(a) By using the Maclaurin series for e^z and referring to Theorem 1 in Sec. 65, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Apply the theorem in Sec. 70 to evaluate the integrals appearing in part *(a)* to arrive at the desired result.

5. Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \ldots, z_n$. Show that

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

6. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0)$$

be such that $m \geq n + 2$. Use the theorem in Sec. 71 to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

[Compare with Exercise 3*(b)*.]

72. THE THREE TYPES OF ISOLATED SINGULAR POINTS

We saw in Sec. 69 that the theory of residues is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots \tag{1}$$

in a punctured disk $0 < |z - z_0| < R_2$. The portion

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots \tag{2}$$

of the series, involving negative powers of $z - z_0$, is called the *principal part* of f at z_0. We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory that appears in following sections.

If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is only finite, then there exists a positive integer m $(m \geq 1)$ such that

$$b_m \neq 0 \quad \text{and} \quad b_{m+1} = b_{m+2} = \cdots = 0.$$

That is, expansion (1) takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m} \quad (0 < |z - z_0| < R_2), \tag{3}$$

where $b_m \neq 0$. In this case, the isolated singular point z_0 is called a *pole of order* m.* A pole of order $m = 1$ is usually referred to as a *simple pole*.

EXAMPLE 1. Observe that the function

$$\frac{z^2 - 2z + 3}{z - 2} = \frac{z(z - 2) + 3}{z - 2} = z + \frac{3}{z - 2} = 2 + (z - 2) + \frac{3}{z - 2} \quad (0 < |z - 2| < \infty)$$

has a simple pole ($m = 1$) at $z_0 = 2$. Its residue b_1 there is 3.

When representation (1) is written in the form (see Sec. 60)

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \quad (0 < |z - z_0| < R_2),$$

the residue of f at z_0 is, of course, the coefficient c_{-1}.

EXAMPLE 2. From the representation

$$\begin{aligned} f(z) &= \frac{1}{z^2(1 + z)} = \frac{1}{z^2} \cdot \frac{1}{1 - (-z)} = \frac{1}{z^2}(1 - z + z^2 - z^3 + z^4 - \cdots) \\ &= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \cdots \quad (0 < |z| < 1), \end{aligned}$$

one can see that f has a pole of order $m = 2$ at the origin and that

$$\operatorname*{Res}_{z=0} f(z) = -1.$$

EXAMPLE 3. The function

$$\frac{\sinh z}{z^4} = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right) = \frac{1}{z^3} + \frac{1}{3!} \cdot \frac{1}{z} + \frac{z}{5!} + \frac{z^3}{7!} + \cdots \quad (0 < |z| < \infty)$$

has a pole of order $m = 3$ at $z_0 = 0$, with residue $B = 1/6$.

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*Reasons for the terminology *pole* are suggested on p. 70 of the book by R. P. Boas that is listed in Appendix 1.

There remain two extremes, the case in which every coefficient in the principal part (2) is zero and the one in which an infinie number of them are nonzero.

When every b_n is zero, so that

$$(4)\quad f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (0 < |z-z_0| < R_2),$$

z_0 is known as a *removable singular point*. Note that the residue at a removable singular point is always zero. If we define, or possibly redefine, f at z_0 so that $f(z_0) = a_0$, expansion (4) becomes valid throughout the entire disk $|z - z_0| < R_2$. Since a power series always represents an analytic function interior to its circle of convergence (Sec. 65), it follows that f is analytic at z_0 when it is assigned the value a_0 there. The singularity z_0 is, therefore, *removed*.

EXAMPLE 4. The point $z_0 = 0$ is a removable singular point of the function

$$f(z) = \frac{1-\cos z}{z^2}$$

because

$$f(z) = \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right] = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots \qquad (0 < |z| < \infty).$$

When the value $f(0) = 1/2$ is assigned, f becomes entire.

If an infinite number of the coefficients b_n in the principal part (2) are nonzero, z_0 is said to be an *essential singular point* of f.

EXAMPLE 5. We recall from Example 1 in Sec. 62 that

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\frac{1}{z^n} = 1 + \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \cdots \qquad (0 < |z| < \infty).$$

From this we see that $e^{1/z}$ has an essential singular point at $z_0 = 0$, where the residue b_1 is unity.

This example can be used to illustrate (see Exercise 4) an important result known as *Picard's theorem*. It concerns the behavior of a function near an essential singular point and states that *in each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception, an infinite number of times*.*

*For a proof of Picard's theorem, see Sec. 51 in Vol. III of the book by Markushevich, cited in Appendix 1.

In the remaining sections of this chapter, we shall develop in greater depth the theory of the three types of isolated singular points just described. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.

EXERCISES

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $z \exp\left(\dfrac{1}{z}\right)$; *(b)* $\dfrac{z^2}{1+z}$; *(c)* $\dfrac{\sin z}{z}$; *(d)* $\dfrac{\cos z}{z}$; *(e)* $\dfrac{1}{(2-z)^3}$.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; *(b)* $\dfrac{1-\exp(2z)}{z^4}$; *(c)* $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans. (a) $m = 1$, $B = -1/2$; *(b)* $m = 3$, $B = -4/3$; *(c)* $m = 2$, $B = 2e^2$.

3. Suppose that a function f is analytic at z_0, and write $g(z) = f(z)/(z - z_0)$. Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 57, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

4. Use the fact (see Sec. 29) that $e^z = -1$ when

$$z = (2n+1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

to show that $e^{1/z}$ assumes the value -1 an infinite number of times in each neighborhood of the origin. More precisely, show that $e^{1/z} = -1$ when

$$z = -\frac{i}{(2n+1)\pi} \quad (n = 0, \pm 1, \pm 2, \ldots);$$

then note that if n is large enough, such points lie in any given ε neighborhood of the origin. Zero is evidently the exceptional value in Picard's theorem, stated in Example 5, Sec. 72.

5. Write the function

$$f(z) = \frac{8a^3 z^2}{(z^2 + a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z - ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3 z^2}{(z + ai)^3}.$$

Point out why $\phi(z)$ has a Taylor series representation about $z = ai$, and then use it to show that the principal part of f at that point is

$$\frac{\phi''(ai)/2}{z - ai} + \frac{\phi'(ai)}{(z - ai)^2} + \frac{\phi(ai)}{(z - ai)^3} = -\frac{i/2}{z - ai} - \frac{a/2}{(z - ai)^2} - \frac{a^2 i}{(z - ai)^3}.$$

73. RESIDUES AT POLES

When a function f has an isolated singularity at a point z_0, the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $1/(z - z_0)$. The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

Theorem. *An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form*

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}, \tag{1}$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \quad \text{if } m = 1 \tag{2}$$

and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if } m \geq 2. \tag{3}$$

Observe that expression (2) need not have been written separately since, with the convention that $\phi^{(0)}(z_0) = \phi(z_0)$ and $0! = 1$, expression (3) reduces to it when $m = 1$.

To prove the theorem, we first assume that $f(z)$ has the form (1) and recall (Sec. 57) that since $\phi(z)$ is analytic at z_0, it has a Taylor series representation

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z - z_0) + \frac{\phi''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z - z_0)^{m-1}$$
$$+ \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z - z_0)^n$$

in some neighborhood $|z - z_0| < \varepsilon$ of z_0; and from expression (1) it follows that

$$f(z) = \frac{\phi(z_0)}{(z - z_0)^m} + \frac{\phi'(z_0)/1!}{(z - z_0)^{m-1}} + \frac{\phi''(z_0)/2!}{(z - z_0)^{m-2}} + \cdots + \frac{\phi^{(m-1)}(z_0)/(m-1)!}{z - z_0}$$
$$+ \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z - z_0)^{n-m}$$

when $0 < |z - z_0| < \varepsilon$. This Laurent series representation, together with the fact that $\phi(z_0) \neq 0$, reveals that z_0 is, indeed, a pole of order m of $f(z)$. The coefficient of $1/(z - z_0)$ tells us, of course, that the residue of $f(z)$ at z_0 is as in the statement of the theorem.

Suppose, on the other hand, that we know only that z_0 is a pole of order m of f, or that $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0)$$

which is valid in a punctured disk $0 < |z - z_0| < R_2$. The function $\phi(z)$ defined by means of the equations

$$\phi(z) = \begin{cases} (z - z_0)^m f(z) & \text{when } z \neq z_0, \\ b_m & \text{when } z = z_0 \end{cases}$$

evidently has the power series representation

$$\phi(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_2(z - z_0)^{m-2} + b_1(z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{m+n}$$

throughout the entire disk $|z - z_0| < R_2$. Consequently, $\phi(z)$ is analytic in that disk (Sec. 65) and, in particular, at z_0. Inasmuch as $\phi(z_0) = b_m \neq 0$, expression (1) is established; and the proof of the theorem is complete.

74. EXAMPLES

The following examples serve to illustrate the use of the theorem in Sec. 73.

EXAMPLE 1. The function

$$f(z) = \frac{z + 1}{z^2 + 9}$$

has an isolated singular point at $z = 3i$ and can be written

$$f(z) = \frac{\phi(z)}{z - 3i} \quad \text{where} \quad \phi(z) = \frac{z + 1}{z + 3i}.$$

Since $\phi(z)$ is analytic at $z = 3i$ and $\phi(3i) \neq 0$, that point is a simple pole of the function f; and the residue there is

$$B_1 = \phi(3i) = \frac{3i + 1}{6i} \cdot \frac{-i}{-i} = \frac{3 - i}{6}.$$

The point $z = -3i$ is also a simple pole of f, with residue

$$B_2 = \frac{3+i}{6}.$$

EXAMPLE 2. If

$$f(z) = \frac{z^3 + 2z}{(z-i)^3},$$

then

$$f(z) = \frac{\phi(z)}{(z-i)^3} \quad \text{where} \quad \phi(z) = z^3 + 2z.$$

The function $\phi(z)$ is entire, and $\phi(i) = i \neq 0$. Hence f has a pole of order 3 at $z = i$, with residue

$$B = \frac{\phi''(i)}{2!} = \frac{6i}{2!} = 3i.$$

The theorem can, of course, be used when branches of multiple-valued functions are involved.

EXAMPLE 3. Suppose that

$$f(z) = \frac{(\log z)^3}{z^2+1},$$

where the branch

$$\log z = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

of the logarithmic function is to be used. To find the residue of f at the singularity $z = i$, we write

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^3}{z+i}.$$

The function $\phi(z)$ is clearly analytic at $z = i$; and, since

$$\phi(i) = \frac{(\log i)^3}{2i} = \frac{(\ln 1 + i\pi/2)^3}{2i} = -\frac{\pi^3}{16} \neq 0,$$

f has a simple pole there. The residue is

$$B = \phi(i) = -\frac{\pi^3}{16}.$$

While the theorem in Sec. 73 can be extremely useful, the identification of an isolated singular point as a pole of a certain order is sometimes done most efficiently by appealing directly to a Laurent series.

EXAMPLE 4. If, for instance, the residue of the function

$$f(z) = \frac{\sinh z}{z^4}$$

is needed at the singularity $z = 0$, it would be incorrect to write

$$f(z) = \frac{\phi(z)}{z^4} \quad \text{where} \quad \phi(z) = \sinh z$$

and to attempt an application of formula (3) in Sec. 73 with $m = 4$. For it is necessary that $\phi(z_0) \neq 0$ if that formula is to be used. In this case, the simplest way to find the residue is to write out a few terms of the Laurent series for $f(z)$, as was done in Example 3 of Sec. 72. There it was shown that $z = 0$ is a pole of the *third* order, with residue $B = 1/6$.

In some cases, the series approach can be effectively combined with the theorem in Sec. 73.

EXAMPLE 5. Since $z(e^z - 1)$ is entire and its zeros are

$$z = 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

the point $z = 0$ is clearly an isolated singular point of the function

$$f(z) = \frac{1}{z(e^z - 1)}.$$

From the Maclaurin series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we see that

$$z(e^z - 1) = z\left(\frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right) \qquad (|z| < \infty).$$

Thus

$$f(z) = \frac{\phi(z)}{z^2} \quad \text{where} \quad \phi(z) = \frac{1}{1 + z/2! + z^2/3! + \cdots}.$$

Since $\phi(z)$ is analytic at $z = 0$ and $\phi(0) = 1 \neq 0$, the point $z = 0$ is a pole of the *second* order; and, according to formula (3) in Sec. 73, the residue is $B = \phi'(0)$. Because

$$\phi'(z) = \frac{-(1/2! + 2z/3! + \cdots)}{(1 + z/2! + z^2/3! + \cdots)^2}$$

in a neighborhood of the origin, then, $B = -1/2$.

This residue can also be found by dividing our series for $z(e^z - 1)$ into 1, or by multiplying the Laurent series for $1/(e^z - 1)$ in Exercise 3, Sec. 67, by $1/z$.

EXERCISES

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z^2+2}{z-1}$; *(b)* $\left(\dfrac{z}{2z+1}\right)^3$; *(c)* $\dfrac{\exp z}{z^2+\pi^2}$.

Ans. *(a)* $m = 1$, $B = 3$; *(b)* $m = 3$, $B = -3/16$; *(c)* $m = 1$, $B = \pm i/2\pi$.

2. Show that

(a) $\operatorname*{Res}_{z=-1} \dfrac{z^{1/4}}{z+1} = \dfrac{1+i}{\sqrt{2}}$ $\quad (|z| > 0, 0 < \arg z < 2\pi)$;

(b) $\operatorname*{Res}_{z=i} \dfrac{\text{Log } z}{(z^2+1)^2} = \dfrac{\pi+2i}{8}$;

(c) $\operatorname*{Res}_{z=i} \dfrac{z^{1/2}}{(z^2+1)^2} = \dfrac{1-i}{8\sqrt{2}}$ $\quad (|z| > 0, 0 < \arg z < 2\pi)$.

3. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz,$$

taken counterclockwise around the circle *(a)* $|z-2| = 2$; *(b)* $|z| = 4$.

Ans. *(a)* πi; *(b)* $6\pi i$.

4. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

taken counterclockwise around the circle *(a)* $|z| = 2$; *(b)* $|z+2| = 3$.

Ans. *(a)* $\pi i/32$; *(b)* 0.

5. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$$

when C is the circle $|z| = 2$, described in the positive sense.

Ans. $4\pi i$.

6. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

(a) $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$; *(b)* $f(z) = \dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$; *(c)* $f(z) = \dfrac{z^3 e^{1/z}}{1+z^3}$.

Ans. *(a)* $9\pi i$; *(b)* $-3\pi i$; *(c)* $2\pi i$.

7. Let z_0 be an isolated singular point of a function f and suppose that

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where m is a positive integer and $\phi(z)$ is analytic and nonzero at z_0. By applying the extended form (6), Sec. 51, of the Cauchy integral formula to the function $\phi(z)$, show that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

as stated in the theorem of Sec. 73.

Suggestion: Since there is a neighborhood $|z - z_0| < \varepsilon$ throughout which $\phi(z)$ is analytic (see Sec. 24), the contour used in the extended Cauchy integral formula can be the positively oriented circle $|z - z_0| = \varepsilon/2$.

75. ZEROS OF ANALYTIC FUNCTIONS

Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions.

Suppose that a function f is analytic at a point z_0. We know from Sec. 52 that all of the derivatives $f^{(n)}(z)$ $(n = 1, 2, \ldots)$ exist at z_0. If $f(z_0) = 0$ and if there is a positive integer m such that $f^{(m)}(z_0) \neq 0$ and each derivative of lower order vanishes at z_0, then f is said to have a *zero of order* m at z_0. Our first theorem here provides a useful alternative characterization of zeros of order m.

Theorem 1. *Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0, such that*

$$f(z) = (z - z_0)^m g(z). \tag{1}$$

Both parts of the proof that follows use the fact (Sec. 57) that if a function is analytic at a point z_0, then it must have a Taylor series representation in powers of $z - z_0$ which is valid throughout a neighborhood $|z - z_0| < \varepsilon$ of z_0.

We start the first part of the proof by assuming that expression (1) holds and noting that since $g(z)$ is analytic at z_0, it has a Taylor series representation

$$g(z) = g(z_0) + \frac{g'(z_0)}{1!}(z - z_0) + \frac{g''(z_0)}{2!}(z - z_0)^2 + \cdots$$

in some neighborhood $|z - z_0| < \varepsilon$ of z_0. Expression (1) thus takes the form

$$f(z) = g(z_0)(z - z_0)^m + \frac{g'(z_0)}{1!}(z - z_0)^{m+1} + \frac{g''(z_0)}{2!}(z - z_0)^{m+2} + \cdots$$

when $|z - z_0| < \varepsilon$. Since this is actually a Taylor series expansion for $f(z)$, according to Theorem 1 in Sec. 66, it follows that

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \tag{2}$$

and that

$$f^{(m)}(z_0) = m!g(z_0) \neq 0. \tag{3}$$

Hence z_0 is a zero of order m of f.

Conversely, if we assume that f has a zero of order m at z_0, the analyticity of f at z_0 and the fact that conditions (2) hold tell us that in some neighborhood $|z - z_0| < \varepsilon$, there is a Taylor series

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \\ &= (z - z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z - z_0)^2 + \cdots \right]. \end{aligned}$$

Consequently, $f(z)$ has the form (1), where

$$g(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z - z_0)^2 + \cdots \qquad (|z - z_0| < \varepsilon).$$

The convergence of this last series when $|z - z_0| < \varepsilon$ ensures that g is analytic in that neighborhood and, in particular, at z_0 (Sec. 65). Moreover,

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

This completes the proof of the theorem.

EXAMPLE 1. The polynomial $f(z) = z^3 - 8 = (z - 2)(z^2 + 2z + 4)$ has a zero of order $m = 1$ at $z_0 = 2$ since

$$f(z) = (z - 2)g(z),$$

where $g(z) = z^2 + 2z + 4$, and because f and g are entire and $g(2) = 12 \neq 0$. Note how the fact that $z_0 = 2$ is a zero of order $m = 1$ of f also follows from the observations that f is entire and that

$$f(2) = 0 \quad \text{and} \quad f'(2) = 12 \neq 0.$$

EXAMPLE 2. The entire function $f(z) = z(e^z - 1)$ has a zero of order $m = 2$ at the point $z_0 = 0$ since

$$f(0) = f'(0) = 0 \quad \text{and} \quad f''(0) = 2 \neq 0.$$

In this case, expression (1) becomes

$$f(z) = (z - 0)^2 g(z),$$

where g is the entire function (see Example 1, Sec. 65) defined by means of the equations

$$g(z) = \begin{cases} (e^z - 1)/z & \text{when } z \neq 0, \\ 1 & \text{when } z = 0. \end{cases}$$

Our next theorem tells us that the zeros of an analytic function are *isolated* when the function is not identically equal to zero.

Theorem 2. *Given a function f and a point z_0, suppose that*

(a) f is analytic at z_0;

(b) $f(z_0) = 0$ but $f(z)$ is not identically equal to zero in any neighborhood of z_0.

Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

To prove this, let f be as stated and observe that not all of the derivatives of f at z_0 are zero. If they were, all of the coefficients in the Taylor series for f about z_0 would be zero; and that would mean that $f(z)$ is identically equal to zero in some neighborhood of z_0. So it is clear from the definition of zeros of order m at the beginning of this section that f must have a zero of some finite order m at z_0. According to Theorem 1, then,

$$f(z) = (z - z_0)^m g(z) \tag{4}$$

where $g(z)$ is analytic and nonzero at z_0.

Now g is continuous, in addition to being nonzero, at z_0 because it is analytic there. Hence there is some neighborhood $|z - z_0| < \varepsilon$ in which equation (4) holds and in which $g(z) \neq 0$ (see Sec. 18). Consequently, $f(z) \neq 0$ in the *deleted* neighborhood $0 < |z - z_0| < \varepsilon$; and the proof is complete.

Our final theorem here concerns functions with zeros that are not all isolated. It was referred to earlier in Sec. 27 and makes an interesting contrast to Theorem 2 just above.

Theorem 3. *Given a function f and a point z_0, suppose that*

(a) f is analytic throughout a neighborhood N_0 of z_0;

(b) $f(z) = 0$ at each point z of a domain D or line segment L containing z_0 (Fig. 90).

Then $f(z) \equiv 0$ in N_0; that is, $f(z)$ is identically equal to zero throughout N_0.

We begin the proof with the observation that under the stated conditions, $f(z) \equiv 0$ in some neighborhood N of z_0. For, otherwise, there would be a deleted neighborhood of z_0 throughout which $f(z) \neq 0$, according to Theorem 2; and that would be inconsistent with the condition that $f(z) = 0$ everywhere in a domain D

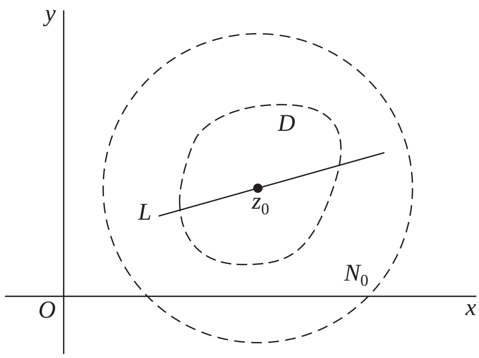

FIGURE 90

or on a line segment L containing z_0. Since $f(z) \equiv 0$ in the neighborhood N, then, it follows that all of the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots)$$

in the Taylor series for $f(z)$ about z_0 must be zero. Thus $f(z) \equiv 0$ in the neighborhood N_0, since the Taylor series also represents $f(z)$ in N_0. This completes the proof.

76. ZEROS AND POLES

The following theorem shows how zeros of order m can create poles of order m.

Theorem 1. *Suppose that*

(a) two functions p and q are analytic at a point z_0;

(b) $p(z_0) \neq 0$ and q has a zero of order m at z_0.

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

The proof is easy. Let p and q be as in the statement of the theorem. Since q has a zero of order m at z_0, we know from Theorem 2 in Sec. 75 that there is a deleted neighborhood of z_0 throughout which $q(z) \neq 0$; and so z_0 is an isolated singular point of the quotient $p(z)/q(z)$. Theorem 1 in Sec. 75 tells us, moreover, that

$$q(z) = (z - z_0)^m g(z),$$

where g is analytic and nonzero at z_0; and this enables us to write

$$\text{(1)} \qquad \frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \quad \text{where} \quad \phi(z) = \frac{p(z)}{g(z)}.$$

Since $\phi(z)$ is analytic and nonzero at z_0, it now follows from the theorem in Sec. 73 that z_0 is a pole of order m of $p(z)/q(z)$.

EXAMPLE 1. The two functions

$$p(z) = 1 \quad \text{and} \quad q(z) = z(e^z - 1)$$

are entire; and we know from Example 2 in Sec. 75 that q has a zero of order $m = 2$ at the point $z_0 = 0$. Hence it follows from Theorem 1 that the quotient

$$\frac{p(z)}{q(z)} = \frac{1}{z(e^z - 1)}$$

has a pole of order 2 at that point. This was demonstrated in another way in Example 5, Sec. 74.

Theorem 1 leads us to another method for identifying *simple* poles and finding the corresponding residues. This method, stated just below as Theorem 2, is sometimes easier to use than the theorem in Sec. 73.

Theorem 2. *Let two functions p and q be analytic at a point z_0. If*

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \textit{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{2}$$

To show this, we assume that p and q are as stated and observe that because of the conditions on q, the point z_0 is a zero of order $m = 1$ of that function. According to Theorem 1 in Sec. 75, then,

$$q(z) = (z - z_0)g(z) \tag{3}$$

where $g(z)$ is analytic and nonzero at z_0. Furthermore, Theorem 1 in this section tells us that z_0 is a simple pole of $p(z)/q(z)$; and expression (1) for $p(z)/q(z)$ in the proof of that theorem becomes

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{z - z_0} \quad \text{where} \quad \phi(z) = \frac{p(z)}{g(z)}.$$

Since this $\phi(z)$ is analytic and nonzero at z_0, we know from the theorem in Sec. 73 that

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)}. \tag{4}$$

But $g(z_0) = q'(z_0)$, as is seen by differentiating each side of equation (3) and then setting $z = z_0$. Expression (4) thus takes the form (2).

EXAMPLE 2. Consider the function

$$f(z) = \cot z = \frac{\cos z}{\sin z},$$

which is a quotient of the entire functions $p(z) = \cos z$ and $q(z) = \sin z$. Its singularities occur at the zeros of q, or at the points

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since

$$p(n\pi) = (-1)^n \neq 0, \quad q(n\pi) = 0, \quad \text{and} \quad q'(n\pi) = (-1)^n \neq 0,$$

each singular point $z = n\pi$ of f is a simple pole, with residue

$$B_n = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1.$$

EXAMPLE 3. The residue of the function

$$f(z) = \frac{\tanh z}{z^2} = \frac{\sinh z}{z^2 \cosh z}$$

at the zero $z = \pi i/2$ of $\cosh z$ (see Sec. 35) is readily found by writing

$$p(z) = \sinh z \quad \text{and} \quad q(z) = z^2 \cosh z.$$

Since

$$p\left(\frac{\pi i}{2}\right) = \sinh\left(\frac{\pi i}{2}\right) = i \sin\frac{\pi}{2} = i \neq 0$$

and

$$q\left(\frac{\pi i}{2}\right) = 0, \qquad q'\left(\frac{\pi i}{2}\right) = \left(\frac{\pi i}{2}\right)^2 \sinh\left(\frac{\pi i}{2}\right) = -\frac{\pi^2}{4} i \neq 0,$$

we find that $z = \pi i/2$ is a simple pole of f and that the residue there is

$$B = \frac{p(\pi i/2)}{q'(\pi i/2)} = -\frac{4}{\pi^2}.$$

EXAMPLE 4. Since the point

$$z_0 = \sqrt{2} e^{i\pi/4} = 1 + i$$

is a zero of the polynomial $z^4 + 4$ (see Exercise 6, Sec. 10), it is also an isolated singularity of the function

$$f(z) = \frac{z}{z^4 + 4}.$$

Writing $p(z) = z$ and $q(z) = z^4 + 4$, we find that

$$p(z_0) = z_0 \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) = 4z_0^3 \neq 0$$

and hence that z_0 is a simple pole of f. The residue there is, moreover,

$$B_0 = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = -\frac{i}{8}.$$

Although this residue can also be found by the method in Sec. 73, the computation is somewhat more involved.

There are formulas similar to formula (2) for residues at poles of higher order, but they are lengthier and, in general, not practical.

EXERCISES

1. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to

(a) Theorem 2 in Sec. 76;

(b) the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 67.

2. Show that

(a) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi}$;

(b) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos(\pi t)$.

3. Show that

(a) $\displaystyle \operatorname*{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n$ where $\displaystyle z_n = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\displaystyle \operatorname*{Res}_{z=z_n}(\tanh z) = 1$ where $\displaystyle z_n = \left(\frac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

4. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

(a) $\displaystyle \int_C \tan z \, dz$; *(b)* $\displaystyle \int_C \frac{dz}{\sinh 2z}$.

Ans. (a) $-4\pi i$; *(b)* $-\pi i$.

5. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity (Exercise 8, Sec. 43), point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

6. Show that

$$\int_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}},$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2-1)^2+3$ are the square roots of the numbers $1 \pm \sqrt{3}i$, show that the reciprocal $1/q(z)$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z_0} = \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

Then apply Theorem 2 in Sec. 76.

7. Consider the function

$$f(z) = \frac{1}{[q(z)]^2}$$

where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Show that z_0 is a pole of order $m = 2$ of the function f, with residue

$$B_0 = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

Suggestion: Note that z_0 is a zero of order $m = 1$ of the function q, so that

$$q(z) = (z - z_0)g(z)$$

where $g(z)$ is analytic and nonzero at z_0. Then write

$$f(z) = \frac{\phi(z)}{(z-z_0)^2} \quad \text{where} \quad \phi(z) = \frac{1}{[g(z)]^2}.$$

The desired form of the residue $B_0 = \phi'(z_0)$ can be obtained by showing that

$$q'(z_0) = g(z_0) \quad \text{and} \quad q''(z_0) = 2g'(z_0).$$

8. Use the result in Exercise 7 to find the residue at $z = 0$ of the function

(a) $f(z) = \csc^2 z$; *(b)* $f(z) = \dfrac{1}{(z+z^2)^2}$.

Ans. (a) 0; *(b)* -2.

9. Let p and q denote functions that are analytic at a point z_0, where $p(z_0) \neq 0$ and $q(z_0) = 0$. Show that if the quotient $p(z)/q(z)$ has a pole of order m at z_0, then z_0 is a zero of order m of q. (Compare with Theorem 1 in Sec. 76.)

Suggestion: Note that the theorem in Sec. 73 enables one to write

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Then solve for $q(z)$.

10. Recall (Sec. 11) that a point z_0 is an accumulation point of a set S if each deleted neighborhood of z_0 contains at least one point of S. One form of the *Bolzano–Weierstrass theorem* can be stated as follows: *an infinite set of points lying in a closed bounded region R has at least one accumulation point in R*.* Use that theorem and Theorem 2 in Sec. 75 to show that if a function f is analytic in the region R consisting of all points inside and on a simple closed contour C, except possibly for poles inside C, and if all the zeros of f in R are interior to C and are of finite order, then those zeros must be finite in number.

11. Let R denote the region consisting of all points inside and on a simple closed contour C. Use the Bolzano–Weierstrass theorem (see Exercise 10) and the fact that poles are isolated singular points to show that if f is analytic in the region R except for poles interior to C, then those poles must be finite in number.

77. BEHAVIOR OF FUNCTIONS NEAR ISOLATED SINGULAR POINTS

As already indicated in Sec. 72, the behavior of a function f near an isolated singular point z_0 varies, depending on whether z_0 is a pole, a removable singular point, or an essential singular point. In this section, we develop the differences in behavior somewhat further. Since the results presented here will not be used elsewhere in the book, the reader who wishes to reach applications of residue theory more quickly may pass directly to Chap. 7 without disruption.

Theorem 1. *If z_0 is a pole of a function f, then*

$$\lim_{z \to z_0} f(z) = \infty. \tag{1}$$

To verify limit (1), we assume that f has a pole of order m at z_0 and use the theorem in Sec. 73. It tells us that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Since

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{\lim\limits_{z \to z_0} (z - z_0)^m}{\lim\limits_{z \to z_0} \phi(z)} = \frac{0}{\phi(z_0)} = 0,$$

*See, for example, A. E. Taylor and W. R. Mann. "Advanced Calculus," 3d ed., pp. 517 and 521, 1983.

then, limit (1) holds, according to the theorem in Sec. 17 regarding limits that involve the point at infinity.

The next theorem emphasizes how the behavior of f near a removable singular point is fundamentally different from behavior near a pole.

Theorem 2. *If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.*

The proof is easy and is based on the fact that the function f here is analytic in a disk $|z - z_0| < R_2$ when $f(z_0)$ is properly defined; f is then continuous in any closed disk $|z - z_0| \le \varepsilon$ where $\varepsilon < R_2$. Consequently, f is bounded in that disk, according to Theorem 3 in Sec. 18; and this means that, in addition to being analytic, f must be bounded in the deleted neighborhood $0 < |z - z_0| < \varepsilon$.

The proof of our final theorem, regarding the behavior of a function near an essential singular point, relies on the following lemma, which is closely related to Theorem 2 and is known as *Riemann's theorem*.

Lemma. *Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of a point z_0. If f is not analytic at z_0, then it has a removable singularity there.*

To prove this, we assume that f is not analytic at z_0. As a consequence, the point z_0 must be an isolated singularity of f; and $f(z)$ is represented by a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{2}$$

throughout the deleted neighborhood $0 < |z - z_0| < \varepsilon$. If C denotes a positively oriented circle $|z - z_0| = \rho$, where $\rho < \varepsilon$ (Fig. 91), we know from Sec. 60 that the

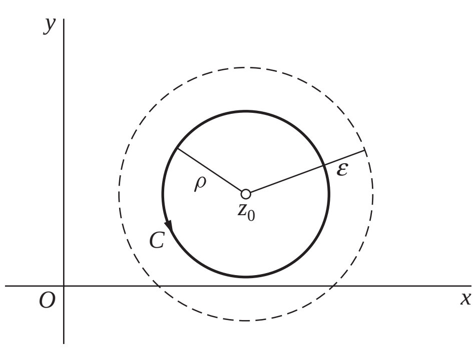

FIGURE 91

coefficients b_n in expansion (2) can be written

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \qquad (n = 1, 2, \ldots). \tag{3}$$

Now the boundedness condition on f tells us that there is a positive constant M such that $|f(z)| \le M$ whenever $0 < |z - z_0| < \varepsilon$. Hence it follows from expression (3) that

$$|b_n| \le \frac{1}{2\pi}\cdot\frac{M}{\rho^{-n+1}}\,2\pi\rho = M\rho^n \qquad (n = 1, 2, \ldots).$$

Since the coefficients b_n are constants and since ρ can be chosen arbitrarily small, we may conclude that $b_n = 0$ $(n = 1, 2, \ldots)$ in the Laurent series (2). This tells us that z_0 is a removable singularity of f, and the proof of the lemma is complete.

We know from Sec. 72 that the behavior of a function near an essential singular point is quite irregular. The next theorem, regarding such behavior, is related to Picard's theorem in that earlier section and is usually referred to as the *Casorati–Weierstrass theorem.* It states that in each deleted neighborhood of an essential singular point, a function assumes values arbitrarily close to any given number.

Theorem 3. *Suppose that z_0 is an essential singularity of a function f, and let w_0 be any complex number. Then, for any positive number ε, the inequality*

$$|f(z) - w_0| < \varepsilon \tag{4}$$

is satisfied at some point z in each deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 (Fig. 92).

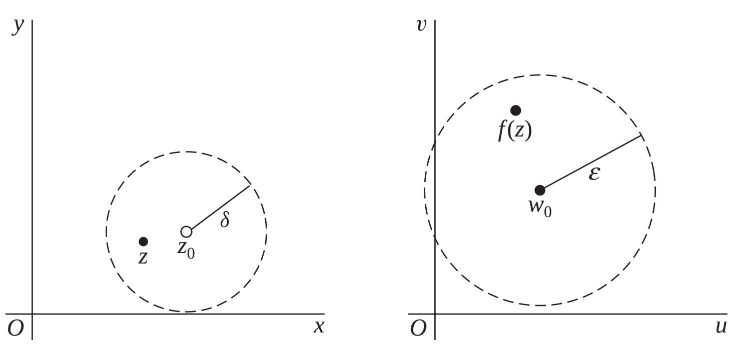

FIGURE 92

The proof is by contradiction. Since z_0 is an isolated singularity of f, there is a deleted neighborhood $0 < |z - z_0| < \delta$ throughout which f is analytic; and we assume that condition (4) is *not* satisfied for any point z there. Thus $|f(z) - w_0| \geq \varepsilon$ when $0 < |z - z_0| < \delta$; and so the function

$$g(z) = \frac{1}{f(z) - w_0} \qquad (0 < |z - z_0| < \delta) \tag{5}$$

is bounded and analytic in its domain of definition. Hence, according to our lemma, z_0 is a removable singularity of g; and we let g be defined at z_0 so that it is analytic there.

If $g(z_0) \neq 0$, the function $f(z)$, which can be written

$$f(z) = \frac{1}{g(z)} + w_0 \tag{6}$$

when $0 < |z - z_0| < \delta$, becomes analytic at z_0 when it is defined there as

$$f(z_0) = \frac{1}{g(z_0)} + w_0.$$

But this means that z_0 is a removable singularity of f, not an essential one, and we have a contradiction.

If $g(z_0) = 0$, the function g must have a zero of some finite order m (Sec. 75) at z_0 because $g(z)$ is not identically equal to zero in the neighborhood $|z - z_0| < \delta$. In view of equation (6), then, f has a pole of order m at z_0 (see Theorem 1 in Sec. 76). So, once again, we have a contradiction; and Theorem 3 here is proved.