(b) Write $z_n = x_n + iy_n$ and recall from the theory of sequences of real numbers that the convergence of x_n and y_n $(n = 1, 2, \ldots)$ implies that $|x_n| \le M_1$ and $|y_n| \le M_2$ $(n = 1, 2, \ldots)$ for some positive numbers M_1 and M_2.

57. TAYLOR SERIES

We turn now to *Taylor's theorem,* which is one of the most important results of the chapter.

Theorem. *Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0* (Fig. 74). *Then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0), \tag{1}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

That is, series (1) *converges to $f(z)$ when z lies in the stated open disk.*

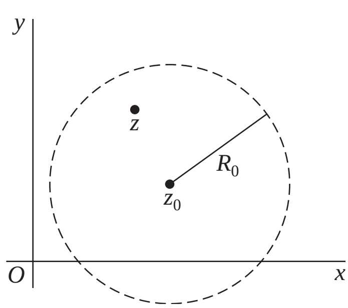

FIGURE 74

This is the expansion of $f(z)$ into a *Taylor series* about the point z_0. It is the familiar Taylor series from calculus, adapted to functions of a complex variable. With the agreement that

$$f^{(0)}(z_0) = f(z_0) \quad \text{and} \quad 0! = 1,$$

series (1) can, of course, be written

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \qquad (|z - z_0| < R_0). \tag{3}$$

Any function which is analytic at a point z_0 must have a Taylor series about z_0. For, if f is analytic at z_0, it is analytic throughout some neighborhood $|z - z_0| < \varepsilon$ of that point (Sec. 24); and ε may serve as the value of R_0 in the statement of Taylor's theorem. Also, if f is entire, R_0 can be chosen arbitrarily large; and the condition of validity becomes $|z - z_0| < \infty$. The series then converges to $f(z)$ at each point z in the finite plane.

When it is known that f is analytic everywhere inside a circle centered at z_0, convergence of its Taylor series about z_0 to $f(z)$ for each point z within that circle is ensured; no test for the convergence of the series is even required. In fact, according to Taylor's theorem, the series converges to $f(z)$ within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 at which f *fails* to be analytic. In Sec. 65, we shall find that this is actually the largest circle centered at z_0 such that the series converges to $f(z)$ for all z interior to it.

In the following section, we shall first prove Taylor's theorem when $z_0 = 0$, in which case f is assumed to be analytic throughout a disk $|z| < R_0$ and series (1) becomes a *Maclaurin series*:

(4) $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad (|z| < R_0).$$

The proof when z_0 is arbitrary will follow as an immediate consequence. A reader who wishes to accept the proof of Taylor's theorem can easily skip to the examples in Sec. 59.

58. PROOF OF TAYLOR'S THEOREM

To begin the derivation of representation (4), Sec. 57, we write $|z| = r$ and let C_0 denote and positively oriented circle $|z| = r_0$, where $r < r_0 < R_0$ (see Fig. 75). Since f is analytic inside and on the circle C_0 and since the point z is interior to

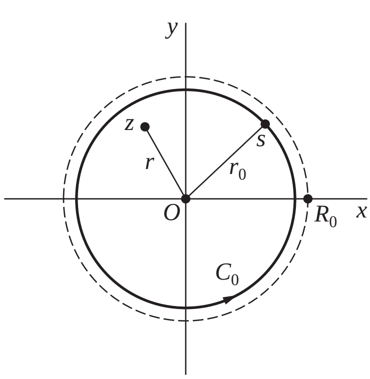

FIGURE 75

C_0, the Cauchy integral formula

(1) $$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s - z}$$

applies.

Now the factor $1/(s - z)$ in the integrand here can be put in the form

(2) $$\frac{1}{s - z} = \frac{1}{s} \cdot \frac{1}{1 - (z/s)};$$

and we know from the example in Sec. 56 that

(3) $$\frac{1}{1 - z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1 - z}$$

when z is any complex number other than unity. Replacing z by z/s in expression (3), then, we can rewrite equation (2) as

(4) $$\frac{1}{s - z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + z^N \frac{1}{(s - z)s^N}.$$

Multiplying through this equation by $f(s)$ and then integrating each side with respect to s around C_0, we find that

$$\int_{C_0} \frac{f(s)\ ds}{s - z} = \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)\ ds}{s^{n+1}} z^n + z^N \int_{C_0} \frac{f(s)\ ds}{(s - z)s^N}.$$

In view of expression (1) and the fact that (Sec. 51)

$$\frac{1}{2\pi i} \int_{C_0} \frac{f(s)\ ds}{s^{n+1}} = \frac{f^{(n)}(0)}{n!} \qquad (n = 0, 1, 2, \ldots),$$

this reduces, after we multiply through by $1/(2\pi i)$, to

(5) $$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z),$$

where

(6) $$\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)\ ds}{(s - z)s^N}.$$

Representation (4) in Sec. 57 now follows once it is shown that

(7) $$\lim_{N\to\infty} \rho_N(z) = 0.$$

To accomplish this, we recall that $|z| = r$ and that C_0 has radius r_0, where $r_0 > r$. Then, if s is a point on C_0, we can see that

$$|s - z| \geq ||s| - |z|| = r_0 - r.$$

Consequently, if M denotes the maximum value of $|f(s)|$ on C_0,

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0^N} 2\pi r_0 = \frac{Mr_0}{r_0 - r}\left(\frac{r}{r_0}\right)^N.$$

Inasmuch as $(r/r_0) < 1$, limit (7) clearly holds.

To verify the theorem when the disk of radius R_0 is centered at an arbitrary point z_0, we suppose that f is analytic when $|z - z_0| < R_0$ and note that the composite function $f(z + z_0)$ must be analytic when $|(z + z_0) - z_0| < R_0$. This last inequality is, of course, just $|z| < R_0$; and, if we write $g(z) = f(z + z_0)$, the analyticity of g in the disk $|z| < R_0$ ensures the existence of a Maclaurin series representation:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \qquad (|z| < R_0).$$

That is,

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \qquad (|z| < R_0).$$

After replacing z by $z - z_0$ in this equation and its condition of validity, we have the desired Taylor series expansion (1) in Sec. 57.

59. EXAMPLES

In Sec. 66, we shall see that if there are constants a_n $(n = 0, 1, 2, \ldots)$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all points z interior to some circle centered at z_0, then the power series here must be *the* Taylor series for f about z_0, regardless of how those constants arise. This observation often allows us to find the coefficients a_n in Taylor series in more efficient ways than by appealing directly to the formula $a_n = f^{(n)}(z_0)/n!$ in Taylor's theorem.

In the following examples, we use the formula in Taylor's theorem to find the Maclaurin series expansions of some fairly simple functions, and we emphasize the use of those expansions in finding other representations. In our examples, we shall freely use expected properties of convergent series, such as those verified in Exercises 7 and 8, Sec. 56.

EXAMPLE 1. Since the function $f(z) = e^z$ is entire, it has a Maclaurin series representation which is valid for all z. Here $f^{(n)}(z) = e^z$ $(n = 0, 1, 2, \ldots)$;

and, because $f^{(n)}(0) = 1$ $(n = 0, 1, 2, \ldots)$, it follows that

(1) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty).$$

Note that if $z = x + i0$, expansion (1) becomes

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty < x < \infty).$$

The entire function $z^2 e^{3z}$ also has a Maclaurin series expansion. The simplest way to obtain it is to replace z by $3z$ on each side of equation (1) and then multiply through the resulting equation by z^2:

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} \qquad (|z| < \infty).$$

Finally, if we replace n by $n - 2$ here, we have

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \qquad (|z| < \infty).$$

EXAMPLE 2. One can use expansion (1) and the definition (Sec. 34)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

to find the Maclaurin series for the entire function $f(z) = \sin z$. To give the details, we refer to expansion (1) and write

$$\sin z = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] = \frac{1}{2i} \sum_{n=0}^{\infty} \left[1 - (-1)^n\right] \frac{i^n z^n}{n!} \qquad (|z| < \infty).$$

But $1 - (-1)^n = 0$ when n is even, and so we can replace n by $2n + 1$ in this last series:

$$\sin z = \frac{1}{2i} \sum_{n=0}^{\infty} \left[1 - (-1)^{2n+1}\right] \frac{i^{2n+1} z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

Inasmuch as

$$1 - (-1)^{2n+1} = 2 \quad \text{and} \quad i^{2n+1} = (i^2)^n i = (-1)^n i,$$

this reduces to

(2) $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

Term by term differentiation will be justified in Sec. 65. Using that procedure here, we differentiate each side of equation (2) and write

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dz} z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)!} z^{2n}.$$

That is,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty). \tag{3}$$

EXAMPLE 3. Because $\sinh z = -i \sin(iz)$ (Sec. 35), we need only replace z by iz on each side of equation (2) and multiply through the result by $-i$ to see that

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty). \tag{4}$$

Likewise, since $\cosh z = \cos(iz)$, it follows from expansion (3) that

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty). \tag{5}$$

Observe that the Taylor series for $\cosh z$ about the point $z_0 = -2\pi i$, for example, is obtained by replacing the variable z by $z + 2\pi i$ on each side of equation (5) and then recalling that $\cosh(z + 2\pi i) = \cosh z$ for all z:

$$\cosh z = \sum_{n=0}^{\infty} \frac{(z+2\pi i)^{2n}}{(2n)!} \qquad (|z| < \infty).$$

EXAMPLE 4. Another Maclaurin series representation is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1), \tag{6}$$

since the derivatives of the function $f(z) = 1/(1-z)$, which fails to be analytic at $z = 1$, are

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \qquad (n = 0, 1, 2, \ldots).$$

In particular, $f^{(n)}(0) = n!$ $(n = 0, 1, 2, \ldots)$. Note that expansion (6) gives us the sum of an infinite *geometric series*, where z is the common ratio of adjacent terms:

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z} \qquad (|z| < 1).$$

This is, of course, the summation formula that was found in another way in the example in Sec. 56.

If we substitute $-z$ for z in equation (6) and its condition of validity, and note that $|z| < 1$ when $|-z| < 1$, we see that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad (|z| < 1).$$

If, on the other hand, we replace the variable z in equation (6) by $1-z$, we have the Taylor series representation

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1| < 1).$$

This condition of validity follows from the one associated with expansion (6) since $|1-z| < 1$ is the same as $|z-1| < 1$.

EXAMPLE 5. For our final example, let us expand the function

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3} \cdot \frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

into a series involving powers of z. We cannot find a Maclaurin series for $f(z)$ since it is not analytic at $z = 0$. But we do know from expansion (6) that

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 - \cdots \qquad (|z| < 1).$$

Hence, when $0 < |z| < 1$,

$$f(z) = \frac{1}{z^3}(2 - 1 + z^2 - z^4 + z^6 - z^8 + \cdots) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots.$$

We call such terms as $1/z^3$ and $1/z$ *negative* powers of z since they can be written z^{-3} and z^{-1}, respectively. The theory of expansions involving negative powers of $z - z_0$ will be discussed in the next section.

EXERCISES*

1. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

*In these and subsequent exercises on series expansions, it is recommended that the reader use, when possible, representations (1) through (6) in Sec. 59.

2. Obtain the Taylor series

$$e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad (|z-1|<\infty)$$

for the function $f(z) = e^z$ by
(a) using $f^{(n)}(1)$ $(n = 0, 1, 2, \ldots)$; *(b)* writing $e^z = e^{z-1}e$.

3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\cdot\frac{1}{1+(z^4/9)}.$$

Ans. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+2}}z^{4n+1} \quad (|z|<\sqrt{3}).$

4. Show that if $f(z) = \sin z$, then

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus give an alternative derivation of the Maclaurin series (2) for $\sin z$ in Sec. 59.

5. Rederive the Maclaurin series (3) in Sec. 59 for the function $f(z) = \cos z$ by
(a) using the definition

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

in Sec. 34 and appealing to the Maclaurin series (1) for e^z in Sec. 59;
(b) showing that

$$f^{(2n)}(0) = (-1)^n \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

6. Use representation (2), Sec. 59, for sin z to write the Maclaurin series for the function

$$f(z) = \sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

7. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i|<\sqrt{2}).$$

Suggestion: Start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\cdot\frac{1}{1-(z-i)/(1-i)}.$$

8. With the aid of the identity (see Sec. 34)

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

9. Use the identity $\sinh(z + \pi i) = -\sinh z$, verified in Exercise 7*(a)*, Sec. 35, and the fact that $\sinh z$ is periodic with period $2\pi i$ to find the Taylor series for $\sinh z$ about the point $z_0 = \pi i$.

$$\textit{Ans. } -\sum_{n=0}^{\infty} \frac{(z - \pi i)^{2n+1}}{(2n+1)!} \qquad (|z - \pi i| < \infty).$$

10. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

11. Show that when $z \neq 0$,

(a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \dfrac{z^2}{4!} + \cdots;$

(b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots.$

12. Derive the expansions

(a) $\dfrac{\sinh z}{z^2} = \dfrac{1}{z} + \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \qquad (0 < |z| < \infty);$

(b) $z^3 \cosh\left(\dfrac{1}{z}\right) = \dfrac{z}{2} + z^3 + \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}} \qquad (0 < |z| < \infty).$

13. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

60. LAURENT SERIES

If a function f fails to be analytic at a point z_0, one cannot apply Taylor's theorem at that point. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$. (See Example 5, Sec. 59, and also Exercises 11, 12, and 13 for that section.) We now present the theory of such representations, and we begin with *Laurent's theorem.*

Theorem. *Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain* (Fig. 76). *Then, at each point in the domain, $f(z)$ has the series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (R_1 < |z - z_0| < R_2), \tag{1}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots) \tag{2}$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \qquad (n = 1, 2, \ldots). \tag{3}$$

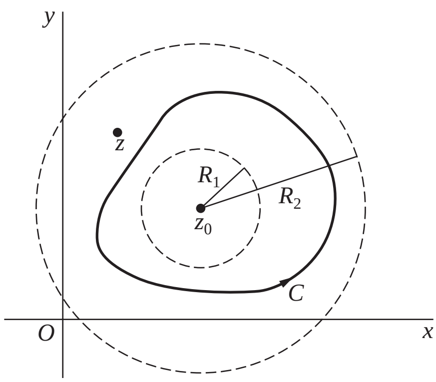

FIGURE 76

Note how replacing n by $-n$ in the second series in representation (1) enables us to write that series as

$$\sum_{n=-\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}},$$

where

$$b_{-n} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n = -1, -2, \ldots).$$

Thus

$$f(z) = \sum_{n=-\infty}^{-1} b_{-n}(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad (R_1 < |z-z_0| < R_2).$$

If

$$c_n = \begin{cases} b_{-n} & \text{when } n \le -1, \\ a_n & \text{when } n \ge 0, \end{cases}$$

this becomes

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n \quad (R_1 < |z-z_0| < R_2), \tag{4}$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

In either one of the forms (1) and (4), the representation of $f(z)$ is called a *Laurent series*.

Observe that the integrand in expression (3) can be written $f(z)(z-z_0)^{n-1}$. Thus it is clear that when f is actually analytic throughout the disk $|z-z_0| < R_2$, this integrand is too. Hence all of the coefficients b_n are zero; and, because (Sec. 51)

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots),$$

expansion (1) reduces to a Taylor series about z_0.

If, however, f fails to be analytic at z_0 but is otherwise analytic in the disk $|z-z_0| < R_2$, the radius R_1 can be chosen arbitrarily small. Representation (1) is then valid in the punctured disk $0 < |z-z_0| < R_2$. Similarly, if f is analytic at each point in the finite plane exterior to the circle $|z-z_0| = R_1$, the condition of validity is $R_1 < |z-z_0| < \infty$. Note that if f is analytic *everywhere* in the finite plane except at z_0, series (1) is valid at each point of analyticity, or when $0 < |z-z_0| < \infty$.

We shall prove Laurent's theorem first when $z_0 = 0$, which means that the annulus is centered at the origin. The verification of the theorem when z_0 is arbitrary will follow readily; and, as was the case with Taylor's theorem, a reader can skip the entire proof without difficulty.

61. PROOF OF LAURENT'S THEOREM

We start the proof by forming a closed annular region $r_1 \le |z| \le r_2$ that is contained in the domain $R_1 < |z| < R_2$ and whose interior contains both the point z and the contour C (Fig. 77). We let C_1 and C_2 denote the circles $|z| = r_1$ and $|z| = r_2$,

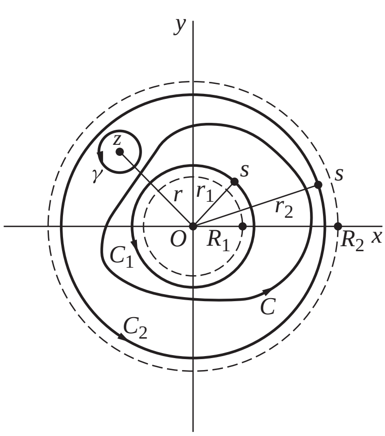

FIGURE 77

respectively, and we assign them a positive orientation. Observe that f is analytic on C_1 and C_2, as well as in the annular domain between them.

Next, we construct a positively oriented circle γ with center at z and small enough to be contained in the interior of the annular region $r_1 \le |z| \le r_2$, as shown in Fig. 77. It then follows from the adaptation of the Cauchy–Goursat theorem to integrals of analytic functions around oriented boundaries of multiply connected domains (Sec. 49) that

$$\int_{C_2} \frac{f(s)\,ds}{s-z} - \int_{C_1} \frac{f(s)\,ds}{s-z} - \int_{\gamma} \frac{f(s)\,ds}{s-z} = 0.$$

But, according to the Cauchy integral formula, the value of the third integral here is $2\pi i f(z)$. Hence

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)\,ds}{s-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)\,ds}{z-s}. \tag{1}$$

Now the factor $1/(s-z)$ in the first of these integrals is the same as in expression (1), Sec. 58, where Taylor's theorem was proved; and we shall need here the expansion

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + z^N \frac{1}{(s-z)s^N}, \tag{2}$$

which was used in that earlier section. As for the factor $1/(z-s)$ in the second integral, an interchange of s and z in equation (2) reveals that

$$\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \cdot \frac{1}{z^{n+1}} + \frac{1}{z^N} \cdot \frac{s^N}{z-s}.$$

If we replace the index of summation n here by $n-1$, this expansion takes the form

$$\frac{1}{z-s} = \sum_{n=1}^{N} \frac{1}{s^{-n+1}} \cdot \frac{1}{z^n} + \frac{1}{z^N} \cdot \frac{s^N}{z-s}, \tag{3}$$

which is to be used in what follows.

Multiplying through equations (2) and (3) by $f(s)/(2\pi i)$ and then integrating each side of the resulting equations with respect to s around C_2 and C_1, respectively, we find from expression (1) that

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + \rho_N(z) + \sum_{n=1}^{N} \frac{b_n}{z^n} + \sigma_N(z), \tag{4}$$

where the numbers a_n $(n = 0, 1, 2, \ldots, N-1)$ and b_n $(n = 1, 2, \ldots, N)$ are given by the equations

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)\,ds}{s^{n+1}}, \quad b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)\,ds}{s^{-n+1}} \tag{5}$$

and where

$$\rho_N(z) = \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)\,ds}{(s-z)s^N}, \quad \sigma_N(z) = \frac{1}{2\pi i\, z^N}\int_{C_1}\frac{s^N f(s)\,ds}{z-s}.$$

As N tends to ∞, expression (4) evidently takes the proper form of a Laurent series in the domain $R_1 < |z| < R_2$, provided that

$$\lim_{N\to\infty}\rho_N(z) = 0 \quad \text{and} \quad \lim_{N\to\infty}\sigma_N(z) = 0. \tag{6}$$

These limits are readily established by a method already used in the proof of Taylor's theorem in Sec. 58. We write $|z| = r$, so that $r_1 < r < r_2$, and let M denote the maximum value of $|f(s)|$ on C_1 and C_2. We also note that if s is a point on C_2, then $|s - z| \geq r_2 - r$; and if s is on C_1, we have $|z - s| \geq r - r_1$. This enables us to write

$$|\rho_N(z)| \leq \frac{Mr_2}{r_2 - r}\left(\frac{r}{r_2}\right)^N \quad \text{and} \quad |\sigma_N(z)| \leq \frac{Mr_1}{r - r_1}\left(\frac{r_1}{r}\right)^N.$$

Since $(r/r_2) < 1$ and $(r_1/r) < 1$, it is now clear that both $\rho_N(z)$ and $\sigma_N(z)$ tend to zero as N tends to infinity.

Finally, we need only recall the corollary in Sec. 49 to see that the contours used in integrals (5) here may be replaced by the contour C. This completes the proof of Laurent's theorem when $z_0 = 0$ since, if z is used instead of s as the variable of integration, expressions (5) for the coefficients a_n and b_n are the same as expressions (2) and (3) in Sec. 60 when $z_0 = 0$ there.

To extend the proof to the general case in which z_0 is an arbitrary point in the finite plane, we let f be a function satisfying the conditions in the theorem; and, just as we did in the proof of Taylor's theorem, we write $g(z) = f(z + z_0)$. Since $f(z)$ is analytic in the annulus $R_1 < |z - z_0| < R_2$, the function $f(z + z_0)$ is analytic when $R_1 < |(z + z_0) - z_0| < R_2$. That is, g is analytic in the annulus $R_1 < |z| < R_2$, which is centered at the origin. Now the simple closed contour C in the statement of the theorem has some parametric representation $z = z(t)$ $(a \leq t \leq b)$, where

$$R_1 < |z(t) - z_0| < R_2 \tag{7}$$

for all t in the interval $a \leq t \leq b$. Hence if Γ denotes the path

$$z = z(t) - z_0 \qquad (a \leq t \leq b), \tag{8}$$

Γ is not only a simple closed contour but, in view of inequalities (7), it lies in the domain $R_1 < |z| < R_2$. Consequently, $g(z)$ has a Laurent series representation

$$g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \qquad (R_1 < |z| < R_2), \tag{9}$$

where

$$a_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{g(z)\,dz}{z^{n+1}} \qquad (n = 0, 1, 2, \ldots), \tag{10}$$

$$b_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{g(z)\,dz}{z^{-n+1}} \qquad (n = 1, 2, \ldots). \tag{11}$$

Representation (1) in Sec. 60 is obtained if we write $f(z + z_0)$ instead of $g(z)$ in equation (9) and then replace z by $z - z_0$ in the resulting equation, as well as in the condition of validity $R_1 < |z| < R_2$. Expression (10) for the coefficients a_n is, moreover, the same as expression (2), Sec. 60, since

$$\int_{\Gamma} \frac{g(z)\,dz}{z^{n+1}} = \int_a^b \frac{f[z(t)]z'(t)}{[z(t) - z_0]^{n+1}}\,dt = \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}.$$

Similarly, the coefficients b_n in expression (11) are the same as those in expression (3), Sec. 60.

62. EXAMPLES

The coefficients in a Laurent series are generally found by means other than appealing directly to their integral representations. This is illustrated in the following examples, where it is always assumed that when the annular domain is specified, a Laurent series for a given function in unique. As was the case with Taylor series, we defer the proof of such uniqueness until Sec. 66.

EXAMPLE 1. Replacing z by $1/z$ in the Maclaurin series expansion (Sec. 59)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we have the Laurent series representation

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \qquad (0 < |z| < \infty).$$

Note that no positive powers of z appear here, the coefficients of the positive powers being zero. Note, too, that the coefficient of $1/z$ is unity; and, according to Laurent's theorem in Sec. 60, that coefficient is the number

$$b_1 = \frac{1}{2\pi i}\int_C e^{1/z}\,dz,$$

where C is any positively oriented simple closed contour around the origin. Since $b_1 = 1$, then,

$$\int_C e^{1/z}\,dz = 2\pi i.$$

This method of evaluating certain integrals around simple closed contours will be developed in considerable detail in Chap. 6.

EXAMPLE 2. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series, where $z_0 = i$. That is,

$$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n(z-i)^n \qquad (0 < |z-i| < \infty)$$

where $c_{-2} = 1$ and all of the other coefficients are zero. From formula (5), Sec. 60, for the coefficients in a Laurent series, we know that

$$c_n = \frac{1}{2\pi i}\int_C \frac{dz}{(z-i)^{n+3}} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

where C is, for instance, any positively oriented circle $|z-i| = R$ about the point $z_0 = i$. Thus [compare with Exercise 10*(b)*, Sec. 42]

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0 & \text{when } n \neq -2, \\ 2\pi i & \text{when } n = -2. \end{cases}$$

The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}, \tag{1}$$

which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains

$$|z| < 1, \quad 1 < |z| < 2, \quad \text{and} \quad 2 < |z| < \infty.$$

In each of those domains, denoted by D_1, D_2, and D_3, respectively, in Fig. 78, $f(z)$ has series representations in powers of z. They can all be found by making the appropriate replacements for z in the expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1) \tag{2}$$

that was obtained in Example 4, Sec. 59. We consider first the domain D_1.

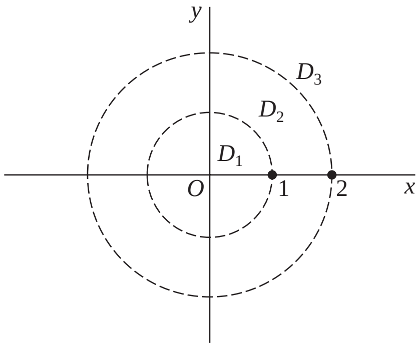

FIGURE 78

EXAMPLE 3. The representation in D_1 is a Maclaurin series. To find it, we observe that

$$|z| < 1 \quad \text{and} \quad |z/2| < 1$$

when z is in D_1; and so we put expression (1) in the form

$$f(z) = -\frac{1}{1-z} + \frac{1}{2} \cdot \frac{1}{1-(z/2)}.$$

This tells us that

$$(3) \qquad f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1)\, z^n \quad (|z| < 1).$$

The representations in D_2 and D_3 are treated in the next two examples.

EXAMPLE 4. Because $1 < |z| < 2$ when z is a point in D_2, we know that

$$|1/z| < 1 \text{ and } |z/2| < 1$$

for such points. This suggests writing expression (1) as

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-(1/z)} + \frac{1}{2} \cdot \frac{1}{1-(z/2)}.$$

In view of expansion (2), then,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad (1 < |z| < 2).$$

If we replace the index of summation n in the first of these series by $n-1$ and then interchange the two series, we arrive at an expansion having the same form as the one in the statement of Laurent's theorem (Sec. 60):

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{z^n}\qquad(1<|z|<2). \tag{4}$$

Since there is only one Laurent series for $f(z)$ in the annulus D_2, expansion (4) is, in fact, *the* Laurent series for $f(z)$ there.

EXAMPLE 5. The representation of the function (1) in the unbounded domain D_3, where $2<|z|<\infty$, is also a Laurent series. Since $|2/z|<1$ when z is in D_3, it is also true that $|1/z|<1$. So if we write expression (1) as

$$f(z)=\frac{1}{2}\cdot\frac{1}{1-(1/z)}-\frac{1}{z}\cdot\frac{1}{1-(2/z)},$$

we find that

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}\qquad(2<|z|<\infty).$$

Replacing n by $n-1$ in this last series then gives the standard form

$$f(z)=\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}\qquad(2<|z|<\infty) \tag{5}$$

used in Laurent's theorem in Sec. 60. Here, of course, all the a_n's in that theorem are zero.

EXERCISES

1. Find the Laurent series that represents the function

$$f(z)=z^2\sin\left(\frac{1}{z^2}\right)$$

in the domain $0<|z|<\infty$.

Ans. $1+\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}}$.

2. Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2}=\frac{1}{e}\left[\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!}+\frac{1}{z+1}+\frac{1}{(z+1)^2}\right]\qquad(0<|z+1|<\infty).$$

3. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Ans. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$.

4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

Ans. $\displaystyle\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2} \quad (0 < |z| < 1); \qquad -\sum_{n=3}^{\infty} \frac{1}{z^n} \quad (1 < |z| < \infty)$.

5. Represent the function

$$f(z) = \frac{z+1}{z-1}$$

(a) by its Maclaurin series, and state where the representation is valid;
(b) by its Laurent series in the domain $1 < |z| < \infty$.

Ans. (a) $\displaystyle -1 - 2\sum_{n=1}^{\infty} z^n \quad (|z| < 1)$; *(b)* $\displaystyle 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$.

6. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

7. Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Ans. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} + \frac{1}{z} \quad (0 < |z| < 1); \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}} \quad (1 < |z| < \infty)$.

8. *(a)* Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

(b) After writing $z = e^{i\theta}$ in the equation obtained in part *(a)*, equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$. (Compare with Exercise 4, Sec. 56.)

9. Suppose that a series

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

converges to an analytic function $X(z)$ in some annulus $R_1 < |z| < R_2$. That sum $X(z)$ is called the *z-transform* of $x[n]$ $(n = 0, \pm 1, \pm 2, \ldots)$.* Use expression (5), Sec. 60, for the coefficients in a Laurent series to show that if the annulus contains the unit circle $|z| = 1$, then the *inverse* z-transform of $X(z)$ can be written

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta})e^{in\theta}\, d\theta \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

10. *(a)* Let z be any complex number, and let C denote the unit circle

$$w = e^{i\phi} \qquad (-\pi \le \phi \le \pi)$$

in the w plane. Then use that contour in expression (5), Sec. 60, for the coefficients in a Laurent series, adapted to such series about the origin in the w plane, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)w^n \qquad (0 < |w| < \infty)$$

where

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin\phi)]\, d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) With the aid of Exercise 5, Sec. 38, regarding certain definite integrals of even and odd complex-valued functions of a real variable, show that the coefficients in part *(a)* here can be written†

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - z\sin\phi)\, d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

---

*The z-transform arises in studies of discrete-time linear systems. See, for instance, the book by Oppenheim, Schafer, and Buck that is listed in Appendix 1.

†These coefficients $J_n(z)$ are called *Bessel functions* of the first kind. They play a prominent role in certain areas of applied mathematics. See, for example, the authors' "Fourier Series and Boundary Value Problems," 7th ed., Chap. 9, 2008.

11. *(a)* Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$. By taking that circle as the path of integration in expressions (2) and (3), Sec. 60, for the coefficients a_n and b_n in a Laurent series in powers of z, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi$$

when z is any point in the annular domain.

(b) Write $u(\theta) = \text{Re}[f(e^{i\theta})]$ and show how it follows from the expansion in part *(a)* that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\,d\phi.$$

This is one form of the *Fourier series* expansion of the real-valued function $u(\theta)$ on the interval $-\pi \le \theta \le \pi$. The restriction on $u(\theta)$ is more severe than is necessary in order for it to be represented by a Fourier series.*

63. ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

This section and the three following it are devoted mainly to various properties of power series. A reader who wishes to simply accept the theorems and the corollary in these sections can easily skip the proofs in order to reach Sec. 67 more quickly.

We recall from Sec. 56 that a series of complex numbers converges *absolutely* if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

Theorem 1. *If a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

converges when $z = z_1$ $(z_1 \ne z_0)$, *then it is absolutely convergent at each point* z *in the open disk* $|z - z_0| < R_1$ *where* $R_1 = |z_1 - z_0|$ (Fig. 79).

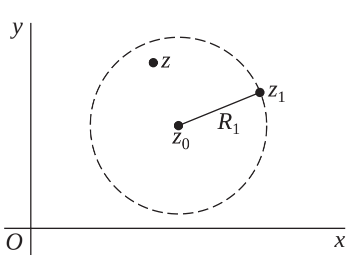

FIGURE 79

*For other sufficient conditions, see Secs. 12 and 13 of the book cited in the footnote to Exercise 10.

We start the proof by assuming that the series

$$\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n \quad (z_1 \neq z_0)$$

converges. The terms $a_n(z_1 - z_0)^n$ are thus bounded; that is,

$$|a_n(z_1 - z_0)^n| \leq M \quad (n = 0, 1, 2, \ldots)$$

for some positive constant M (see Sec. 56). If $|z - z_0| < R_1$ and if we write

$$\rho = \frac{|z - z_0|}{|z_1 - z_0|},$$

we can see that

$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \left(\frac{|z - z_0|}{|z_1 - z_0|} \right)^n \leq M\rho^n \quad (n = 0, 1, 2, \ldots).$$

Now the series

$$\sum_{n=0}^{\infty} M\rho^n$$

is a geometric series, which converges since $\rho < 1$. Hence, by the comparison test for series of real numbers,

$$\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$$

converges in the open disk $|z - z_0| < R_1$. This completes the proof.

The theorem tells us that the set of all points inside some circle centered at z_0 is a region of convergence for the power series (1), provided it converges at some point other than z_0. The greatest circle centered at z_0 such that series (1) converges at each point inside is called the *circle of convergence* of series (1). The series cannot converge at any point z_2 outside that circle, according to the theorem; for if it did, it would converge everywhere inside the circle centered at z_0 and passing through z_2. The first circle could not, then, be the circle of convergence.

Our next theorem involves terminology that we must first define. Suppose that the power series (1) has circle of convergence $|z - z_0| = R$, and let $S(z)$ and $S_N(z)$ represent the sum and partial sums, respectively, of that series:

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad S_N(z) = \sum_{n=0}^{N-1} a_n(z - z_0)^n \qquad (|z - z_0| < R).$$

Then write the remainder function (see Sec. 56)

$$\rho_N(z) = S(z) - S_N(z) \qquad (|z - z_0| < R). \tag{2}$$

Since the power series converges for any fixed value of z when $|z - z_0| < R$, we know that the remainder $\rho_N(z)$ approaches zero for any such z as N tends to infinity. According to definition (2), Sec. 55, of the limit of a sequence, this means that corresponding to each positive number ε, there is a positive integer N_ε such that

$$|\rho_N(z)| < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon. \tag{3}$$

When the choice of N_ε depends only on the value of ε and is independent of the point z taken in a specified region within the circle of convergence, the convergence is said to be *uniform* in that region.

Theorem 2. *If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n, \tag{4}$$

then that series must be uniformly convergent in the closed disk $|z - z_0| \leq R_1$, where $R_1 = |z_1 - z_0|$ (Fig. 80).

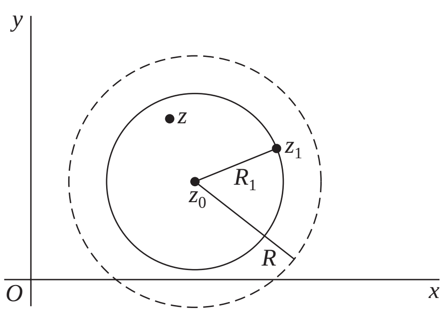

FIGURE 80

Our proof of this theorem depends on Theorem 1. Given that z_1 is a point lying inside the circle of convergence of series (4), we note that there are points inside that circle and farther from z_0 than z_1 for which the series converges. So, according to Theorem 1,

$$\sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n| \tag{5}$$

converges. Letting m and N denote positive integers, where $m > N$, one can write the remainders of series (4) and (5) as

$$\rho_N(z) = \lim_{m\to\infty} \sum_{n=N}^{m} a_n(z - z_0)^n \tag{6}$$

and

$$\sigma_N = \lim_{m\to\infty} \sum_{n=N}^{m} |a_n(z_1 - z_0)^n|, \tag{7}$$

respectively.

Now, in view of Exercise 3, Sec. 56,

$$|\rho_N(z)| = \lim_{m\to\infty} \left| \sum_{n=N}^{m} a_n(z - z_0)^n \right|;$$

and, when $|z - z_0| \le |z_1 - z_0|$,

$$\left| \sum_{n=N}^{m} a_n(z - z_0)^n \right| \le \sum_{n=N}^{m} |a_n||z - z_0|^n \le \sum_{n=N}^{m} |a_n||z_1 - z_0|^n = \sum_{n=N}^{m} |a_n(z_1 - z_0)^n|.$$

Consequently,

$$|\rho_N(z)| \le \sigma_N \quad \text{when} \quad |z - z_0| \le R_1. \tag{8}$$

Since σ_N are the remainders of a convergent series, they tend to zero as N tends to infinity. That is, for each positive number ε, an integer N_ε exists such that

$$\sigma_N < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon. \tag{9}$$

Because of conditions (8) and (9), then, condition (3) holds for all points z in the disk $|z - z_0| \le R_1$; and the value of N_ε is independent of the choice of z. Hence the convergence of series (4) is uniform in that disk.

64. CONTINUITY OF SUMS OF POWER SERIES

Our next theorem is an important consequence of uniform convergence, discussed in the previous section.

Theorem. *A power series*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{1}$$

represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$.

Another way to state this theorem is to say that if $S(z)$ denotes the sum of series (1) within its circle of convergence $|z - z_0| = R$ and if z_1 is a point inside that circle, then for each positive number ε there is a positive number δ such that

$$|S(z) - S(z_1)| < \varepsilon \quad \text{whenever} \quad |z - z_1| < \delta. \tag{2}$$

[See definition (4), Sec. 18, of continuity.] The number δ here is small enough so that z lies in the domain of definition $|z - z_0| < R$ of $S(z)$ (Fig. 81).

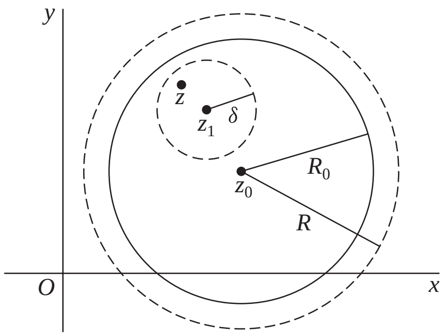

FIGURE 81

To prove the theorem, we let $S_n(z)$ denote the sum of the first N terms of series (1) and write the remainder function

$$\rho_N(z) = S(z) - S_N(z) \qquad (|z - z_0| < R).$$

Then, because

$$S(z) = S_N(z) + \rho_N(z) \qquad (|z - z_0| < R),$$

one can see that

$$|S(z) - S(z_1)| = |S_N(z) - S_N(z_1) + \rho_N(z) - \rho_N(z_1)|,$$

or

$$|S(z) - S(z_1)| \le |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)|. \tag{3}$$

If z is any point lying in some closed disk $|z - z_0| \le R_0$ whose radius R_0 is greater than $|z_1 - z_0|$ but less than the radius R of the circle of convergence of series (1) (see Fig. 81), the uniform convergence stated in Theorem 2, Sec. 63, ensures that there is a positive integer N_ε such that

$$|\rho_N(z)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad N > N_\varepsilon. \tag{4}$$

In particular, condition (4) holds for each point z in some neighborhood $|z - z_1| < \delta$ of z_1 that is small enough to be contained in the disk $|z - z_0| \le R_0$.

Now the partial sum $S_N(z)$ is a polynomial and is, therefore, continuous at z_1 for each value of N. In particular, when $N = N_\varepsilon + 1$, we can choose our δ so small that

$$|S_N(z) - S_N(z_1)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta. \tag{5}$$

By writing $N = N_\varepsilon + 1$ in inequality (3) and using the fact that statements (4) and (5) are true when $N = N_\varepsilon + 1$, we now find that

$$|S(z) - S(z_1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta.$$

This is statement (2), and the theorem is now established.

By writing $w = 1/(z - z_0)$, one can modify the two theorems in the previous section and the theorem here so as to apply to series of the type

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{6}$$

If, for instance, series (6) converges at a point z_1 ($z_1 \neq z_0$), the series

$$\sum_{n=1}^{\infty} b_n w^n$$

must converge absolutely to a continuous function when

$$|w| < \frac{1}{|z_1 - z_0|}. \tag{7}$$

Thus, since inequality (7) is the same as $|z - z_0| > |z_1 - z_0|$, series (6) must converge absolutely to a continuous function in the domain *exterior to* the circle $|z - z_0| = R_1$, where $R_1 = |z_1 - z_0|$. Also, we know that if a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid in an annulus $R_1 < |z - z_0| < R_2$, then *both* of the series on the right converge uniformly in any closed annulus which is concentric to and interior to that region of validity.

65. INTEGRATION AND DIFFERENTIATION OF POWER SERIES

We have just seen that a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

represents a continuous function at each point interior to its circle of convergence. In this section, we prove that the sum $S(z)$ is actually analytic within that circle. Our proof depends on the following theorem, which is of interest in itself.