain D, then the real-valued functions u and v are harmonic in that domain. That is, they have continuous partial derivatives of the first and second order in D and satisfy Laplace's equation there:

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0. \tag{1}$$

We had seen earlier that the first-order partial derivatives of u and v satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x\,; \tag{2}$$

and, as pointed out in Sec. 26, v is called a harmonic conjugate of u.

Suppose now that $u(x, y)$ is any given harmonic function defined on a *simply connected* (Sec. 48) domain D. In this section, we show that $u(x, y)$ always has a harmonic conjugate $v(x, y)$ in D by deriving an expression for $v(x, y)$.

To accomplish this, we first recall some important facts about line integrals in advanced calculus.* Suppose that $P(x, y)$ and $Q(x, y)$ have continuous first-order partial derivatives in a simply connected domain D of the xy plane, and let (x_0, y_0) and (x, y) be any two points in D. If $P_y = Q_x$ everywhere in D, then the line integral

$$\int_C P(s, t)\,ds + Q(s, t)\,dt$$

from (x_0, y_0) to (x, y) is independent of the contour C that is taken as long as the contour lies entirely in D. Furthermore, when the point (x_0, y_0) is kept fixed and (x, y) is allowed to vary throughout D, the integral represents a single-valued function

$$F(x, y) = \int_{(x_0, y_0)}^{(x, y)} P(s, t)\,ds + Q(s, t)\,dt \tag{3}$$

of x and y whose first-order partial derivatives are given by the equations

$$F_x(x, y) = P(x, y), \quad F_y(x, y) = Q(x, y). \tag{4}$$

Note that the value of F is changed by an additive constant when a different starting point (x_0, y_0) is taken.

Returning to the given harmonic function $u(x, y)$, observe how it follows from Laplace's equation $u_{xx} + u_{yy} = 0$ that

$$(-u_y)_y = (u_x)_x$$

everywhere in D. Also, the second-order partial derivatives of u are continuous in D; and this means that the first-order partial derivatives of $-u_y$ and u_x are continuous there. Thus, if (x_0, y_0) is a fixed point in D, the function

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} -u_t(s, t)\,ds + u_s(s, t)\,dt \tag{5}$$

is well defined for all (x, y) in D; and, according to equations (4),

$$v_x(x, y) = -u_y(x, y), \quad v_y(x, y) = u_x(x, y). \tag{6}$$

*See, for example, W. Kaplan, "Advanced Mathematics for Engineers," pp. 546–550, 1992.

These are the Cauchy–Riemann equations. Since the first-order partial derivatives of u are continuous, it is evident from equations (6) that those derivatives of v are also continuous. Hence (Sec. 22) $u(x, y) + iv(x, y)$ is an analytic function in D; and v is, therefore, a harmonic conjugate of u.

The function v defined by equation (5) is, of course, not the only harmonic conjugate of u. The function $v(x, y) + c$, where c is any real constant, is also a harmonic conjugate of u. [Recall Exercise 2, Sec. 26.]

EXAMPLE. Consider the function $u(x, y) = xy$, which is harmonic throughout the entire xy plane. According to equation (5), the function

$$v(x, y) = \int_{(0,0)}^{(x,y)} -s\,ds + t\,dt$$

is a harmonic conjugate of $u(x, y)$. The integral here is readily evaluated by inspection. It can also be evaluated by integrating first along the horizontal path from the point $(0, 0)$ to the point $(x, 0)$ and then along the vertical path from $(x, 0)$ to the point (x, y). The result is

$$v(x, y) = -\frac{1}{2}x^2 + \frac{1}{2}y^2,$$

and the corresponding analytic function is

$$f(z) = xy - \frac{i}{2}(x^2 - y^2) = -\frac{i}{2}z^2.$$

105. TRANSFORMATIONS OF HARMONIC FUNCTIONS

The problem of finding a function that is harmonic in a specified domain and satisfies prescribed conditions on the boundary of the domain is prominent in applied mathematics. If the values of the function are prescribed along the boundary, the problem is known as a boundary value problem of the first kind, or a *Dirichlet problem.* If the values of the normal derivative of the function are prescribed on the boundary, the boundary value problem is one of the second kind, or a *Neumann problem.* Modifications and combinations of those types of boundary conditions also arise.

The domains most frequently encountered in the applications are simply connected; and, since a function that is harmonic in a simply connected domain always has a harmonic conjugate (Sec. 104), solutions of boundary value problems for such domains are the real or imaginary components of analytic functions.

EXAMPLE 1. In Example 1, Sec. 26, we saw that the function

$$T(x, y) = e^{-y} \sin x$$

satisfies a certain Dirichlet problem for the strip $0 < x < \pi$, $y > 0$ and noted that it represents a solution of a temperature problem. The function $T(x, y)$, which is actually harmonic throughout the xy plane, is the real component of the entire function

$$-ie^{iz} = e^{-y} \sin x - ie^{-y} \cos x.$$

It is also the imaginary component of the entire function e^{iz}.

Sometimes a solution of a given boundary value problem can be *discovered* by identifying it as the real or imaginary component of an analytic function. But the success of that procedure depends on the simplicity of the problem and on one's familiarity with the real and imaginary components of a variety of analytic functions. The following theorem is an important aid.

Theorem. *Suppose that an analytic function*

$$w = f(z) = u(x, y) + iv(x, y) \tag{1}$$

maps a domain D_z *in the* z *plane onto a domain* D_w *in the* w *plane. If* $h(u, v)$ *is a harmonic function defined on* D_w, *then the function*

$$H(x, y) = h[u(x, y), v(x, y)] \tag{2}$$

is harmonic in D_z.

We first prove the theorem for the case in which the domain D_w is simply connected. According to Sec. 104, that property of D_w ensures that the given harmonic function $h(u, v)$ has a harmonic conjugate $g(u, v)$. Hence the function

$$\Phi(w) = h(u, v) + ig(u, v) \tag{3}$$

is analytic in D_w. Since the function $f(z)$ is analytic in D_z, the composite function $\Phi[f(z)]$ is also analytic in D_z. Consequently, the real part $h[u(x, y), v(x, y)]$ of this composition is harmonic in D_z.

If D_w is *not* simply connected, we observe that each point w_0 in D_w has a neighborhood $|w - w_0| < \varepsilon$ lying entirely in D_w. Since that neighborhood is simply connected, a function of the type (3) is analytic in it. Furthermore, since f is continuous at a point z_0 in D_z whose image is w_0, there is a neighborhood $|z - z_0| < \delta$ whose image is contained in the neighborhood $|w - w_0| < \varepsilon$. Hence it follows that the composition $\Phi[f(z)]$ is analytic in the neighborhood $|z - z_0| < \delta$, and we may conclude that $h[u(x, y), v(x, y)]$ is harmonic there. Finally, since w_0 was arbitrarily chosen in D_w and since each point in D_z is mapped onto such a point under the transformation $w = f(z)$, the function $h[u(x, y), v(x, y)]$ must be harmonic throughout D_z.

The proof of the theorem for the general case in which D_w is not necessarily simply connected can also be accomplished directly by means of the chain rule for partial derivatives. The computations are, however, somewhat involved (see Exercise 8, Sec. 106).

EXAMPLE 2. The function $h(u, v) = e^{-v} \sin u$ is harmonic in the domain D_w consisting of all points in the upper half plane $v > 0$ (see Example 1). If the transformation is $w = z^2$, we have $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$; moreover, the domain D_z consisting of the points in the first quadrant $x > 0$, $y > 0$ of the z plane is mapped onto the domain D_w, as shown in Example 3, Sec. 13. Hence the function

$$H(x, y) = e^{-2xy} \sin(x^2 - y^2)$$

is harmonic in D_z.

EXAMPLE 3. A minor modification of Fig. 114 in Example 3, Sec. 95, reveals that as a point $z = r \exp(i\Theta_0)$ $(-\pi/2 < \Theta_0 < \pi/2)$ travels outward from the origin along a ray $\Theta = \Theta_0$ in the z plane, its image under the transformation

$$w = \text{Log } z = \ln r + i\Theta \quad (r > 0, -\pi < \Theta < \pi)$$

travels along the entire length of the horizontal line $v = \Theta_0$ in the w plane. So the right half plane $x > 0$ is mapped onto the horizontal strip $-\pi/2 < v < \pi/2$. By considering the function

$$h(u, v) = \text{Im } w = v,$$

which is harmonic in the strip, and writing

$$\text{Log } z = \ln \sqrt{x^2 + y^2} + i \arctan \frac{y}{x},$$

where $-\pi/2 < \arctan t < \pi/2$, we find that

$$H(x, y) = \arctan \frac{y}{x}$$

is harmonic in the half plane $x > 0$.

106. TRANSFORMATIONS OF BOUNDARY CONDITIONS

The conditions that a function or its normal derivative have prescribed values along the boundary of a domain in which it is harmonic are the most common, although not the only, important types of boundary conditions. In this section, we show that certain of these conditions remain unaltered under the change of variables associated with a conformal transformation. These results will be used in Chap. 10 to solve boundary value problems. The basic technique there is to transform a given boundary value problem in the xy plane into a simpler one in the uv plane and then to use the theorems of this and Sec. 105 to write the solution of the original problem in terms of the solution obtained for the simpler one.

Theorem. *Suppose that a transformation*

$$w = f(z) = u(x, y) + iv(x, y) \tag{1}$$

is conformal on a smooth arc C, and let Γ be the image of C under that transformation. If a function $h(u, v)$ satisfies either of the conditions

$$h = h_0 \quad or \quad \frac{dh}{dn} = 0 \tag{2}$$

along Γ, where h_0 is a real constant and dh/dn denotes derivatives normal to Γ, then the function

$$H(x, y) = h[u(x, y), v(x, y)] \tag{3}$$

satisfies the corresponding condition

$$H = h_0 \quad or \quad \frac{dH}{dN} = 0 \tag{4}$$

along C, where dH/dN denotes derivatives normal to C.

To show that the condition $h = h_0$ on Γ implies that $H = h_0$ on C, we note from equation (3) that the value of H at any point (x, y) on C is the same as the value of h at the image (u, v) of (x, y) under transformation (1). Since the image point (u, v) lies on Γ and since $h = h_0$ along that curve, it follows that $H = h_0$ along C.

Suppose, on the other hand, that $dh/dn = 0$ on Γ. From calculus, we know that

$$\frac{dh}{dn} = (\text{grad } h) \cdot \mathbf{n}, \tag{5}$$

where grad h denotes the gradient of h at a point (u, v) on Γ and $\mathbf{n}$ is a unit vector normal to Γ at (u, v). Since $dh/dn = 0$ at (u, v), equation (5) tells us that grad h is orthogonal to $\mathbf{n}$ at (u, v). That is, grad h is tangent to Γ there (Fig. 137). But gradients are orthogonal to level curves; and, because grad h is tangent to Γ, we see that Γ is orthogonal to a level curve $h(u, v) = c$ passing through (u, v).

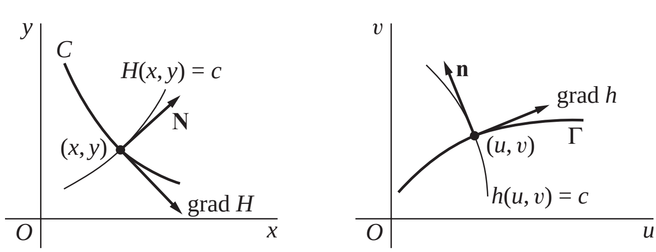

FIGURE 137

Now, according to equation (3), the level curve $H(x, y) = c$ in the z plane can be written

$$h[u(x, y), v(x, y)] = c;$$

and so it is evidently transformed into the level curve $h(u, v) = c$ under transformation (1). Furthermore, since C is transformed into Γ and Γ is orthogonal to the level curve $h(u, v) = c$, as demonstrated in the preceding paragraph, it follows from the conformality of transformation (1) that C is orthogonal to the level curve $H(x, y) = c$ at the point (x, y) corresponding to (u, v). Because gradients are orthogonal to level curves, this means that grad H is tangent to C at (x, y) (see Fig. 137). Consequently, if $\mathbf{N}$ denotes a unit vector normal to C at (x, y), grad H is orthogonal to $\mathbf{N}$. That is,

$$(\text{grad } H) \cdot \mathbf{N} = 0. \tag{6}$$

Finally, since

$$\frac{dH}{dN} = (\text{grad } H) \cdot \mathbf{N},$$

we may conclude from equation (6) that $dH/dN = 0$ at points on C.

In this discussion, we have tacitly assumed that grad $h \neq \mathbf{0}$. If grad $h = \mathbf{0}$, it follows from the identity

$$|\text{grad } H(x, y)| = |\text{grad } h(u, v)||f'(z)|,$$

derived in Exercise 10*(a)* of this section, that grad $H = \mathbf{0}$; hence dh/dn and the corresponding normal derivative dH/dN are both zero. We have also assumed that

(a) grad h and grad H always exist;

(b) the level curve $H(x, y) = c$ is smooth when grad $h \neq \mathbf{0}$ at (u, v).

Condition *(b)* ensures that angles between arcs are preserved by transformation (1) when it is conformal. In all of our applications, both conditions *(a)* and *(b)* will be satisfied.

EXAMPLE. Consider, for instance, the function $h(u, v) = v + 2$. The transformation

$$w = iz^2 = -2xy + i(x^2 - y^2)$$

is conformal when $z \neq 0$. It maps the half line $y = x$ $(x > 0)$ onto the negative u axis, where $h = 2$, and the positive x axis onto the positive v axis, where the normal derivative h_u is 0 (Fig. 138). According to the above theorem, the function

$$H(x, y) = x^2 - y^2 + 2$$

must satisfy the condition $H = 2$ along the half line $y = x$ $(x > 0)$ and $H_y = 0$ along the positive x axis, as one can verify directly.

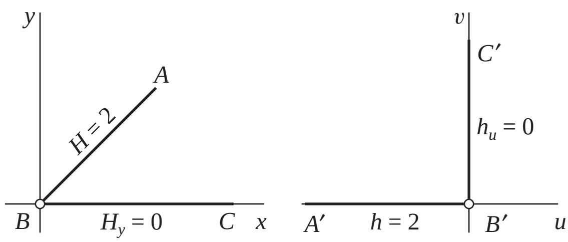

FIGURE 138

A boundary condition that is not of one of the two types mentioned in the theorem may be transformed into a condition that is substantially different from the original one (see Exercise 6). New boundary conditions for the transformed problem can be obtained for a particular transformation in any case. It is interesting to note that under a conformal transformation, the ratio of a directional derivative of H along a smooth arc C in the z plane to the directional derivative of h along the image curve Γ at the corresponding point in the w plane is $|f'(z)|$; usually, this ratio is not constant along a given arc. (See Exercise 10.)

EXERCISES

1. Use expression (5), Sec. 104, to find a harmonic conjugate of the harmonic function $u(x, y) = x^3 - 3xy^2$. Write the resulting analytic function in terms of the complex variable z.

2. Let $u(x, y)$ be harmonic in a simply connected domain D. By appealing to results in Secs. 104 and 52, show that its partial derivatives of all orders are continuous throughout that domain.

3. The transformation $w = \exp z$ maps the horizontal strip $0 < y < \pi$ onto the upper half plane $v > 0$, as shown in Fig. 6 of Appendix 2; and the function

$$h(u, v) = \text{Re}(w^2) = u^2 - v^2$$

is harmonic in that half plane. With the aid of the theorem in Sec. 105, show that the function $H(x, y) = e^{2x} \cos 2y$ is harmonic in the strip. Verify this result directly.

4. Under the transformation $w = \exp z$, the image of the segment $0 \le y \le \pi$ of the y axis is the semicircle $u^2 + v^2 = 1, v \ge 0$ (see Sec. 14). Also, the function

$$h(u, v) = \text{Re}\left(2 - w + \frac{1}{w}\right) = 2 - u + \frac{u}{u^2 + v^2}$$

is harmonic everywhere in the w plane except for the origin; and it assumes the value $h = 2$ on the semicircle. Write an explicit expression for the function $H(x, y)$ in the theorem of Sec. 106. Then illustrate the theorem by showing directly that $H = 2$ along the segment $0 \le y \le \pi$ of the y axis.

5. The transformation $w = z^2$ maps the positive x and y axes and the origin in the z plane onto the u axis in the w plane. Consider the harmonic function

$$h(u, v) = \text{Re}(e^{-w}) = e^{-u} \cos v,$$

and observe that its normal derivative h_v along the u axis is zero. Then illustrate the theorem in Sec. 106 when $f(z) = z^2$ by showing directly that the normal derivative of the function $H(x, y)$ defined in that theorem is zero along both positive axes in the z plane. (Note that the transformation $w = z^2$ is not conformal at the origin.)

6. Replace the function $h(u, v)$ in Exercise 5 by the harmonic function

$$h(u, v) = \text{Re}(-2iw + e^{-w}) = 2v + e^{-u} \cos v.$$

Then show that $h_v = 2$ along the u axis but that $H_y = 4x$ along the positive x axis and $H_x = 4y$ along the positive y axis. This illustrates how a condition of the type

$$\frac{dh}{dn} = h_0 \neq 0$$

is *not necessarily* transformed into a condition of the type $dH/dN = h_0$.

7. Show that if a function $H(x, y)$ is a solution of a Neumann problem (Sec. 105), then $H(x, y) + A$, where A is any real constant, is also a solution of that problem.

8. Suppose that an analytic function $w = f(z) = u(x, y) + iv(x, y)$ maps a domain D_z in the z plane onto a domain D_w in the w plane; and let a function $h(u, v)$, with continuous partial derivatives of the first and second order, be defined on D_w. Use the chain rule for partial derivatives to show that if $H(x, y) = h[u(x, y), v(x, y)]$, then

$$H_{xx}(x, y) + H_{yy}(x, y) = [h_{uu}(u, v) + h_{vv}(u, v)]\,|f'(z)|^2.$$

Conclude that the function $H(x, y)$ is harmonic in D_z when $h(u, v)$ is harmonic in D_w. This is an alternative proof of the theorem in Sec. 105, even when the domain D_w is multiply connected.

Suggestion: In the simplifications, it is important to note that since f is analytic, the Cauchy–Riemann equations $u_x = v_y$, $u_y = -v_x$ hold and that the functions u and v both satisfy Laplace's equation. Also, the continuity conditions on the derivatives of h ensure that $h_{vu} = h_{uv}$.

9. Let $p(u, v)$ be a function that has continuous partial derivatives of the first and second order and satisfies *Poisson's equation*

$$p_{uu}(u, v) + p_{vv}(u, v) = \Phi(u, v)$$

in a domain D_w of the w plane, where Φ is a prescribed function. Show how it follows from the identity obtained in Exercise 8 that if an analytic function

$$w = f(z) = u(x, y) + iv(x, y)$$

maps a domain D_z onto the domain D_w, then the function

$$P(x, y) = p[u(x, y), v(x, y)]$$

satisfies the Poisson equation

$$P_{xx}(x, y) + P_{yy}(x, y) = \Phi[u(x, y), v(x, y)]\,|f'(z)|^2$$

in D_z.

10. Suppose that $w = f(z) = u(x, y) + iv(x, y)$ is a conformal mapping of a smooth arc C onto a smooth arc Γ in the w plane. Let the function $h(u, v)$ be defined on Γ, and write

$$H(x, y) = h[u(x, y), v(x, y)].$$

(a) From calculus, we know that the x and y components of grad H are the partial derivatives H_x and H_y, respectively; likewise, grad h has components h_u and h_v. By applying the chain rule for partial derivatives and using the Cauchy–Riemann equations, show that if (x, y) is a point on C and (u, v) is its image on Γ, then

$$|\text{grad } H(x, y)| = |\text{grad } h(u, v)||f'(z)|.$$

(b) Show that the angle from the arc C to grad H at a point (x, y) on C is equal to the angle from Γ to grad h at the image (u, v) of the point (x, y).

(c) Let s and σ denote distance along the arcs C and Γ, respectively; and let $\mathbf{t}$ and $\boldsymbol{\tau}$ denote unit tangent vectors at a point (x, y) on C and its image (u, v), in the direction of increasing distance. With the aid of the results in parts *(a)* and *(b)* and using the fact that

$$\frac{dH}{ds} = (\text{grad } H) \cdot \mathbf{t} \quad \text{and} \quad \frac{dh}{d\sigma} = (\text{grad } h) \cdot \boldsymbol{\tau},$$

show that the directional derivative along the arc Γ is transformed as follows:

$$\frac{dH}{ds} = \frac{dh}{d\sigma}\,|f'(z)|.$$