46. CAUCHY–GOURSAT THEOREM

In Sec. 44, we saw that when a continuous function f has an antiderivative in a domain D, the integral of $f(z)$ around any given closed contour C lying entirely in D has value zero. In this section, we present a theorem giving other conditions on a function f which ensure that the value of the integral of $f(z)$ around a *simple* closed contour (Sec. 39) is zero. The theorem is central to the theory of functions of a complex variable; and some modifications of it, involving certain special types of domains, will be given in Secs. 48 and 49.

We let C denote a simple closed contour $z = z(t)$ $(a \le t \le b)$, described in the *positive sense* (counterclockwise), and we assume that f is analytic at each point interior to and on C. According to Sec. 40,

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt; \tag{1}$$

and if

$$f(z) = u(x, y) + iv(x, y) \quad \text{and} \quad z(t) = x(t) + iy(t),$$

the integrand $f[z(t)]z'(t)$ in expression (1) is the product of the functions

$$u[x(t), y(t)] + iv[x(t), y(t)], \qquad x'(t) + iy'(t)$$

of the real variable t. Thus

$$\int_C f(z)\,dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt. \tag{2}$$

In terms of line integrals of real-valued functions of two real variables, then,

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy. \tag{3}$$

Observe that expression (3) can be obtained formally by replacing $f(z)$ and dz on the left with the binomials

$$u + iv \quad \text{and} \quad dx + i\,dy,$$

respectively, and expanding their product. Expression (3) is, of course, also valid when C is any contour, not necessarily a simple closed one, and when $f[z(t)]$ is only piecewise continuous on it.

We next recall a result from calculus that enables us to express the line integrals on the right in equation (3) as double integrals. Suppose that two real-valued functions $P(x, y)$ and $Q(x, y)$, together with their first-order partial derivatives, are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C. According to *Green's theorem*,

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.$$

Now f is continuous in R, since it is analytic there. Hence the functions u and v are also continuous in R. Likewise, if the derivative f' of f is continuous in R, so are the first-order partial derivatives of u and v. Green's theorem then enables us to rewrite equation (3) as

$$\int_C f(z)\,dz = \int\!\!\int_R (-v_x - u_y)\,dA + i\int\!\!\int_R (u_x - v_y)\,dA. \tag{4}$$

But, in view of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

the integrands of these two double integrals are zero throughout R. So *when f is analytic in R and f' is continuous there,*

$$\int_C f(z)\,dz = 0. \tag{5}$$

This result was obtained by Cauchy in the early part of the nineteenth century.

Note that once it has been established that the value of this integral is zero, the orientation of C is immaterial. That is, statement (5) is also true if C is taken in the clockwise direction, since then

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz = 0.$$

EXAMPLE. If C is any simple closed contour, in either direction, then

$$\int_C \exp(z^3)\,dz = 0.$$

This is because the composite function $f(z) = \exp(z^3)$ is analytic everywhere and its derivative $f'(z) = 3z^2\exp(z^3)$ is continuous everywhere.

Goursat* was the first to prove that *the condition of continuity on f' can be omitted*. Its removal is important and will allow us to show, for example, that the derivative f' of an analytic function f is analytic without having to assume the continuity of f', which follows as a consequence. We now state the revised form of Cauchy's result, known as the *Cauchy–Goursat theorem*.

Theorem. *If a function f is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\,dz = 0.$$

---

*E. Goursat (1858–1936), pronounced *gour-sah′*.

The proof is presented in the next section, where, to be specific, we assume that C is positively oriented. The reader who wishes to accept this theorem without proof may pass directly to Sec. 48.

47. PROOF OF THE THEOREM

We preface the proof of the Cauchy–Goursat theorem with a lemma. We start by forming subsets of the region R which consists of the points on a positively oriented simple closed contour C together with the points interior to C. To do this, we draw equally spaced lines parallel to the real and imaginary axes such that the distance between adjacent vertical lines is the same as that between adjacent horizontal lines. We thus form a finite number of closed square subregions, where each point of R lies in at least one such subregion and each subregion contains points of R. We refer to these square subregions simply as *squares*, always keeping in mind that by a square we mean a boundary together with the points interior to it. If a particular square contains points that are not in R, we remove those points and call what remains a *partial square*. We thus *cover* the region R with a finite number of squares and partial squares (Fig. 55), and our proof of the following lemma starts with this covering.

Lemma. *Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any positive number ε, the region R can be covered with a finite number of squares and partial squares, indexed by $j = 1, 2, \ldots, n$, such that in each one there is a fixed point z_j for which the inequality*

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \tag{1}$$

is satisfied by all points other than z_j in that square or partial square.

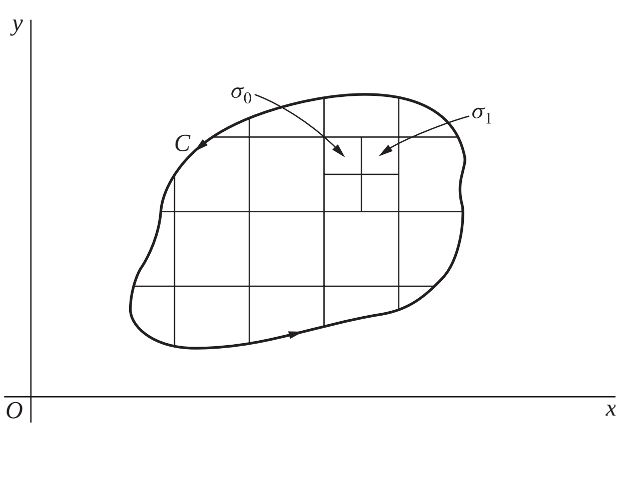

FIGURE 55

To start the proof, we consider the possibility that in the covering constructed just prior to the statement of the lemma, there is some square or partial square in which no point z_j exists such that inequality (1) holds for all other points z in it. If that subregion is a square, we construct four smaller squares by drawing line segments joining the midpoints of its opposite sides (Fig. 55). If the subregion is a partial square, we treat the whole square in the same manner and then let the portions that lie outside of R be discarded. If in any one of these smaller subregions, no point z_j exists such that inequality (1) holds for all other points z in it, we construct still smaller squares and partial squares, etc. When this is done to each of the original subregions that requires it, we find that *after a finite number of steps*, the region R can be covered with a finite number of squares and partial squares such that the lemma is true.

To verify this, we suppose that the needed points z_j do *not* exist after subdividing one of the original subregions a finite number of times and reach a contradiction. We let σ_0 denote that subregion if it is a square; if it is a partial square, we let σ_0 denote the entire square of which it is a part. After we subdivide σ_0, at least one of the four smaller squares, denoted by σ_1, must contain points of R but no appropriate point z_j. We then subdivide σ_1 and continue in this manner. It may be that after a square σ_{k-1} $(k = 1, 2, \ldots)$ has been subdivided, more than one of the four smaller squares constructed from it can be chosen. To make a specific choice, we take σ_k to be the one lowest and then furthest to the left.

In view of the manner in which the nested infinite sequence

$$\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k, \ldots \tag{2}$$

of squares is constructed, it is easily shown (Exercise 9, Sec. 49) that there is a point z_0 common to each σ_k; also, each of these squares contains points of R other than possibly z_0. Recall how the sizes of the squares in the sequence are decreasing, and note that any δ neighborhood $|z - z_0| < \delta$ of z_0 contains such squares when their diagonals have lengths less than δ. Every δ neighborhood $|z - z_0| < \delta$ therefore contains points of R distinct from z_0, and this means that z_0 is an accumulation point of R. Since the region R is a closed set, it follows that z_0 is a point in R. (See Sec. 11.)

Now the function f is analytic throughout R and, in particular, at z_0. Consequently, $f'(z_0)$ exists, According to the definition of derivative (Sec. 19), there is, for each positive number ε, a δ neighborhood $|z - z_0| < \delta$ such that the inequality

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

is satisfied by all points distinct from z_0 in that neighborhood. But the neighborhood $|z - z_0| < \delta$ contains a square σ_K when the integer K is large enough that the length of a diagonal of that square is less than δ (Fig. 56). Consequently, z_0 serves as the point z_j in inequality (1) for the subregion consisting of the square σ_K or a part of σ_K. Contrary to the way in which the sequence (2) was formed, then, it is not necessary to subdivide σ_K. We thus arrive at a contradiction, and the proof of the lemma is complete.

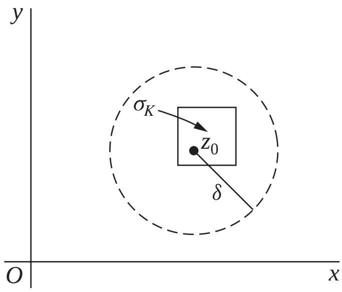

FIGURE 56

Continuing with a function f which is analytic throughout a region R consisting of a positively oriented simple closed contour C and points interior to it, we are now ready to prove the Cauchy–Goursat theorem, namely that

$$\int_C f(z)\,dz = 0. \tag{3}$$

Given an arbitrary positive number ε, we consider the covering of R in the statement of the lemma. We then define on the jth square or partial square a function $\delta_j(z)$ whose values are $\delta_j(z_j) = 0$, where z_j is the fixed point in inequality (1), and

$$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \quad \text{when } z \neq z_j. \tag{4}$$

According to inequality (1),

$$|\delta_j(z)| < \varepsilon \tag{5}$$

at all points z in the subregion on which $\delta_j(z)$ is defined. Also, the function $\delta_j(z)$ is continuous throughout the subregion since $f(z)$ is continuous there and

$$\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0.$$

Next, we let C_j $(j = 1, 2, \ldots, n)$ denote the positively oriented boundaries of the above squares or partial squares covering R. In view of our definition of $\delta_j(z)$, the value of f at a point z on any particular C_j can be written

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z);$$

and this means that

$$\int_{C_j} f(z)\,dz = [f(z_j) - z_j f'(z_j)] \int_{C_j} dz + f'(z_j) \int_{C_j} z\,dz \tag{6}$$
$$+ \int_{C_j} (z - z_j)\delta_j(z)\,dz.$$

But

$$\int_{C_j} dz = 0 \quad \text{and} \quad \int_{C_j} z\, dz = 0$$

since the functions 1 and z possess antiderivatives everywhere in the finite plane. So equation (6) reduces to

$$\int_{C_j} f(z)\, dz = \int_{C_j} (z - z_j)\delta_j(z)\, dz \qquad (j = 1, 2, \ldots, n). \tag{7}$$

The sum of all n integrals on the left in equations (7) can be written

$$\sum_{j=1}^{n} \int_{C_j} f(z)\, dz = \int_C f(z)\, dz$$

since the two integrals along the common boundary of every pair of adjacent subregions cancel each other, the integral being taken in one sense along that line segment in one subregion and in the opposite sense in the other (Fig. 57). Only the integrals along the arcs that are parts of C remain. Thus, in view of equations (7),

$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)\, dz\,;$$

and so

$$\left| \int_C f(z)\, dz \right| \le \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)\, dz \right|. \tag{8}$$

We now use the theorem in Sec. 43 to find an upper bound for each modulus on the right in inequality (8). To do this, we first recall that each C_j coincides either

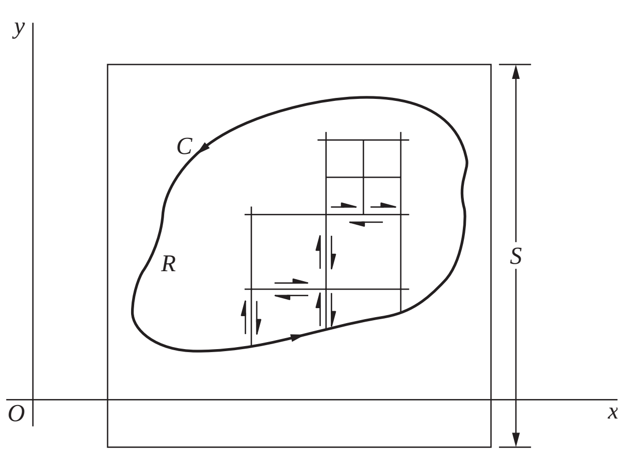

FIGURE 57

entirely or partially with the boundary of a square. In either case, we let s_j denote the length of a side of the square. Since, in the jth integral, both the variable z and the point z_j lie in that square,

$$|z - z_j| \leq \sqrt{2}s_j.$$

In view of inequality (5), then, we know that each integrand on the right in inequality (8) satisfies the condition

$$|(z - z_j)\delta_j(z)| = |z - z_j|\,|\delta_j(z)| < \sqrt{2}s_j\varepsilon. \tag{9}$$

As for the length of the path C_j, it is $4s_j$ if C_j is the boundary of a square. In that case, we let A_j denote the area of the square and observe that

$$\left|\int_{C_j} (z - z_j)\delta_j(z)\, dz\right| < \sqrt{2}s_j\varepsilon 4s_j = 4\sqrt{2}A_j\varepsilon. \tag{10}$$

If C_j is the boundary of a partial square, its length does not exceed $4s_j + L_j$, where L_j is the length of that part of C_j which is also a part of C. Again letting A_j denote the area of the full square, we find that

$$\left|\int_{C_j} (z - z_j)\delta_j(z)\, dz\right| < \sqrt{2}s_j\varepsilon(4s_j + L_j) < 4\sqrt{2}A_j\varepsilon + \sqrt{2}SL_j\varepsilon, \tag{11}$$

where S is the length of a side of some square that encloses the entire contour C as well as all of the squares originally used in covering R (Fig. 57). Note that the sum of all the A_j's does not exceed S^2.

If L denotes the length of C, it now follows from inequalities (8), (10), and (11) that

$$\left|\int_C f(z)\, dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon.$$

Since the value of the positive number ε is arbitrary, we can choose it so that the right-hand side of this last inequality is as small as we please. The left-hand side, which is independent of ε, must therefore be equal to zero; and statement (3) follows. This completes the proof of the Cauchy–Goursat theorem.

48. SIMPLY CONNECTED DOMAINS

A *simply connected* domain D is a domain such that every simple closed contour within it encloses only points of D. The set of points interior to a simple closed contour is an example. The annular domain between two concentric circles is, however, not simply connected. Domains that are not simply connected are discussed in the next section.

The closed contour in the Cauchy–Goursat theorem (Sec. 46) need not be simple when the theorem is adapted to simply connected domains. More precisely, the contour can actually cross itself. The following theorem allows for this possibility.

Theorem. *If a function f is analytic throughout a simply connected domain D, then*

$$\int_C f(z)\,dz = 0 \tag{1}$$

for every closed contour C lying in D.

The proof is easy if C is a *simple* closed contour or if it is a closed contour that intersects itself a *finite* number of times. For if C is simple and lies in D, the function f is analytic at each point interior to and on C; and the Cauchy–Goursat theorem ensures that equation (1) holds. Furthermore, if C is closed but intersects itself a finite number of times, it consists of a finite number of simple closed contours. This is illustrated in Fig. 58, where the simple closed contours C_k $(k = 1, 2, 3, 4)$ make up C. Since the value of the integral around each C_k is zero, according to the Cauchy–Goursat theorem, it follows that

$$\int_C f(z)\,dz = \sum_{k=1}^{4} \int_{C_k} f(z)\,dz = 0.$$

Subtleties arise if the closed contour has an *infinite* number of self-intersection points. One method that can sometimes be used to show that the theorem still applies is illustrated in Exercise 5, Sec. 49.*

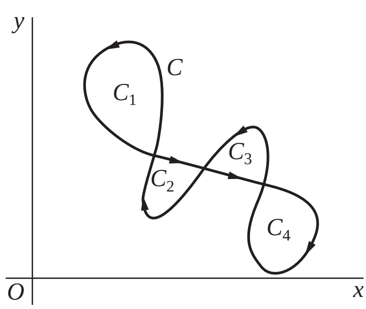

FIGURE 58

EXAMPLE. If C denotes any closed contour lying in the open disk $|z| < 2$ (Fig. 59), then

$$\int_C \frac{z\,e^z}{(z^2+9)^5}\,dz = 0.$$

*For a proof of the theorem involving more general paths of finite length, see, for example, Secs. 63–65 in Vol. I of the book by Markushevich that is cited in Appendix 1.

This is because the disk is a simply connected domain and the two singularities $z = \pm 3i$ of the integrand are exterior to the disk.

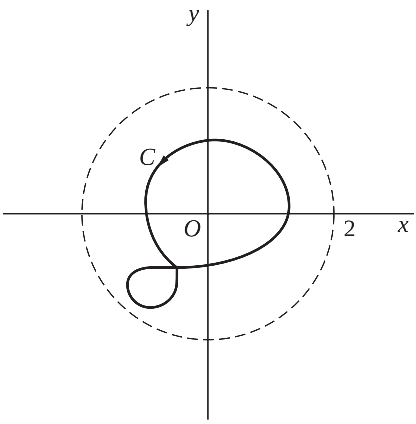

FIGURE 59

Corollary. *A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.*

We begin the proof of this corollary with the observation that a function f is continuous on a domain D when it is analytic there. Consequently, since equation (1) holds for the function in the hypothesis of this corollary and for each closed contour C in D, f has an antiderivative throughout D, according to the theorem in Sec. 44. Note that since the finite plane is simply connected, the corollary tells us that *entire functions always possess antiderivatives*.

49. MULTIPLY CONNECTED DOMAINS

A domain that is not simply connected (Sec. 48) is said to be *multiply connected*. The following theorem is an adaptation of the Cauchy–Goursat theorem to multiply connected domains.

Theorem. *Suppose that*

(a) C is a simple closed contour, described in the counterclockwise direction;

(b) C_k $(k = 1, 2, \ldots, n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common (Fig. 60).

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0. \tag{1}$$

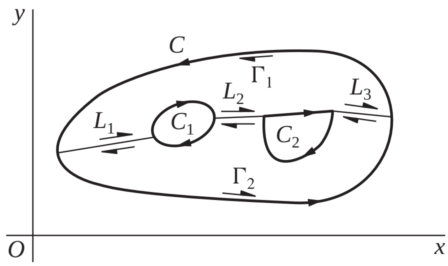

FIGURE 60

Note that in equation (1), the direction of each path of integration is such that the multiply connected domain lies to the *left* of that path.

To prove the theorem, we introduce a polygonal path L_1, consisting of a finite number of line segments joined end to end, to connect the outer contour C to the inner contour C_1. We introduce another polygonal path L_2 which connects C_1 to C_2; and we continue in this manner, with L_{n+1} connecting C_n to C. As indicated by the single-barbed arrows in Fig. 60, two simple closed contours Γ_1 and Γ_2 can be formed, each consisting of polygonal paths L_k or $-L_k$ and pieces of C and C_k and each described in such a direction that the points enclosed by them lie to the left. The Cauchy–Goursat theorem can now be applied to f on Γ_1 and Γ_2, and the sum of the values of the integrals over those contours is found to be zero. Since the integrals in opposite directions along each path L_k cancel, only the integrals along C and the C_k remain; and we arrive at statement (1).

Corollary. *Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2* (Fig. 61). *If a function f is analytic in the closed region consisting of those contours and all points between them, then*

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz. \tag{2}$$

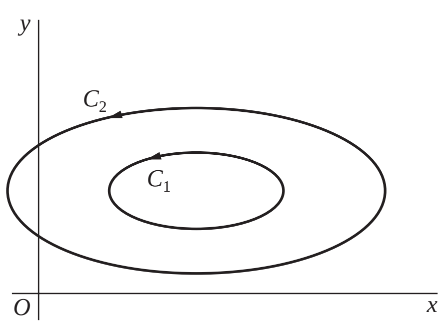

FIGURE 61

This corollary is known as the *principle of deformation of paths* since it tells us that if C_1 is continuously deformed into C_2, always passing through points at which f is analytic, then the value of the integral of f over C_1 never changes. To verify the corollary, we need only write equation (2) as

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

and apply the theroem.

EXAMPLE. When C is any positively oriented simple closed contour surrounding the origin, the corollary can be used to show that

$$\int_C \frac{dz}{z} = 2\pi i.$$

This is done by constructing a positively oriented circle C_0 with center at the origin and radius so small that C_0 lies entirely inside C (Fig. 62). Since (see Example 2, Sec. 42)

$$\int_{C_0} \frac{dz}{z} = 2\pi i$$

and since $1/z$ is analytic everywhere except at $z = 0$, the desired result follows.

Note that the radius of C_0 could equally well have been so large that C lies entirely inside C_0.

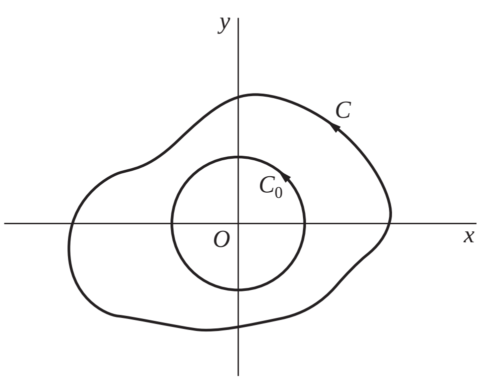

FIGURE 62

EXERCISES

1. Apply the Cauchy–Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z-3}$; *(b)* $f(z) = z\,e^{-z}$; *(c)* $f(z) = \dfrac{1}{z^2 + 2z + 2}$;

(d) $f(z) = \text{sech}\, z$; *(e)* $f(z) = \tan z$; *(f)* $f(z) = \text{Log}\,(z + 2)$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 63). With the aid of the corollary in Sec. 49, point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2 + 1}$; *(b)* $f(z) = \dfrac{z + 2}{\sin(z/2)}$; *(c)* $f(z) = \dfrac{z}{1 - e^z}$.

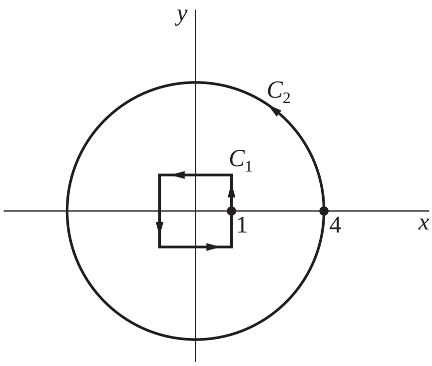

FIGURE 63

3. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z - z_0)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 10*(b)*, Sec. 42. Use that result and the corollary in Sec. 49 to show that if C is the boundary of the rectangle $0 \le x \le 3, 0 \le y \le 2$, described in the positive sense, then

$$\int_C (z - 2 - i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 64 can be written

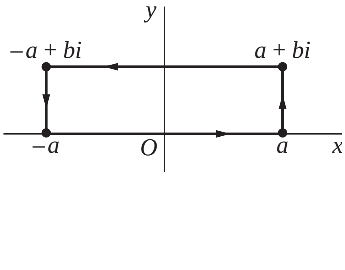

FIGURE 64

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\ dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy–Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\, dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\ dy.$$

(b) By accepting the fact that*

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\, dy\right| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part *(a)*.

5. According to Exercise 6, Sec. 39, the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0 \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (Fig. 65). Apply the Cauchy–Goursat theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)\ dz = \int_{C_3} f(z)\ dz \quad \text{and} \quad \int_{C_2} f(z)\ dz = -\int_{C_3} f(z)\ dz.$$

Conclude that even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\int_C f(z)\ dz = 0.$$

*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 680–681, 1983.

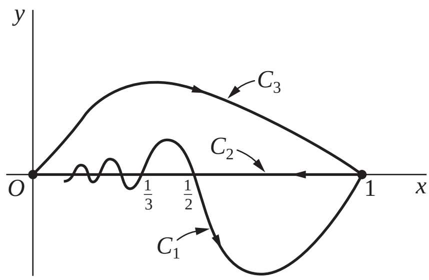

FIGURE 65

6. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)\, dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy–Goursat theorem not apply here?

7. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \overline{z}\, dz.$$

Suggestion: Note that expression (4), Sec. 46, can be used here even though the function $f(z) = \overline{z}$ is not analytic anywhere [see Example 2, Sec. 19].

8. *Nested Intervals*. An infinite sequence of closed intervals $a_n \le x \le b_n$ $(n = 0, 1, 2, \ldots)$ is formed in the following way. The interval $a_1 \le x \le b_1$ is either the left-hand or right-hand half of the first interval $a_0 \le x \le b_0$, and the interval $a_2 \le x \le b_2$ is then one of the two halves of $a_1 \le x \le b_1$, etc. Prove that there is a point x_0 which belongs to every one of the closed intervals $a_n \le x \le b_n$.

Suggestion: Note that the left-hand end points a_n represent a bounded nondecreasing sequence of numbers, since $a_0 \le a_n \le a_{n+1} < b_0$; hence they have a limit A as n tends to infinity. Show that the end points b_n also have a limit B. Then show that $A = B$, and write $x_0 = A = B$.

9. *Nested Squares*. A square $\sigma_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. One of those four smaller squares $\sigma_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected according to some rule. It, in turn, is divided into four equal squares one of which, called σ_2, is selected, etc. (see Sec. 47). Prove that there is a point (x_0, y_0) which belongs to each of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the result in Exercise 8 to each of the sequences of closed intervals $a_n \le x \le b_n$ and $c_n \le y \le d_n$ $(n = 0, 1, 2, \ldots)$.

50. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}. \tag{1}$$

Formula (1) is called the *Cauchy integral formula*. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

When the Cauchy integral formula is written as

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0), \tag{2}$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, formula (2) tells us that

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)}\,dz = 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 66). Since the quotient $f(z)/(z - z_0)$ is analytic between and on the contours C_ρ and C, it follows from the principle of deformation of paths (Sec. 49) that

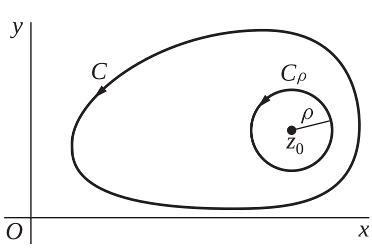

FIGURE 66

$$\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}.$$

This enables us to write

$$\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{3}$$

But [see Exercise 10*(b)*, Sec. 42]

$$\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i;$$

and so equation (3) becomes

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{4}$$

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, however small, there is a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta. \tag{5}$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z - z_0| = \rho < \delta$ when z is on C_ρ, it follows that the *first* of inequalities (5) holds when z is such a point; and the theorem in Sec. 43, giving upper bounds for the moduli of contour integrals, tells us that

$$\left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| < \frac{\varepsilon}{\rho}\,2\pi\rho = 2\pi\varepsilon.$$

In view of equation (4), then,

$$\left| \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right| < 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it must be equal to zero. Hence equation (2) is valid, and the theorem is proved.

51. AN EXTENSION OF THE CAUCHY INTEGRAL FORMULA

The Cauchy integral formula in the theorem in Sec. 50 can be extended so as to provide an integral representation for derivatives of f at z_0. To obtain the extension, we consider a function f that is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. We then write the Cauchy integral formula as

(1) $$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z},$$

where z is interior to C and where s denotes points on C. Differentiating *formally* with respect to z under the integral sign here, without rigorous justification, we find that

(2) $$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}.$$

To verify that $f'(z)$ exists and that expression (2) is in fact valid, we led d denote the smallest distance from z to points s on C and use expression (1) to write

$$\begin{aligned}\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds\\ &= \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)},\end{aligned}$$

where $0 < |\Delta z| < d$ (see Fig. 67). Evidently, then,

(3) $$\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2}.$$

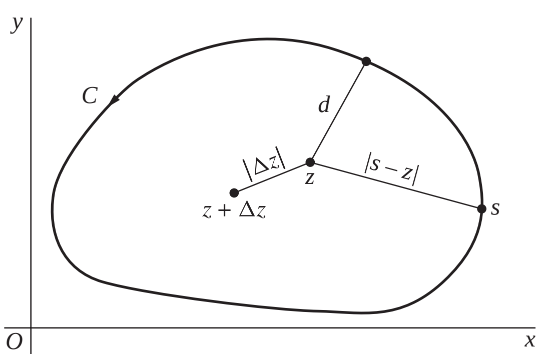

FIGURE 67

Next, we let M denote the maximum value of $|f(s)|$ on C and observe that since $|s-z| \ge d$ and $|\Delta z| < d$,

$$|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \geq d - |\Delta z| > 0.$$

Thus

$$\left| \int_C \frac{\Delta z\, f(s)\, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|)d^2} L,$$

where L is the length of C. Upon letting Δz tend to zero, we find from this inequality that the right-hand side of equation (3) also tends to zero. Consequently,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2} = 0\,;$$

and the desired expression for $f'(z)$ is established.

The same technique can be used to suggest and verify the expression

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s - z)^3}. \tag{4}$$

The details, which are outlined in Exercise 9, Sec. 52, are left to the reader. Mathematical induction can, moreover, be used to obtain the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^{n+1}} \qquad (n = 1, 2, \ldots). \tag{5}$$

The verification is considerably more involved than for just $n = 1$ and $n = 2$, and we refer the interested reader to other texts for it.* Note that with the agreement that

$$f^{(0)}(z) = f(z) \quad \text{and} \quad 0! = 1,$$

expression (5) is also valid when $n = 0$, in which case it becomes the Cauchy integral formula (1).

When written in the form

$$\int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0, 1, 2, \ldots), \tag{6}$$

expressions (1) and (5) can be useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 50 when $n = 0$.

EXAMPLE 1. If C is the positively oriented unit circle $|z| = 1$ and

$$f(z) = \exp(2z),$$

---

*See, for example, pp. 299–301 in Vol. I of the book by Markushevich, cited in Appendix 1.

then

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

EXAMPLE 2. Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z) = 1$, expression (6) shows that

$$\int_C \frac{dz}{z - z_0} = 2\pi i$$

and

$$\int_C \frac{dz}{(z - z_0)^{n+1}} = 0 \qquad (n = 1, 2, \ldots).$$

(Compare with Exercise 10*(b)*, Sec. 42.)

52. SOME CONSEQUENCES OF THE EXTENSION

We turn now to some important consequences of the extension of the Cauchy integral formula in the previous section.

Theorem 1. *If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.*

To prove this remarkable theorem, we assume that a function f is analytic at a point z_0. There must, then, be a neighborhood $|z - z_0| < \varepsilon$ of z_0 throughout which f is analytic (see Sec. 24). Consequently, there is a positively oriented circle C_0, centered at z_0 and with radius $\varepsilon/2$, such that f is analytic inside and on C_0 (Fig. 68). From expression (4), Sec. 51, we know that

$$f''(z) = \frac{1}{\pi i} \int_{C_0} \frac{f(s)\,ds}{(s - z)^3}$$

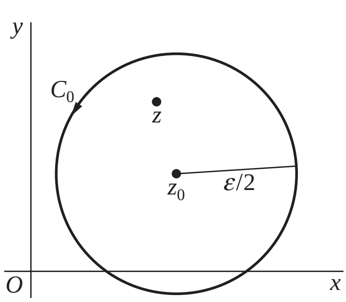

FIGURE 68

at each point z interior to C_0, and the existence of $f''(z)$ throughout the neighborhood $|z - z_0| < \varepsilon/2$ means that f' is analytic at z_0. One can apply the same argument to the analytic function f' to conclude that its derivative f'' is analytic, etc. Theorem 1 is now established.

As a consequence, when a function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic at a point $z = (x, y)$, the differentiability of f' ensures the continuity of f' there (Sec. 19). Then, since (Sec. 21)

$$f'(z) = u_x + iv_x = v_y - iu_y,$$

we may conclude that the first-order partial derivatives of u and v are continuous at that point. Furthermore, since f'' is analytic and continuous at z and since

$$f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx},$$

etc., we arrive at a corollary that was anticipated in Sec. 26, where harmonic functions were introduced.

Corollary. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.*

The proof of the next theorem, due to E. Morera (1856–1909), depends on the fact that the derivative of an analytic function is itself analytic, as stated in Theorem 1.

Theorem 2. *Let f be continuous on a domain D. If*

$$\int_C f(z)\,dz = 0 \tag{1}$$

for every closed contour C in D, then f is analytic throughout D.

In particular, when D is *simply connected*, we have for the class of continuous functions defined on D the converse of the theorem in Sec. 48, which is the adaptation of the Cauchy–Goursat theorem to such domains.

To prove the theorem here, we observe that when its hypothesis is satisfied, the theorem in Sec. 44 ensures that f has an antiderivative in D; that is, there exists an analytic function F such that $F'(z) = f(z)$ at each point in D. Since f is the derivative of F, it then follows from Theorem 1 that f is analytic in D.

Our final theorem here will be essential in the next section.

Theorem 3. *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R* (Fig. 69). *If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

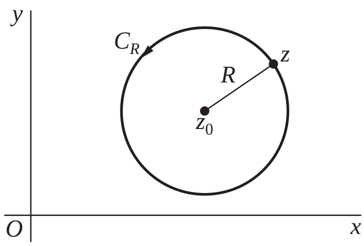

FIGURE 69

Inequality (2) is called *Cauchy's inequality* and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

which is a slightly different form of equation (6), Sec. 51, when n is a positive integer. We need only apply the theorem in Sec. 43, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} 2\pi R \quad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; *(b)* $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\,dz$; *(c)* $\displaystyle\int_C \frac{z\,dz}{2z + 1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; *(e)* $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; *(b)* $\pi i/4$; *(c)* $-\pi i/2$; *(d)* 0; *(e)* $i\pi \sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2 + 4}$; *(b)* $g(z) = \dfrac{1}{(z^2 + 4)^2}$.

Ans. (a) $\pi/2$; *(b)* $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\, dz}{z - z_0} = \int_C \frac{f(z)\, dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following a procedure used in Sec. 51, prove that the function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

8. *(a)* With the aid of the binomial formula (Sec. 3), show that for each value of n, the function

$$P_n(z) = \frac{1}{n!\, 2^n} \frac{d^n}{dz^n}(z^2 - 1)^n \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial of degree n.*

*These are Legendre polynomials, which appear in Exercise 7, Sec. 43, when $z = x$. See the footnote to that exercise.

(b) Let C denote any positively oriented simple closed contour surrounding a fixed point z. With the aid of the integral representation (5), Sec. 51, for the nth derivative of a function, show that the polynomials in part *(a)* can be expressed in the form

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\, ds \qquad (n = 0, 1, 2, \ldots).$$

(c) Point out how the integrand in the representation for $P_n(z)$ in part *(b)* can be written $(s+1)^n/(s-1)$ if $z = 1$. Then apply the Cauchy integral formula to show that

$$P_n(1) = 1 \qquad (n = 0, 1, 2, \ldots).$$

Similarly, show that

$$P_n(-1) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

9. Follow these steps below to verify the expression

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s-z)^3}$$

in Sec. 51.

(a) Use expression (2) in Sec. 51 for $f'(z)$ to show that

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s-z)^3} = \frac{1}{2\pi i} \int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2 (s-z)^3} f(s)\, ds.$$

(b) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 51 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part *(a)* is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d - |\Delta z|)^2 d^3} L.$$

(c) Use the results in parts *(a)* and *(b)* to obtain the desired expression for $f''(z)$.

10. Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 52) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

53. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality in Theorem 3 of Sec. 52 can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here, which is known as *Liouville's theorem*, states this result in a somewhat different way.

Theorem 1. *If a function f is entire and bounded in the complex plane, then $f(z)$ is constant througout the plane.*

To start the proof, we assume that f is as stated and note that since f is entire, Theorem 3 in Sec. 52 can be applied with any choice of z_0 and R. In particular, Cauchy's inequality (2) in that theorem tells us that when $n = 1$,

$$|f'(z_0)| \leq \frac{M_R}{R}. \tag{1}$$

Moreover, the boundedness condition on f tells us that a nonnegative constant M exists such that $|f(z)| \leq M$ for all z; and, because the constant M_R in inequality (1) is always less than or equal to M, it follows that

$$\left|f'(z_0)\right| \leq \frac{M}{R}, \tag{2}$$

where R can be arbitrarily large. Now the number M in inequality (2) is independent of the value of R that is taken. Hence that inequality holds for arbitrarily large values of R only if $f'(z_0) = 0$. Since the choice of z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function, according to the theorem in Sec. 24.

The following theorem, called the *fundamental theorem of algebra*, follows readily from Liouville's theorem.

Theorem 2. *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n ($n \geq 1$) has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

The proof here is by contradiction. Suppose that $P(z)$ is *not* zero for any value of z. Then the reciprocal

$$f(z) = \frac{1}{P(z)}$$

is clearly entire, and it is also bounded in the complex plane.

To show that its is bounded, we first write

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}, \tag{3}$$

so that

$$P(z) = (a_n + w)z^n. \tag{4}$$

Next, we observe that a sufficiently large positive number R can be found such that the modulus of each of the quotients in expression (3) is less than the number $|a_n|/(2n)$ when $|z| > R$. The generalized triangle inequality (10), Sec. 4, which applies to n complex numbers, thus shows that

$$|w| < \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R.$$

Consequently,

$$|a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R.$$

This inequality and expression (4) enable us to write

$$(5) \qquad |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n \quad \text{whenever} \quad |z| > R.$$

Evidently, then,

$$|f(z)| = \frac{1}{|P(z)|} < \frac{2}{|a_n|R^n} \quad \text{whenever} \quad |z| > R.$$

So f is bounded in the region *exterior to* the disk $|z| \leq R$. But f is continuous in that closed disk, and this means that f is bounded there too (Sec. 18). Hence f is bounded in the entire plane.

It now follows from Liouville's theorem that $f(z)$, and consequently $P(z)$, is constant. But $P(z)$ is not constant, and we have reached a contradiction.*

The fundamental theorem tells us that any polynomial $P(z)$ of degree n $(n \geq 1)$ can be expressed as a product of linear factors:

$$(6) \qquad P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n),$$

where c and z_k $(k = 1, 2, \ldots, n)$ are complex constants. More precisely, the theorem ensures that $P(z)$ has a zero z_1. Then, according to Exercise 9, Sec. 54,

$$P(z) = (z - z_1)Q_1(z),$$

where $Q_1(z)$ is a polynomial of degree $n - 1$. The same argument, applied to $Q_1(z)$, reveals that there is a number z_2 such that

$$P(z) = (z - z_1)(z - z_2)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n - 2$. Continuing in this way, we arrive at expression (6). Some of the constants z_k in expression (6) may, of course, appear more than once, and it is clear that $P(z)$ can have no more than n *distinct* zeros.

---

*For an interesting proof of the fundamental theorem using the Cauchy–Goursat theorem, see R. P. Boas, Jr., *Amer. Math. Monthly*, Vol. 71, No. 2, p. 180, 1964.