## 54. MAXIMUM MODULUS PRINCIPLE

In this section, we derive an important result involving maximum values of the moduli of analytic functions. We begin with a needed lemma.

**Lemma.** *Suppose that* $|f(z)| \le |f(z_0)|$ *at each point* $z$ *in some neighborhood* $|z - z_0| < \varepsilon$ *in which* $f$ *is analytic. Then* $f(z)$ *has the constant value* $f(z_0)$ *throughout that neighborhood.*

To prove this, we assume that $f$ satisfies the stated conditions and let $z_1$ be any point other than $z_0$ in the given neighborhood. We then let $\rho$ be the distance between $z_1$ and $z_0$. If $C_\rho$ denotes the positively oriented circle $|z - z_0| = \rho$, centered at $z_0$ and passing through $z_1$ (Fig. 70), the Cauchy integral formula tells us that

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)\,dz}{z - z_0}; \tag{1}$$

and the parametric representation

$$z = z_0 + \rho e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

for $C_\rho$ enables us to write equation (1) as

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta. \tag{2}$$

We note from expression (2) that when a function is analytic within and on a given circle, its value at the center is the arithmetic mean of its values on the circle. This result is called *Gauss's mean value theorem*.

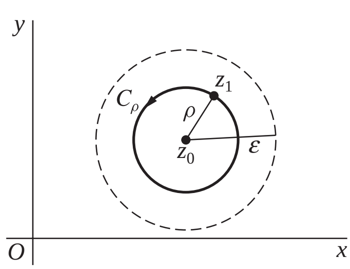


**FIGURE 70**

From equation (2), we obtain the inequality

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta. \tag{3}$$

On the other hand, since

$$|f(z_0 + \rho e^{i\theta})| \le |f(z_0)| \qquad (0 \le \theta \le 2\pi), \tag{4}$$

we find that

$$\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \le \int_0^{2\pi} |f(z_0)|\, d\theta = 2\pi |f(z_0)|.$$

Thus

$$|f(z_0)| \ge \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta. \tag{5}$$

It is now evident from inequalities (3) and (5) that

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta,$$

or

$$\int_0^{2\pi} [|f(z_0)| - |f(z_0 + \rho e^{i\theta})|]\, d\theta = 0.$$

The integrand in this last integral is continuous in the variable $\theta$; and, in view of condition (4), it is greater than or equal to zero on the entire interval $0 \le \theta \le 2\pi$. Because the value of the integral is zero, then, the integrand must be identically equal to zero. That is,

$$|f(z_0 + \rho e^{i\theta})| = |f(z_0)| \qquad (0 \le \theta \le 2\pi). \tag{6}$$

This shows that $|f(z)| = |f(z_0)|$ *for all points* $z$ *on the circle* $|z - z_0| = \rho$.

Finally, since $z_1$ is any point in the deleted neighborhood $0 < |z - z_0| < \varepsilon$, we see that the equation $|f(z)| = |f(z_0)|$ is, in fact, satisfied by all points $z$ lying on any circle $|z - z_0| = \rho$, where $0 < \rho < \varepsilon$. Consequently, $|f(z)| = |f(z_0)|$ everywhere in the neighborhood $|z - z_0| < \varepsilon$. But we know from Example 4. Sec. 25, that when the modulus of an analytic function is constant in a domain, the function itself is constant there. Thus $f(z) = f(z_0)$ for each point $z$ in the neighborhood, and the proof of the lemma is complete.

This lemma can be used to prove the following theorem, which is known as the *maximum modulus principle*.

***Theorem.*** *If a function $f$ is analytic and not constant in a given domain $D$, then $|f(z)|$ has no maximum value in $D$. That is, there is no point $z_0$ in the domain such that $|f(z)| \le |f(z_0)|$ for all points $z$ in it.*

Given that $f$ is analytic in $D$, we shall prove the theorem by assuming that $|f(z)|$ *does* have a maximum value at some point $z_0$ in $D$ and then showing that $f(z)$ must be constant throughout $D$.

The general approach here is similar to that taken in the proof of the lemma in Sec. 27. We draw a polygonal line $L$ lying in $D$ and extending from $z_0$ to any other point $P$ in $D$. Also, $d$ represents the shortest distance from points on $L$ to the boundary of $D$. When $D$ is the entire plane, $d$ may have any positive value. Next, we observe that there is a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along $L$ such that $z_n$ coincides with the point $P$ and

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$

In forming a finite sequence of neighborhoods (Fig. 71)

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n$$

where each $N_k$ has center $z_k$ and radius $d$, we see that $f$ is analytic in each of these neighborhoods, which are all contained in $D$, and that the center of each neighborhood $N_k$ $(k = 1, 2, \ldots, n)$ lies in the neighborhood $N_{k-1}$.

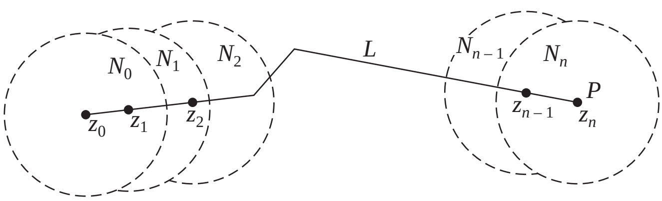


**FIGURE 71**

Since $|f(z)|$ was assumed to have a maximum value in $D$ at $z_0$, it also has a maximum value in $N_0$ at that point. Hence, according to the preceding lemma, $f(z)$ has the constant value $f(z_0)$ throughout $N_0$. In particular, $f(z_1) = f(z_0)$. This means that $|f(z)| \le |f(z_1)|$ for each point $z$ in $N_1$; and the lemma can be applied again, this time telling us that

$$f(z) = f(z_1) = f(z_0)$$

when $z$ is in $N_1$. Since $z_2$ is in $N_1$, then, $f(z_2) = f(z_0)$. Hence $|f(z)| \le |f(z_2)|$ when $z$ is in $N_2$; and the lemma is once again applicable, showing that

$$f(z) = f(z_2) = f(z_0)$$

when $z$ is in $N_2$. Continuing in this manner, we eventually reach the neighborhood $N_n$ and arrive at the fact that $f(z_n) = f(z_0)$.

Recalling that $z_n$ coincides with the point $P$, which is any point other than $z_0$ in $D$, we may conclude that $f(z) = f(z_0)$ for *every* point $z$ in $D$. Inasmuch as $f(z)$ has now been shown to be constant throughout $D$, the theorem is proved.

If a function $f$ that is analytic at each point in the interior of a closed bounded region $R$ is also continuous throughout $R$, then the modulus $|f(z)|$ has a maximum value somewhere in $R$ (Sec. 18). That is, there exists a nonnegative constant $M$ such that $|f(z)| \le M$ for all points $z$ in $R$, and equality holds for at least one such point.

If $f$ is a constant function, then $|f(z)| = M$ for all $z$ in $R$. If, however, $f(z)$ is not constant, then, according to the theorem just proved, $|f(z)| \neq M$ for any point $z$ in the interior of $R$. We thus arrive at an important corollary.

***Corollary.*** *Suppose that a function $f$ is continuous on a closed bounded region $R$ and that it is analytic and not constant in the interior of $R$. Then the maximum value of $|f(z)|$ in $R$, which is always reached, occurs somewhere on the boundary of $R$ and never in the interior.*

**EXAMPLE.** Let $R$ denote the rectangular region $0 \le x \le \pi$, $0 \le y \le 1$. The corollary tells us that the modulus of the entire function $f(z) = \sin z$ has a maximum value in $R$ that occurs somewhere on the boundary of $R$ and not in its interior. This can be verified directly by writing (see Sec. 34)

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$$

and noting that the term $\sin^2 x$ is greatest when $x = \pi/2$ and that the increasing function $\sinh^2 y$ is greatest when $y = 1$. Thus the maximum value of $|f(z)|$ in $R$ occurs at the boundary point $z = (\pi/2, 1)$ and at no other point in $R$ (Fig. 72).

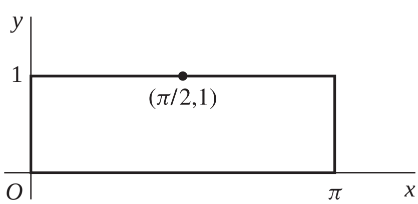


**FIGURE 72**

When the function $f$ in the corollary is written $f(z) = u(x, y) + iv(x, y)$, *the component function $u(x, y)$ also has a maximum value in $R$ which is assumed on the boundary of $R$ and never in the interior*, where it is harmonic (Sec. 26). This is because the composite function $g(z) = \exp[f(z)]$ is continuous in $R$ and analytic and not constant in the interior. Hence its modulus $|g(z)| = \exp[u(x, y)]$, which is continuous in $R$, must assume its maximum value in $R$ on the boundary. In view of the increasing nature of the exponential function, it follows that the maximum value of $u(x, y)$ also occurs on the boundary.

Properties of *minimum* values of $|f(z)|$ and $u(x, y)$ are treated in the exercises.

## EXERCISES

**1.** Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound $u_0$; that is, $u(x, y) \le u_0$ for all points $(x, y)$ in the $xy$ plane. Show that $u(x, y)$ must be constant throughout the plane.

*Suggestion:* Apply Liouville's theorem (Sec. 53) to the function $g(z) = \exp[f(z)]$.

**2.** Show that for $R$ sufficiently large, the polynomial $P(z)$ in Theorem 2, Sec. 53, satisfies the inequality

$$|P(z)| < 2|a_n||z|^n \quad \text{whenever} \quad |z| \geq R.$$

[Compare with the first of inequalities (5), Sec. 53.]

*Suggestion:* Observe that there is a positive number $R$ such that the modulus of each quotient in expression (3), Sec. 53, is less than $|a_n|/n$ when $|z| > R$.

**3.** Let a function $f$ be continuous on a closed bounded region $R$, and let it be analytic and not constant throughout the interior of $R$. Assuming that $f(z) \neq 0$ anywhere in $R$, prove that $|f(z)|$ has a *minimum value* $m$ in $R$ which occurs on the boundary of $R$ and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 54) to the function $g(z) = 1/f(z)$.

**4.** Use the function $f(z) = z$ to show that in Exercise 3 the condition $f(z) \neq 0$ anywhere in $R$ is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ *can* reach its minimum value at an interior point when the minimum value is zero.

**5.** Consider the function $f(z) = (z+1)^2$ and the closed triangular region $R$ with vertices at the points $z = 0$, $z = 2$, and $z = i$. Find points in $R$ where $|f(z)|$ has its maximum and minimum values, thus illustrating results in Sec. 54 and Exercise 3.

*Suggestion:* Interpret $|f(z)|$ as the square of the distance between $z$ and $-1$.

*Ans.* $z = 2, z = 0$.

**6.** Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region $R$ and analytic and not constant throughout the interior of $R$. Prove that the component function $u(x, y)$ has a minimum value in $R$ which occurs on the boundary of $R$ and never in the interior. (See Exercise 3.)

**7.** Let $f$ be the function $f(z) = e^z$ and $R$ the rectangular region $0 \leq x \leq 1, 0 \leq y \leq \pi$. Illustrate results in Sec. 54 and Exercise 6 by finding points in $R$ where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.

*Ans.* $z = 1, z = 1 + \pi i$.

**8.** Let the function $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed bounded region $R$, and suppose that it is analytic and not constant in the interior of $R$. Show that the component function $v(x, y)$ has maximum and minimum values in $R$ which are reached on the boundary of $R$ and never in the interior, where it is harmonic.

*Suggestion:* Apply results in Sec. 54 and Exercise 6 to the function $g(z) = -if(z)$.

**9.** Let $z_0$ be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree $n$ $(n \geq 1)$. Show in the following way that

$$P(z) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$.

*(a)* Verify that

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z\,z_0^{k-2} + z_0^{k-1}) \qquad (k = 2, 3, \ldots).$$

*(b)* Use the factorization in part *(a)* to show that

$$P(z) - P(z_0) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$, and deduce the desired result from this.

CHAPTER

5

# SERIES

This chapter is devoted mainly to series representations of analytic functions. We present theorems that guarantee the existence of such representations, and we develop some facility in manipulating series.

## 55. CONVERGENCE OF SEQUENCES

An infinite *sequence*

(1) $$z_1, z_2, \ldots, z_n, \ldots$$

of complex numbers has a *limit* $z$ if, for each positive number $\varepsilon$, there exists a positive integer $n_0$ such that

(2) $$|z_n - z| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

Geometrically, this means that for sufficiently large values of $n$, the points $z_n$ lie in any given $\varepsilon$ neighborhood of $z$ (Fig. 73). Since we can choose $\varepsilon$ as small as we please,

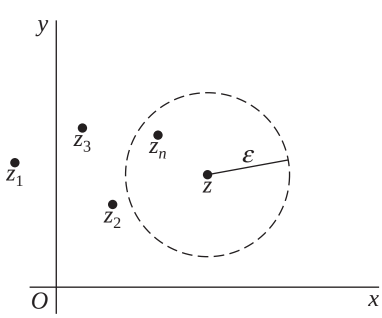


**FIGURE 73**

it follows that the points $z_n$ become arbitrarily close to $z$ as their subscripts increase. Note that the value of $n_0$ that is needed will, in general, depend on the value of $\varepsilon$.

The sequence (1) can have at most one limit. That is, a limit $z$ is unique if it exists (Exercise 5, Sec. 56). When that limit exists, the sequence is said to *converge* to $z$; and we write

$$\lim_{n\to\infty} z_n = z. \tag{3}$$

If the sequence has no limit, it *diverges*.

***Theorem.*** *Suppose that* $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ *and* $z = x + iy$. *Then*

$$\lim_{n\to\infty} z_n = z \tag{4}$$

*if and only if*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y. \tag{5}$$

To prove this theorem, we first assume that conditions (5) hold and obtain condition (4) from it. According to conditions (5), there exist, for each positive number $\varepsilon$, positive integers $n_1$ and $n_2$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_1$$

and

$$|y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_2.$$

Hence if $n_0$ is the larger of the two integers $n_1$ and $n_2$,

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_0.$$

Since

$$|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \le |x_n - x| + |y_n - y|,$$

then,

$$|z_n - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{whenever} \quad n > n_0.$$

Condition (4) thus holds.

Conversely, if we start with condition (4), we know that for each positive number $\varepsilon$, there exists a positive integer $n_0$ such that

$$|(x_n + iy_n) - (x + iy)| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

But

$$|x_n - x| \le |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

and

$$|y_n - y| \le |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|;$$

and this means that

$$|x_n - x| < \varepsilon \quad \text{and} \quad |y_n - y| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

That is, conditions (5) are satisfied.

Note how the theorem enables us to write

$$\lim_{n\to\infty}(x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

whenever we know that both limits on the right exist or that the one on the left exists.

**EXAMPLE 1.** The sequence

$$z_n = \frac{1}{n^3} + i \qquad (n = 1, 2, \ldots)$$

converges to $i$ since

$$\lim_{n\to\infty}\left(\frac{1}{n^3} + i\right) = \lim_{n\to\infty}\frac{1}{n^3} + i \lim_{n\to\infty} 1 = 0 + i\cdot 1 = i.$$

Definition (2) can also be used to obtain this result. More precisely, for each positive number $\varepsilon$,

$$|z_n - i| = \frac{1}{n^3} < \varepsilon \quad \text{whenever} \quad n > \frac{1}{\sqrt[3]{\varepsilon}}.$$

One must be careful when adapting our theorem to polar coordinates, as the following example shows.

**EXAMPLE 2.** When

$$z_n = -2 + i\frac{(-1)^n}{n^2} \quad (n = 1, 2, \ldots),$$

the theorem tells us that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-2) + i \lim_{n\to\infty}\frac{(-1)^n}{n^2} = -2 + i \cdot 0 = -2.$$

If, using polar coordinates, we write

$$r_n = |z_n| \quad \text{and} \quad \Theta_n = \operatorname{Arg} z_n \qquad (n = 1, 2, \ldots),$$

where Arg $z_n$ denotes principal arguments $(-\pi < \Theta \leq \pi)$ of $z_n$, we find that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \sqrt{4 + \frac{1}{n^4}} = 2$$

but that

$$\lim_{n\to\infty} \Theta_{2n} = \pi \quad \text{and} \quad \lim_{n\to\infty} \Theta_{2n-1} = -\pi \qquad (n = 1, 2, \ldots).$$

Evidently, then, the limit of $\Theta_n$ does not exist as $n$ tends to infinity. (See also Exercise 2, Sec. 56.)

## 56. CONVERGENCE OF SERIES

An infinite *series*

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots \tag{1}$$

of complex numbers *converges* to the *sum* $S$ if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N \qquad (N = 1, 2, \ldots) \tag{2}$$

of *partial sums* converges to $S$; we then write

$$\sum_{n=1}^{\infty} z_n = S.$$

Note that since a sequence can have at most one limit, a series can have at most one sum. When a series does not converge, we say that it *diverges.*

***Theorem.*** *Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $S = X + iY$. Then*

$$\sum_{n=1}^{\infty} z_n = S \tag{3}$$

*if and only if*

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y. \tag{4}$$

This theorem tells us, of course, that one can write

$$\sum_{n=1}^{\infty}(x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n$$

whenever it is known that the two series on the right converge or that the one on the left does.

To prove the theorem, we first write the partial sums (2) as

$$S_N = X_N + iY_N, \tag{5}$$

where

$$X_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad Y_N = \sum_{n=1}^{N} y_n.$$

Now statement (3) is true if and only if

$$\lim_{N\to\infty} S_N = S\,; \tag{6}$$

and, in view of relation (5) and the theorem on sequences in Sec. 55, limit (6) holds if and only if

$$\lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y. \tag{7}$$

Limits (7) therefore imply statement (3), and conversely. Since $X_N$ and $Y_N$ are the partial sums of the series (4), the theorem here is proved.

This theorem can be useful in showing that a number of familiar properties of series in calculus carry over to series whose terms are complex numbers. To illustrate how this is done, we include here two such properties and present them as corollaries.

***Corollary 1.*** *If a series of complex numbers converges, the nth term converges to zero as n tends to infinity.*

Assuming that series (1) converges, we know from the theorem that if

$$z_n = x_n + iy_n \quad (n = 1, 2, \ldots),$$

then each of the series

$$\sum_{n=1}^{\infty} x_n \quad \text{and} \quad \sum_{n=1}^{\infty} y_n \tag{8}$$

converges. We know, moreover, from calculus that the $n$th term of a convergent series of real numbers approaches zero as $n$ tends to infinity. Thus, by the theorem in Sec. 55,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + 0 \cdot i = 0\,;$$

and the proof of Corollary 1 is complete.

It follows from this corollary that the terms of convergent series are *bounded*. That is, when series (1) converges, there exists a positive constant $M$ such that $|z_n| \leq M$ for each positive integer $n$. (See Exercise 9.)

For another important property of series of complex numbers that follows from a corresponding property in calculus, series (1) is said to be *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \qquad (z_n = x_n + iy_n)$$

of real numbers $\sqrt{x_n^2 + y_n^2}$ converges.

***Corollary 2.*** *The absolute convergence of a series of complex numbers implies the convergence of that series.*

To prove Corollary 2, we assume that series (1) converges absolutely. Since

$$|x_n| \leq \sqrt{x_n^2 + y_n^2} \quad \text{and} \quad |y_n| \leq \sqrt{x_n^2 + y_n^2},$$

we know from the comparison test in calculus that the two series

$$\sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n|$$

must converge. Moreover, since the absolute convergence of a series of real numbers implies the convergence of the series itself, it follows that the series (8) both converge. In view of the theorem in this section, then, series (1) converges. This finishes the proof of Corollary 2.

In establishing the fact that the sum of a series is a given number $S$, it is often convenient to define the *remainder* $\rho_N$ after $N$ terms, using the partial sums (2):

$$\rho_N = S - S_N. \tag{9}$$

Thus $S = S_N + \rho_N$; and, since $|S_N - S| = |\rho_N - 0|$, we see that *a series converges to a number $S$ if and only if the sequence of remainders tends to zero.* We shall make considerable use of this observation in our treatment of *power series*. They are series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots,$$

where $z_0$ and the coefficients $a_n$ are complex constants and $z$ may be any point in a stated region containing $z_0$. In such series, involving a variable $z$, we shall denote sums, partial sums, and remainders by $S(z)$, $S_N(z)$, and $\rho_N(z)$, respectively.

**EXAMPLE.** With the aid of remainders, it is easy to verify that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever} \quad |z| < 1. \tag{10}$$

We need only recall the identity (Exercise 9, Sec. 8)

$$1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z} \qquad (z \neq 1)$$

to write the partial sums

$$S_N(z) = \sum_{n=0}^{N-1} z^n = 1 + z + z^2 + \cdots + z^{N-1} \qquad (z \neq 1)$$

as

$$S_N(z) = \frac{1-z^N}{1-z}.$$

If

$$S(z) = \frac{1}{1-z},$$

then,

$$\rho_N(z) = S(z) - S_N(z) = \frac{z^N}{1-z} \qquad (z \neq 1).$$

Thus

$$|\rho_N(z)| = \frac{|z|^N}{|1-z|},$$

and it is clear from this that the remainders $\rho_N(z)$ tend to zero when $|z| < 1$ but not when $|z| \geq 1$. Summation formula (10) is, therefore, established.

## EXERCISES

**1.** Use definition (2), Sec. 55, of limits of sequences to verify the limit of the sequence $z_n$ $(n = 1, 2, \ldots)$ found in Example 2, Sec. 55.

**2.** Let $\Theta_n$ $(n = 1, 2, \ldots)$ denote the principal arguments of the numbers

$$z_n = 2 + i\frac{(-1)^n}{n^2} \quad (n = 1, 2, \ldots).$$

Point out why

$$\lim_{n\to\infty} \Theta_n = 0,$$

and compare with Example 2, Sec. 55.

**3.** Use the inequality (see Sec. 4) $||z_n| - |z|| \le |z_n - z|$ to show that

$$\text{if} \quad \lim_{n\to\infty} z_n = z\,, \quad \text{then} \quad \lim_{n\to\infty} |z_n| = |z|.$$

**4.** Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 56. Then, with the aid of the theorem in Sec. 56, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

**5.** Show that a limit of a convergent sequence of complex numbers is unique by appealing to the corresponding result for a sequence of real numbers.

**6.** Show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

**7.** Let $c$ denote any complex number and show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} cz_n = cS.$$

**8.** By recalling the corresponding result for series of real numbers and referring to the theorem in Sec. 56, show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S \quad \text{and} \quad \sum_{n=1}^{\infty} w_n = T, \qquad \text{then} \quad \sum_{n=1}^{\infty} (z_n + w_n) = S + T.$$

**9.** Let a sequence $z_n$ $(n = 1, 2, \ldots)$ converge to a number $z$. Show that there exists a positive number $M$ such that the inequality $|z_n| \le M$ holds for all $n$. Do this in each of the following ways.

*(a)* Note that there is a positive integer $n_0$ such that

$$|z_n| = |z + (z_n - z)| < |z| + 1$$

whenever $n > n_0$.

*(b)* Write $z_n = x_n + iy_n$ and recall from the theory of sequences of real numbers that the convergence of $x_n$ and $y_n$ $(n = 1, 2, \ldots)$ implies that $|x_n| \le M_1$ and $|y_n| \le M_2$ $(n = 1, 2, \ldots)$ for some positive numbers $M_1$ and $M_2$.

## 57. TAYLOR SERIES

We turn now to *Taylor's theorem,* which is one of the most important results of the chapter.

***Theorem.*** *Suppose that a function $f$ is analytic throughout a disk $|z - z_0| < R_0$, centered at $z_0$ and with radius $R_0$* (Fig. 74). *Then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0), \tag{1}$$

*where*

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

*That is, series* (1) *converges to $f(z)$ when $z$ lies in the stated open disk.*

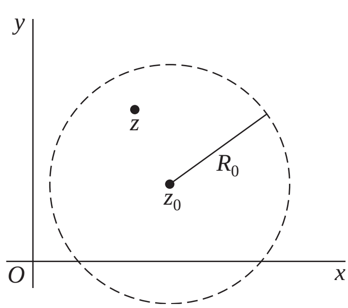


**FIGURE 74**

This is the expansion of $f(z)$ into a *Taylor series* about the point $z_0$. It is the familiar Taylor series from calculus, adapted to functions of a complex variable. With the agreement that

$$f^{(0)}(z_0) = f(z_0) \quad \text{and} \quad 0! = 1,$$

series (1) can, of course, be written

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \qquad (|z - z_0| < R_0). \tag{3}$$