By writing $N = N_\varepsilon + 1$ in inequality (3) and using the fact that statements (4) and (5) are true when $N = N_\varepsilon + 1$, we now find that

$$|S(z) - S(z_1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta.$$

This is statement (2), and the theorem is now established.

By writing $w = 1/(z - z_0)$, one can modify the two theorems in the previous section and the theorem here so as to apply to series of the type

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{6}$$

If, for instance, series (6) converges at a point z_1 ($z_1 \neq z_0$), the series

$$\sum_{n=1}^{\infty} b_n w^n$$

must converge absolutely to a continuous function when

$$|w| < \frac{1}{|z_1 - z_0|}. \tag{7}$$

Thus, since inequality (7) is the same as $|z - z_0| > |z_1 - z_0|$, series (6) must converge absolutely to a continuous function in the domain *exterior to* the circle $|z - z_0| = R_1$, where $R_1 = |z_1 - z_0|$. Also, we know that if a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid in an annulus $R_1 < |z - z_0| < R_2$, then *both* of the series on the right converge uniformly in any closed annulus which is concentric to and interior to that region of validity.

65. INTEGRATION AND DIFFERENTIATION OF POWER SERIES

We have just seen that a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

represents a continuous function at each point interior to its circle of convergence. In this section, we prove that the sum $S(z)$ is actually analytic within that circle. Our proof depends on the following theorem, which is of interest in itself.

Theorem 1. *Let C denote any contour interior to the circle of convergence of the power series* (1)*, and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; that is,*

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\,dz. \tag{2}$$

To prove this theorem, we note that since both $g(z)$ and the sum $S(z)$ of the power series are continuous on C, the integral over C of the product

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n\, g(z)(z-z_0)^n + g(z)\rho_N(z),$$

where $\rho_N(z)$ is the remainder of the given series after N terms, exists. The terms of the finite sum here are also continuous on the contour C, and so their integrals over C exist. Consequently, the integral of the quantity $g(z)\rho_N(z)$ must exist; and we may write

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z)\rho_N(z)\,dz. \tag{3}$$

Now let M be the maximum value of $|g(z)|$ on C, and let L denote the length of C. In view of the uniform convergence of the given power series (Sec. 63), we know that for each positive number ε there exists a positive integer N_ε such that, for all points z on C,

$$|\rho_N(z)| < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon.$$

Since N_ε is independent of z, we find that

$$\left| \int_C g(z)\rho_N(z)\,dz \right| < M\varepsilon L \quad \text{whenever} \quad N > N_\varepsilon;$$

that is,

$$\lim_{N\to\infty} \int_C g(z)\rho_N(z)\,dz = 0.$$

It follows, therefore, from equation (3) that

$$\int_C g(z)S(z)\,dz = \lim_{N\to\infty} \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz.$$

This is the same as equation (2), and Theorem 1 is proved.

If $g(z) = 1$ for each value of z in the open disk bounded by the circle of convergence of power series (1), the fact that $(z - z_0)^n$ is entire when $n = 0, 1, 2, \ldots$ ensures that

$$\int_C g(z)(z - z_0)^n \, dz = \int_C (z - z_0)^n \, dz = 0 \qquad (n = 0, 1, 2, \ldots)$$

for every *closed* contour C lying in that domain. According to equation (2), then,

$$\int_C S(z) \, dz = 0$$

for every such contour; and, by Morera's theorem (Sec. 52), the function $S(z)$ is analytic throughout the domain. We state this result as a corollary.

Corollary. *The sum* $S(z)$ *of power series* (1) *is analytic at each point* z *interior to the circle of convergence of that series.*

This corollary is often helpful in establishing the analyticity of functions and in evaluating limits.

EXAMPLE 1. To illustrate, let us show that the function defined by means of the equations

$$f(z) = \begin{cases} (e^z - 1)/z & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases}$$

is entire. Since the Maclaurin series expansion

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} \tag{4}$$

represents $e^z - 1$ for every value of z, the representation

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots, \tag{5}$$

obtained by dividing each side of equation (4) by z, is valid when $z \neq 0$. But series (5) clearly converges to $f(0)$ when $z = 0$. Hence representation (5) is valid for all z; and f is, therefore, an entire function. Note that since $(e^z - 1)/z = f(z)$ when $z \neq 0$ and since f is continuous at $z = 0$,

$$\lim_{z \to 0} \frac{e^z - 1}{z} = \lim_{z \to 0} f(z) = f(0) = 1.$$

The first limit here is, of course, also evident if we write it in the form

$$\lim_{z\to 0} \frac{(e^z - 1) - 0}{z - 0},$$

which is the definition of the derivative of $e^z - 1$ at $z = 0$.

We observed in Sec. 57 that the Taylor series for a function f about a point z_0 converges to $f(z)$ at each point z interior to the circle centered at z_0 and passing through the nearest point z_1 where f fails to be analytic. In view of our corollary to Theorem 1, we now know that *there is no larger circle* about z_0 such that at each point z interior to it the Taylor series converges to $f(z)$. For if there were such a circle, f would be analytic at z_1; but f is not analytic at z_1.

We now present a companion to Theorem 1.

Theorem 2. *The power series* (1) *can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,*

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{6}$$

To prove this, let z denote any point interior to the circle of convergence of series (1). Then let C be some positively oriented simple closed contour surrounding z and interior to that circle. Also, define the function

$$g(s) = \frac{1}{2\pi i} \cdot \frac{1}{(s - z)^2} \tag{7}$$

at each point s on C. Since $g(s)$ is continuous on C, Theorem 1 tells us that

$$\int_C g(s)S(s)\, ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s - z_0)^n\, ds. \tag{8}$$

Now $S(z)$ is analytic inside and on C, and this enables us to write

$$\int_C g(s)S(s)\, ds = \frac{1}{2\pi i} \int_C \frac{S(s)\, ds}{(s - z)^2} = S'(z)$$

with the aid of the integral representation for derivatives in Sec. 51. Furthermore,

$$\int_C g(s)(s - z_0)^n\, ds = \frac{1}{2\pi i} \int_C \frac{(s - z_0)^n}{(s - z)^2}\, ds = \frac{d}{dz}(z - z_0)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus equation (8) reduces to

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z - z_0)^n,$$

which is the same as equation (6). This completes the proof.

EXAMPLE 2. In Example 4, Sec. 59, we saw that

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1| < 1).$$

Differentiation of each side of this equation reveals that

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty} (-1)^n n (z-1)^{n-1} \qquad (|z-1| < 1),$$

or

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n \qquad (|z-1| < 1).$$

66. UNIQUENESS OF SERIES REPRESENTATIONS

The uniqueness of Taylor and Laurent series representations, anticipated in Secs. 59 and 62, respectively, follows readily from Theorem 1 in Sec. 65. We consider first the uniqueness of Taylor series representations.

Theorem 1. *If a series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{1}$$

converges to $f(z)$ *at all points interior to some circle* $|z - z_0| = R$, *then it is the Taylor series expansion for* f *in powers of* $z - z_0$.

To start the proof, we write the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (|z-z_0| < R) \tag{2}$$

in the hypothesis of the theorem using the index of summation m:

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m \qquad (|z-z_0| < R).$$

Then, by appealing to Theorem 1 in Sec. 65, we may write

$$\int_C g(z) f(z)\, dz = \sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m\, dz, \tag{3}$$

where $g(z)$ is any one of the functions

$$g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z - z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots) \tag{4}$$

and C is some circle centered at z_0 and with radius less than R.

In view of the extension (6), Sec. 51, of the Cauchy integral formula (see also the corollary in Sec. 65), we find that

$$\int_C g(z) f(z)\, dz = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}; \tag{5}$$

and, since (see Exercise 10, Sec. 42)

$$\int_C g(z)(z - z_0)^m\, dz = \frac{1}{2\pi i} \int_C \frac{dz}{(z - z_0)^{n-m+1}} = \begin{cases} 0 & \text{when } m \neq n, \\ 1 & \text{when } m = n, \end{cases} \tag{6}$$

it is clear that

$$\sum_{m=0}^{\infty} a_m \int_C g(z)(z - z_0)^m\, dz = a_n. \tag{7}$$

Because of equations (5) and (7), equation (3) now reduces to

$$\frac{f^{(n)}(z_0)}{n!} = a_n.$$

This shows that series (2) is, in fact, the Taylor series for f about the point z_0.

Note how it follows from Theorem 1 that if series (1) converges to zero throughout some neighborhood of z_0, then the coefficients a_n must all be zero.

Our second theorem here concerns the uniqueness of Laurent series representations.

Theorem 2. *If a series*

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{8}$$

converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

The method of proof here is similar to the one used in proving Theorem 1. The hypothesis of this theorem tells us that there is an annular domain about z_0 such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

for each point z in it. Let $g(z)$ be as defined by equation (4), but now allow n to be a negative integer too. Also, let C be any circle around the annulus, centered at z_0 and taken in the positive sense. Then, using the index of summation m and adapting Theorem 1 in Sec. 65 to series involving both nonnegative *and* negative powers of $z - z_0$ (Exercise 10), write

$$\int_C g(z) f(z)\, dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz,$$

or

$$\frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz. \tag{9}$$

Since equations (6) are also valid when the integers m and n are allowed to be negative, equation (9) reduces to

$$\frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = c_n, \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which is expression (5), Sec. 60, for coefficients in the Laurent series for f in the annulus.

EXERCISES

1. By differentiating the Maclaurin series representation

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

obtain the expansions

$$\frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)\, z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1 - z)^3} = \sum_{n=0}^{\infty} (n + 1)(n + 2)\, z^n \qquad (|z| < 1).$$

2. By substituting $1/(1 - z)$ for z in the expansion

$$\frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)\, z^n \qquad (|z| < 1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n - 1)}{(z - 1)^n} \qquad (1 < |z - 1| < \infty).$$

(Compare with Example 2, Sec. 65.)

3. Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2}\cdot\frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

4. With the aid of series, show that the function f defined by means of the equations

$$f(z) = \begin{cases} (\sin z)/z & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases}$$

is entire. Use that result to establish the limit

$$\lim_{z\to 0}\frac{\sin z}{z} = 1.$$

(See Example 1, Sec. 65.)

5. Prove that if

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2-(\pi/2)^2} & \text{when } z \neq \pm\pi/2, \\ -\dfrac{1}{\pi} & \text{when } z = \pm\pi/2, \end{cases}$$

then f is an entire function.

6. In the w plane, integrate the Taylor series expansion (see Example 4, Sec. 59)

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1| < 1)$$

along a contour interior to the circle of convergence from $w = 1$ to $w = z$ to obtain the representation

$$\text{Log } z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1| < 1).$$

7. Use the result in Exercise 6 to show that if

$$f(z) = \frac{\text{Log } z}{z-1} \qquad \text{when } z \neq 1$$

and $f(1) = 1$, then f is analytic throughout the domain

$$0 < |z| < \infty, \ -\pi < \text{Arg } z < \pi.$$

8. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \dfrac{f(z)}{(z - z_0)^{m+1}} & \text{when } z \neq z_0, \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.

9. Suppose that a function $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

inside some circle $|z - z_0| = R$. Use Theorem 2 in Sec. 65, regarding term by term differentiation of such a series, and mathematical induction to show that

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k} (z - z_0)^k \qquad (n = 0, 1, 2, \ldots)$$

when $|z - z_0| < R$. Then, by setting $z = z_0$, show that the coefficients a_n $(n = 0, 1, 2, \ldots)$ are the coefficients in the Taylor series for f about z_0. Thus give an alternative proof of Theorem 1 in Sec. 66.

10. Consider two series

$$S_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad S_2(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

which converge in some annular domain centered at z_0. Let C denote any contour lying in that annulus, and let $g(z)$ be a function which is continuous on C. Modify the proof of Theorem 1, Sec. 65, which tells us that

$$\int_C g(z) S_1(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n \, dz,$$

to prove that

$$\int_C g(z) S_2(z)\, dz = \sum_{n=1}^{\infty} b_n \int_C \frac{g(z)}{(z - z_0)^n} \, dz.$$

Conclude from these results that if

$$S(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

then

$$\int_C g(z) S(z)\, dz = \sum_{n=-\infty}^{\infty} c_n \int_C g(z)(z - z_0)^n \, dz.$$

11. Show that the function

$$f_2(z) = \frac{1}{z^2 + 1} \qquad (z \neq \pm i)$$

is the analytic continuation (Sec. 27) of the function

$$f_1(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad (|z| < 1)$$

into the domain consisting of all points in the z plane except $z = \pm i$.

12. Show that the function $f_2(z) = 1/z^2$ $(z \neq 0)$ is the analytic continuation (Sec. 27) of the function

$$f_1(z) = \sum_{n=0}^{\infty} (n+1)(z+1)^n \qquad (|z+1| < 1)$$

into the domain consisting of all points in the z plane except $z = 0$.

67. MULTIPLICATION AND DIVISION OF POWER SERIES

Suppose that each of the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n (z - z_0)^n \tag{1}$$

converges within some circle $|z - z_0| = R$. Their sums $f(z)$ and $g(z)$, respectively, are then analytic functions in the disk $|z - z_0| < R$ (Sec. 65), and the product of those sums has a Taylor series expansion which is valid there:

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad (|z - z_0| < R). \tag{2}$$

According to Theorem 1 in Sec. 66, the series (1) are themselves Taylor series. Hence the first three coefficients in series (2) are given by the equations

$$c_0 = f(z_0)g(z_0) = a_0 b_0,$$

$$c_1 = \frac{f(z_0)g'(z_0) + f'(z_0)g(z_0)}{1!} = a_0 b_1 + a_1 b_0,$$

and

$$c_2 = \frac{f(z_0)g''(z_0) + 2f'(z_0)g'(z_0) + f''(z_0)g(z_0)}{2!} = a_0 b_2 + a_1 b_1 + a_2 b_0.$$

The general expression for any coefficient c_n is easily obtained by referring to *Leibniz's rule* (Exercise 6)

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z) \qquad (n = 1, 2, \ldots), \tag{3}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n),$$

for the nth derivative of the product of two differentiable functions. As usual, $f^{(0)}(z) = f(z)$ and $0! = 1$. Evidently,

$$c_n = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k};$$

and so expansion (2) can be written

$$\begin{aligned} (4) \qquad f(z)g(z) = a_0 b_0 &+ (a_0 b_1 + a_1 b_0)(z - z_0) \\ &+ (a_0 b_2 + a_1 b_1 + a_2 b_0)(z - z_0)^2 + \cdots \\ &+ \left(\sum_{k=0}^{n} a_k b_{n-k} \right)(z - z_0)^n + \cdots \qquad (|z - z_0| < R). \end{aligned}$$

Series (4) is the same as the series obtained by formally multiplying the two series (1) term by term and collecting the resulting terms in like powers of $z - z_0$; it is called the *Cauchy product* of the two given series.

EXAMPLE 1. The function $e^z/(1+z)$ has a singular point at $z = -1$, and so its Maclaurin series representation is valid in the open disk $|z| < 1$. The first three nonzero terms are easily found by writing

$$\frac{e^z}{1+z} = e^z \frac{1}{1-(-z)} = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)(1 - z + z^2 - z^3 + \cdots)$$

and multiplying these two series term by term. To be precise, we may multiply each term in the first series by 1, then each term in that series by $-z$, etc. The following systematic approach is suggested, where like powers of z are assembled vertically so that their coefficients can be readily added:

$$\begin{array}{llllllll} 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots \\ \quad -z - \quad z^2 - \frac{1}{2}z^3 - \frac{1}{6}z^4 - \cdots \\ \qquad\qquad z^2 + \quad z^3 + \frac{1}{2}z^4 + \frac{1}{6}z^5 + \cdots \\ \qquad\qquad\quad - \quad z^3 - \quad z^4 - \frac{1}{2}z^5 - \frac{1}{6}z^6 - \cdots \\ \qquad\qquad\qquad \vdots \end{array}$$

The desired result is

$$(5) \qquad \frac{e^z}{1+z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \cdots \qquad (|z| < 1).$$

Continuing to let $f(z)$ and $g(z)$ denote the sums of series (1), suppose that $g(z) \neq 0$ when $|z - z_0| < R$. Since the quotient $f(z)/g(z)$ is analytic throughout the disk $|z - z_0| < R$, it has a Taylor series representation

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z - z_0)^n \qquad (|z - z_0| < R), \tag{6}$$

where the coefficients d_n can be found by differentiating $f(z)/g(z)$ successively and evaluating the derivatives at $z = z_0$. The results are the same as those found by formally carrying out the division of the first of series (1) by the second. Since it is usually only the first few terms that are needed in practice, this method is not difficult.

EXAMPLE 2. As pointed out in Sec. 35, the zeros of the entire function $\sinh z$ are the numbers $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$. So the quotient

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^2(z + z^3/3! + z^5/5! + \cdots)},$$

which can be written

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right), \tag{7}$$

has a Laurent series representation in the punctured disk $0 < |z| < \pi$. The denominator of the fraction in parentheses on the right-hand side of equation (7) is a power series that converges to $(\sinh z)/z$ when $z \neq 0$ and to 1 when $z = 0$. Thus the sum of that series is not zero anywhere in the disk $|z| < \pi$; and a power series representation of the fraction in parentheses can be found by dividing the series into unity as follows:

$$
\begin{array}{r|l}
 & 1 - \dfrac{1}{3!}z^2 + \left[\dfrac{1}{(3!)^2} - \dfrac{1}{5!}\right] z^4 + \cdots \\
\hline
1 + \dfrac{1}{3!}z^2 + \dfrac{1}{5!}z^4 + \cdots & 1 \\
 & 1 + \dfrac{1}{3!}z^2 \qquad + \dfrac{1}{5!}z^4 + \cdots \\
\cline{2-2}
 & -\dfrac{1}{3!}z^2 \qquad - \dfrac{1}{5!}z^4 + \cdots \\
 & -\dfrac{1}{3!}z^2 \qquad - \dfrac{1}{(3!)^2}z^4 - \cdots \\
\cline{2-2}
 & \left[\dfrac{1}{(3!)^2} - \dfrac{1}{5!}\right] z^4 + \cdots \\
 & \left[\dfrac{1}{(3!)^2} - \dfrac{1}{5!}\right] z^4 + \cdots \\
\cline{2-2}
 & \vdots
\end{array}
$$

That is,

$$\frac{1}{1 + z^2/3! + z^4/5! + \cdots} = 1 - \frac{1}{3!}z^2 + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots,$$

or

$$\frac{1}{1 + z^2/3! + z^4/5! + \cdots} = 1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots \qquad (|z| < \pi). \tag{8}$$

Hence

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360}z + \cdots \qquad (0 < |z| < \pi). \tag{9}$$

Although we have given only the first three nonzero terms of this Laurent series, any number of terms can, of course, be found by continuing the division.

EXERCISES

1. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \qquad (0 < |z| < 1).$$

2. By writing $\csc z = 1/\sin z$ and then using division, show that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots \qquad (0 < |z| < \pi).$$

3. Use division to obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \qquad (0 < |z| < 2\pi).$$

4. Use the expansion

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360}z + \cdots \qquad (0 < |z| < \pi)$$

in Example 2, Sec. 67, and the method illustrated in Example 1, Sec. 62, to show that

$$\int_C \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3},$$

when C is the positively oriented unit circle $|z| = 1$.

5. Follow these steps, which illustrate an alternative to straightforward division, to obtain representation (8) in Example 2, Sec. 67.

(a) Write

$$\frac{1}{1 + z^2/3! + z^4/5! + \cdots} = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots,$$

where the coefficients in the power series on the right are to be determined by multiplying the two series in the equation

$$1 = \left(1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots\right)(d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots).$$

Perform this multiplication to show that

$$(d_0 - 1) + d_1 z + \left(d_2 + \frac{1}{3!}d_0\right)z^2 + \left(d_3 + \frac{1}{3!}d_1\right)z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right)z^4 + \cdots = 0$$

when $|z| < \pi$.

(b) By setting the coefficients in the last series in part *(a)* equal to zero, find the values of d_0, d_1, d_2, d_3, and d_4. With these values, the first equation in part *(a)* becomes equation (8), Sec. 67.

6. Use mathematical induction to establish Leibniz' rule (Sec. 67)

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \qquad (n = 1, 2, \ldots)$$

for the n^{th} derivative of the product of two differentiable functions $f(z)$ and $g(z)$.

Suggestion: Note that the rule is valid when $n = 1$. Then, assuming that it is valid when $n = m$ where m is any positive integer, show that

$$\begin{aligned}(fg)^{(m+1)} &= (fg')^{(m)} + (f'g)^{(m)} \\ &= fg^{(m+1)} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(k)} g^{(m+1-k)} + f^{(m+1)}g.\end{aligned}$$

Finally, with the aid of the identify

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$$

that was used in Exercise 8, Sec. 3, show that

$$\begin{aligned}(fg)^{(m+1)} &= fg^{(m+1)} + \sum_{k=1}^{m} \binom{m+1}{k} f^{(k)} g^{(m+1-k)} + f^{(m+1)}g \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{(m+1-k)}.\end{aligned}$$

7. Let $f(z)$ be an entire function that is represented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (|z| < \infty).$$

(a) By differentiating the composite function $g(z) = f[f(z)]$ successively, find the first three nonzero terms in the Maclaurin series for $g(z)$ and thus show that

$$f[f(z)] = z + 2\,a_2 z^2 + 2\,(a_2^2 + a_3)z^3 + \cdots \qquad (|z| < \infty).$$

(b) Obtain the result in part *(a)* in a *formal* manner by writing

$$f[f(z)] = f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \cdots,$$

replacing $f(z)$ on the right-hand side here by its series representation, and then collecting terms in like powers of z.

(c) By applying the result in part *(a)* to the function $f(z) = \sin z$, show that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \cdots \qquad (|z| < \infty).$$

8. The *Euler numbers* are the numbers E_n $(n = 0, 1, 2, \ldots)$ in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \qquad (|z| < \pi/2).$$

Point out why this representation is valid in the indicated disk and why

$$E_{2n+1} = 0 \qquad (n = 0, 1, 2, \ldots).$$

Then show that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$