CHAPTER

4

INTEGRALS

Integrals are extremely important in the study of functions of a complex variable. The theory of integration, to be developed in this chapter, is noted for its mathematical elegance. The theorems are generally concise and powerful, and many of the proofs are short.

37. DERIVATIVES OF FUNCTIONS $w(t)$

In order to introduce integrals of $f(z)$ in a fairly simple way, we need to first consider derivatives of complex-valued functions w of a *real* variable t. We write

(1) $$w(t) = u(t) + iv(t),$$

where the functions u and v are *real-valued* functions of t. The derivative

$$w'(t), \text{ or } \frac{d}{dt}w(t),$$

of the function (1) at a point t is defined as

(2) $$w'(t) = u'(t) + iv'(t),$$

provided each of the derivatives u' and v' exists at t.

From definition (2), it follows that for every complex constant $z_0 = x_0 + iy_0$,

$$\frac{d}{dt}[z_0 w(t)] = [(x_0 + iy_0)(u + iv)]' = [(x_0 u - y_0 v) + i(y_0 u + x_0 v)]'$$
$$= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)' = (x_0 u' - y_0 v') + i(y_0 u' + x_0 v').$$

But

$$(x_0u' - y_0v') + i(y_0u' + x_0v') = (x_0 + iy_0)(u' + iv') = z_0w'(t),$$

and so

$$\frac{d}{dt}[z_0w(t)] = z_0w'(t). \tag{3}$$

Another expected rule that we shall often use is

$$\frac{d}{dt}e^{z_0t} = z_0e^{z_0t}, \tag{4}$$

where $z_0 = x_0 + iy_0$. To verify this, we write

$$e^{z_0t} = e^{x_0t}e^{iy_0t} = e^{x_0t}\cos y_0t + ie^{x_0t}\sin y_0t$$

and refer to definition (2) to see that

$$\frac{d}{dt}e^{z_0t} = (e^{x_0t}\cos y_0t)' + i(e^{x_0t}\sin y_0t)'.$$

Familiar rules from calculus and some simple algebra then lead us to the expression

$$\frac{d}{dt}e^{z_0t} = (x_0 + iy_0)(e^{x_0t}\cos y_0t + ie^{x_0t}\sin y_0t),$$

or

$$\frac{d}{dt}e^{z_0t} = (x_0 + iy_0)e^{x_0t}e^{iy_0t}.$$

This is, of course, the same as equation (4).

Various other rules learned in calculus, such as the ones for differentiating sums and products, apply just as they do for real-valued functions of t. As was the case with property (3) and formula (4), verifications may be based on corresponding rules in calculus. It should be pointed out, however, that not every such rule carries over to functions of type (1). The following example illustrates this.

EXAMPLE. Suppose that $w(t)$ is continuous on an interval $a \le t \le b$; that is, its component functions $u(t)$ and $v(t)$ are continuous there. Even if $w'(t)$ exists when $a < t < b$, the mean value theorem for derivatives no longer applies. To be precise, it is not necessarily true that there is a number c in the interval $a < t < b$ such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}.$$

To see this, consider the function $w(t) = e^{it}$ on the interval $0 \le t \le 2\pi$. When that function is used, $|w'(t)| = |ie^{it}| = 1$; and this means that the derivative $w'(t)$ is never zero, while $w(2\pi) - w(0) = 0$.

38. DEFINITE INTEGRALS OF FUNCTIONS $w(t)$

When $w(t)$ is a complex-valued function of a real variable t and is written

$$w(t) = u(t) + iv(t), \tag{1}$$

where u and v are real-valued, the definite integral of $w(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt, \tag{2}$$

provided the individual integrals on the right exist. Thus

$$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b \operatorname{Re}[w(t)]\,dt \quad \text{and} \quad \operatorname{Im}\int_a^b w(t)\,dt = \int_a^b \operatorname{Im}[w(t)]\,dt. \tag{3}$$

EXAMPLE 1. For an illustration of definition (2),

$$\int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt = \frac{2}{3} + i.$$

Improper integrals of $w(t)$ over unbounded intervals are defined in a similar way.

The existence of the integrals of u and v in definition (2) is ensured if those functions are *piecewise continuous* on the interval $a \le t \le b$. Such a function is continuous everywhere in the stated interval except possibly for a finite number of points where, although discontinuous, it has one-sided limits. Of course, only the right-hand limit is required at a; and only the left-hand limit is required at b. When both u and v are piecewise continuous, the function w is said to have that property.

Anticipated rules for integrating a complex constant times a function $w(t)$, for integrating sums of such functions, and for interchanging limits of integration are all valid. Those rules, as well as the property

$$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt,$$

are easy to verify by recalling corresponding results in calculus.

The *fundamental theorem of calculus*, involving antiderivatives, can, moreover, be extended so as to apply to integrals of the type (2). To be specific, suppose that the functions

$$w(t) = u(t) + iv(t) \quad \text{and} \quad W(t) = U(t) + iV(t)$$

are continuous on the interval $a \le t \le b$. If $W'(t) = w(t)$ when $a \le t \le b$, then $U'(t) = u(t)$ and $V'(t) = v(t)$. Hence, in view of definition (2),

$$\int_a^b w(t)\,dt = U(t)\Big]_a^b + iV(t)\Big]_a^b = [U(b) + iV(b)] - [U(a) + iV(a)].$$

That is,

$$\int_a^b w(t)\,dt = W(b) - W(a) = W(t)\Big]_a^b. \tag{4}$$

EXAMPLE 2. Since (see Sec. 37)

$$\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{i}\frac{d}{dt}e^{it} = \frac{1}{i}\,ie^{it} = e^{it},$$

one can see that

$$\int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{\pi/4} = \frac{e^{i\pi/4}}{i} - \frac{1}{i} = \frac{1}{i}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - 1\right)$$
$$= \frac{1}{i}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - 1\right) = \frac{1}{\sqrt{2}} + \frac{1}{i}\left(\frac{1}{\sqrt{2}} - 1\right).$$

Then, because $1/i = -i$,

$$\int_0^{\pi/4} e^{it}\,dt = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

We recall from the example in Sec. 37 how the mean value theorem for derivatives in calculus does not carry over to complex-valued functions $w(t)$. Our final example here shows that the mean value theorem for *integrals* does not carry over either. Thus special care must continue to be used in applying rules from calculus.

EXAMPLE 3. Let $w(t)$ be a continuous complex-valued function of t defined on an interval $a \le t \le b$. In order to show that it is not necessarily true that there is a number c in the interval $a < t < b$ such that

$$\int_a^b w(t)\,dt = w(c)(b-a),$$

we write $a = 0$, $b = 2\pi$ and use the same function $w(t) = e^{it}\,(0 \le t \le 2\pi)$ as in the example in Sec. 37. It is easy to see that

$$\int_a^b w(t)\,dt = \int_0^{2\pi} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{2\pi} = 0.$$

But, for any number c such that $0 < c < 2\pi$,

$$|w(c)(b-a)| = |e^{ic}|\,2\pi = 2\pi;$$

and this means that $w(c)(b-a)$ is *not* zero.

EXERCISES

1. Use rules in calculus to establish the following rules when

$$w(t) = u(t) + iv(t)$$

is a complex-valued function of a real variable t and $w'(t)$ exists:

(a) $\dfrac{d}{dt} w(-t) = -w'(-t)$ where $w'(-t)$ denotes the derivative of $w(t)$ with respect to t, evaluated at $-t$;

(b) $\dfrac{d}{dt}[w(t)]^2 = 2\,w(t)w'(t)$.

2. Evaluate the following integrals:

(a) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt\,;$ *(b)* $\displaystyle\int_0^{\pi/6} e^{i2t}\,dt\,;$ *(c)* $\displaystyle\int_0^{\infty} e^{-zt}\,dt$ $(\operatorname{Re} z > 0)$.

Ans. *(a)* $-\dfrac{1}{2} - i\ln 4$; *(b)* $\dfrac{\sqrt{3}}{4} + \dfrac{i}{4}$; *(c)* $\dfrac{1}{z}$.

3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

4. According to definition (2), Sec. 38, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\,dx = \int_0^{\pi} e^x \cos x\,dx + i\int_0^{\pi} e^x \sin x\,dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans. $-(1 + e^{\pi})/2, \quad (1 + e^{\pi})/2$.

5. Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \le t \le a$.

(a) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)\,dt = 2\int_0^{a} w(t)\,dt.$$

(b) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point t in the given interval, then

$$\int_{-a}^{a} w(t)\,dt = 0.$$

Suggestion: In each part of this exercise, use the corresponding property of integrals of *real-valued* functions of t, which is graphically evident.

39. CONTOURS

Integrals of complex-valued functions of a *complex* variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

A set of points $z = (x, y)$ in the complex plane is said to be an *arc* if

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b), \tag{1}$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. This definition establishes a continuous mapping of the interval $a \le t \le b$ into the xy, or z, plane; and the image points are ordered according to increasing values of t. It is convenient to describe the points of C by means of the equation

$$z = z(t) \qquad (a \le t \le b), \tag{2}$$

where

$$z(t) = x(t) + iy(t). \tag{3}$$

The arc C is a *simple arc*, or a Jordan arc,* if it does not cross itself; that is, C is simple if $z(t_1) \ne z(t_2)$ when $t_1 \ne t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a *simple closed curve*, or a Jordan curve. Such a curve is *positively oriented* when it is in the counterclockwise direction.

The geometric nature of a particular arc often suggests different notation for the parameter t in equation (2). This is, in fact, the case in the following examples.

EXAMPLE 1. The polygonal line (Sec. 11) defined by means of the equations

$$z = \begin{cases} x + ix & \text{when } 0 \le x \le 1, \\ x + i & \text{when } 1 \le x \le 2 \end{cases} \tag{4}$$

and consisting of a line segment from 0 to $1 + i$ followed by one from $1 + i$ to $2 + i$ (Fig. 36) is a simple arc.

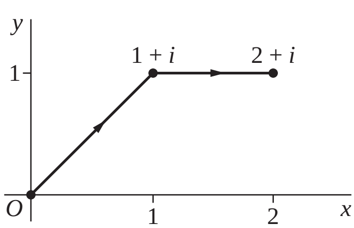

FIGURE 36

*Named for C. Jordan (1838–1922), pronounced *jor-don′*.

EXAMPLE 2. The unit circle

$$z = e^{i\theta} \qquad (0 \le \theta \le 2\pi) \tag{5}$$

about the origin is a simple closed curve, oriented in the counterclockwise direction. So is the circle

$$z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi), \tag{6}$$

centered at the point z_0 and with radius R (see Sec. 6).

The same set of points can make up different arcs.

EXAMPLE 3. The arc

$$z = e^{-i\theta} \qquad (0 \le \theta \le 2\pi) \tag{7}$$

is not the same as the arc described by equation (5). The set of points is the same, but now the circle is traversed in the *clockwise* direction.

EXAMPLE 4. The points on the arc

$$z = e^{i2\theta} \qquad (0 \le \theta \le 2\pi) \tag{8}$$

are the same as those making up the arcs (5) and (7). The arc here differs, however, from each of those arcs since the circle is traversed *twice* in the counterclockwise direction.

The parametric representation used for any given arc C is, of course, not unique. It is, in fact, possible to change the interval over which the parameter ranges to any other interval. To be specific, suppose that

$$t = \phi(\tau) \qquad (\alpha \le \tau \le \beta), \tag{9}$$

where ϕ is a real-valued function mapping an interval $\alpha \le \tau \le \beta$ onto the interval $a \le t \le b$ in representation (2). (See Fig. 37.) We assume that ϕ is continuous with

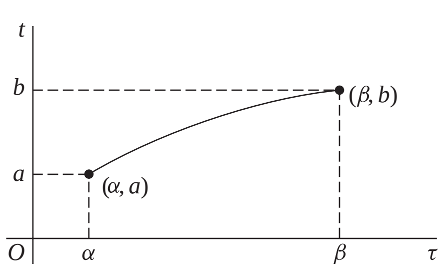

FIGURE 37
$t = \phi\ (\tau)$

a continuous derivative. We also assume that $\phi'(\tau) > 0$ for each τ; this ensures that t increases with τ. Representation (2) is then transformed by equation (9) into

$$z = Z(\tau) \qquad (\alpha \le \tau \le \beta), \tag{10}$$

where

$$Z(\tau) = z[\phi(\tau)]. \tag{11}$$

This is illustrated in Exercise 3.

Suppose now that the components $x'(t)$ and $y'(t)$ of the derivative (Sec. 37)

$$z'(t) = x'(t) + iy'(t) \tag{12}$$

of the function (3), used to represent C, are continuous on the entire interval $a \le t \le b$. The arc is then called a *differentiable arc,* and the real-valued function

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over the interval $a \le t \le b$. In fact, according to the definition of arc length in calculus, the length of C is the number

$$L = \int_a^b |z'(t)|\, dt. \tag{13}$$

The value of L is invariant under certain changes in the representation for C that is used, as one would expect. More precisely, with the change of variable indicated in equation (9), expression (13) takes the form [see Exercise 1*(b)*]

$$L = \int_\alpha^\beta |z'[\phi(\tau)]|\phi'(\tau)\, d\tau.$$

So, if representation (10) is used for C, the derivative (Exercise 4)

$$Z'(\tau) = z'[\phi(\tau)]\phi'(\tau) \tag{14}$$

enables us to write expression (13) as

$$L = \int_\alpha^\beta |Z'(\tau)|\, d\tau.$$

Thus the same length of C would be obtained if representation (10) were to be used.

If equation (2) represents a differentiable arc and if $z'(t) \ne 0$ anywhere in the interval $a < t < b$, then the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all t in that open interval, with angle of inclination $\arg z'(t)$. Also, when $\mathbf{T}$ turns, it does so continuously as the parameter t varies over the entire interval

$a < t < b$. This expression for $\mathbf{T}$ is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be *smooth*. In referring to a smooth arc $z = z(t)$ $(a \leq t \leq b)$, then, we agree that the derivative $z'(t)$ is continuous on the closed interval $a \leq t \leq b$ and nonzero throughout the open interval $a < t < b$.

A *contour*, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line (4) is, for example, a contour. When only the initial and final values of $z(t)$ are the same, a contour C is called a *simple closed contour*. Examples are the circles (5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. It will be convenient to accept this statement, known as the *Jordan curve theorem*, as geometrically evident; the proof is not easy.*

EXERCISES

1. Show that if $w(t) = u(t) + iv(t)$ is continuous on an interval $a \leq t \leq b$, then

(a) $\displaystyle\int_{-b}^{-a} w(-t)\,dt = \int_a^b w(\tau)\,d\tau;$

(b) $\displaystyle\int_a^b w(t)\,dt = \int_\alpha^\beta w[\phi(\tau)]\phi'(\tau)\,d\tau,$ where $\phi(\tau)$ is the function in equation (9), Sec. 39.

Suggestion: These identities can be obtained by noting that they are valid for *real-valued* functions of t.

2. Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for C are

$$z = z(\theta) = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \qquad (-2 \leq y \leq 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan\frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

*See pp. 115–116 of the book by Newman or Sec. 13 of the one by Thron, both of which are cited in Appendix 1. The special case in which C is a simple closed polygon is proved on pp. 281–285 of Vol. 1 of the work by Hille, also cited in Appendix 1.

Also, show that this function ϕ has a positive derivative, as required in the conditions following equation (9), Sec. 39.

3. Derive the equation of the line through the points (α, a) and (β, b) in the τt plane that are shown in Fig. 37. Then use it to find the linear function $\phi(\tau)$ which can be used in equation (9), Sec. 39, to transform representation (2) in that section into representation (10) there.

Ans. $\phi(\tau) = \dfrac{b-a}{\beta-\alpha}\,\tau + \dfrac{a\beta - b\alpha}{\beta - \alpha}.$

4. Verify expression (14), Sec. 39, for the derivative of $Z(\tau) = z[\phi(\tau)]$.

Suggestion: Write $Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)]$ and apply the chain rule for real-valued functions of a real variable.

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

6. Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

(b) Verify that the arc C in part *(a)* is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

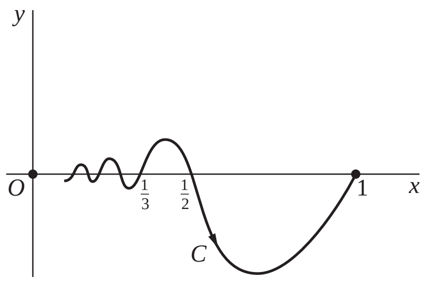

FIGURE 38

40. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the contour C as well as on the function f. It is written

$$\int_C f(z)\,dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\,dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral may be defined directly as the limit of a sum, we choose to define it in terms of a definite integral of the type introduced in Sec. 38.

Suppose that the equation

$$z = z(t) \quad (a \le t \le b) \tag{1}$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 38) on the interval $a \le t \le b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or *contour integral*, of f along C in terms of the parameter t:

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt. \tag{2}$$

Note that since C is a contour, $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral (2) is ensured.

The value of a contour integral is invariant under a change in the representation of its contour when the change is of the type (11), Sec. 39. This can be seen by following the same general procedure that was used in Sec. 39 to show the invariance of arc length.

It follows immediately from definition (2) and properties of integrals of complex-valued functions $w(t)$ mentioned in Sec. 38 that

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz, \tag{3}$$

for any complex constant z_0, and

$$\int_C \big[f(z) + g(z)\big]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz. \tag{4}$$

Associated with the contour C used in integral (2) is the contour $-C$, consisting of the same set of points but with the order reversed so that the new contour extends from the point z_2 to the point z_1 (Fig. 39). The contour $-C$ has parametric representation

$$z = z(-t) \qquad (-b \le t \le -a).$$

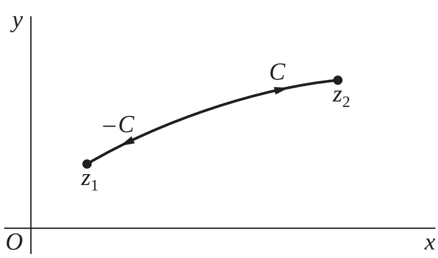

FIGURE 39

Hence, in view of Exercise 1*(a)*, Sec. 38,

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\,[z(-t)]\,\frac{d}{dt}z(-t)\,dt = -\int_{-b}^{-a} f\,[z(-t)]\,z'(-t)\,dt$$

where $z'(-t)$ denotes the derivative of $z(t)$ with respect to t, evaluated at $-t$. Making the substitution $\tau = -t$ in this last integral and referring to Exercise 1*(a)*, Sec. 39, we obtain the expression

$$\int_{-C} f(z)\,dz = -\int_a^b f[z(\tau)]z'(\tau)\,d\tau,$$

which is the same as

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz. \tag{5}$$

Consider now a path C, with representation (1), that consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3, the initial point of C_2 being the final point of C_1 (Fig. 40). There is a value c of t, where $a < c < b$, such that $z(c) = z_2$. Consequently, C_1 is represented by

$$z = z(t) \qquad (a \le t \le c)$$

and C_2 is represented by

$$z = z(t) \qquad (c \le t \le b).$$

Also, by a rule for integrals of functions $w(t)$ that was noted in Sec. 38,

$$\int_a^b f[z(t)]z'(t)\,dt = \int_a^c f[z(t)]z'(t)\,dt + \int_c^b f[z(t)]z'(t)\,dt.$$

Evidently, then,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{6}$$

Sometimes the contour C is called the *sum* of its legs C_1 and C_2 and is denoted by $C_1 + C_2$. The sum of two contours C_1 and $-C_2$ is well defined when C_1 and C_2 have the same final points, and it is written $C_1 - C_2$.

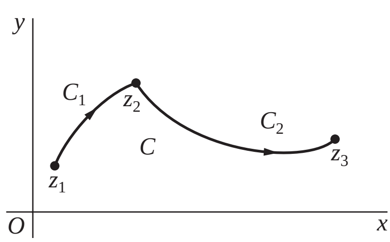

FIGURE 40
$C = C_1 + C_2$

Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.

41. SOME EXAMPLES

The purpose of this and the next section is to provide examples of the definition in Sec. 40 of contour integrals and to illustrate various properties that were mentioned there. We defer development of the concept of antiderivatives of the integrands $f(z)$ of contour integrals until Sec. 44.

EXAMPLE 1. Let us find the value of the integral

$$I = \int_C \overline{z}\,dz \tag{1}$$

when C is the right-hand half

$$z = 2\,e^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$ (Fig. 41). According to definition (2), Sec. 40,

$$I = \int_{-\pi/2}^{\pi/2} \overline{2\,e^{i\theta}}(2\,e^{i\theta})'\,d\theta = 4\int_{-\pi/2}^{\pi/2} \overline{e^{i\theta}}(e^{i\theta})'\,d\theta;$$

and, since

$$\overline{e^{i\theta}} = e^{-i\theta} \quad \text{and} \quad (e^{i\theta})' = ie^{i\theta},$$

this means that

$$I = 4\int_{-\pi/2}^{\pi/2} e^{-i\theta} ie^{i\theta}\,d\theta = 4i\int_{-\pi/2}^{\pi/2} d\theta = 4\pi i.$$

Note that $z\overline{z} = |z|^2 = 4$ when z is a point on the semicircle C. Hence the result

(2) $$\int_C \overline{z}\,dz = 4\pi i$$

can also be written

$$\int_C \frac{dz}{z} = \pi i.$$

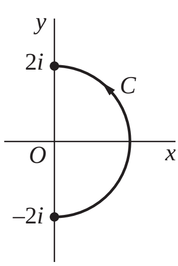

FIGURE 41

If $f(z)$ is given in the form $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, one can sometimes apply definition (2), Sec. 40, using one of the variables x and y as the parameter.

EXAMPLE 2. Here we first let C_1 denote the polygonal line OAB shown in Fig. 42 and evaluate the integral

(3) $$I_1 = \int_{C_1} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz,$$

where

$$f(z) = y - x - i3x^2 \quad (z = x + iy).$$

The leg *OA* may be represented parametrically as $z = 0 + iy$ $(0 \le y \le 1)$; and, since $x = 0$ at points on that line segment, the values of f there vary with the parameter y according to the equation $f(z) = y$ $(0 \le y \le 1)$. Consequently,

$$\int_{OA} f(z)\,dz = \int_0^1 yi\,dy = i\int_0^1 y\,dy = \frac{i}{2}.$$

On the leg AB, the points are $z = x + i$ $(0 \le x \le 1)$; and, since $y = 1$ on this segment,

$$\int_{AB} f(z)\,dz = \int_0^1 (1 - x - i3x^2)\cdot 1\,dx = \int_0^1 (1 - x)\,dx - 3i\int_0^1 x^2\,dx = \frac{1}{2} - i.$$

In view of equation (3), we now see that

$$I_1 = \frac{1-i}{2}. \tag{4}$$

If C_2 denotes the segment OB of the line $y = x$ in Fig. 42, with parametric representation $z = x + ix$ $(0 \le x \le 1)$, the fact that $y = x$ on *OB* enables us to write

$$I_2 = \int_{C_2} f(z)\,dz = \int_0^1 -i3x^2(1+i)\,dx = 3(1-i)\int_0^1 x^2\,dx = 1 - i.$$

Evidently, then, the integrals of $f(z)$ along the two paths C_1 and C_2 have *different values* even though those paths have the same initial and the same final points.

Observe how it follows that the integral of $f(z)$ over the simple closed contour *OABO*, or $C_1 - C_2$, has the *nonzero value*

$$I_1 - I_2 = \frac{-1+i}{2}.$$

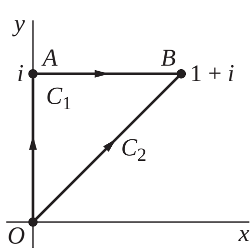

FIGURE 42

EXAMPLE 3. We begin here by letting C denote an arbitrary *smooth* arc (Sec. 39)

$$z = z(t) \quad (a \le t \le b)$$

from a fixed point z_1 to a fixed point z_2 (Fig. 43). In order to evaluate the integral

$$\int_C z\,dz = \int_a^b z(t)z'(t)\,dt,$$

we note that according to Exercise 1*(b)*, Sec. 38,

$$\frac{d}{dt}\frac{[z(t)]^2}{2} = z(t)z'(t).$$

Then, because $z(a) = z_1$ and $z(b) = z_2$, we have

$$\int_C z\,dz = \frac{[z(t)]^2}{2}\Bigg]_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$

Inasmuch as the value of this integral depends only on the end points of C and is otherwise independent of the arc that is taken, we may write

$$\int_{z_1}^{z_2} z\,dz = \frac{z_2^2 - z_1^2}{2}. \tag{5}$$

(Compare with Example 2, where the value of an integral from one fixed point to another depended on the path that was taken.)

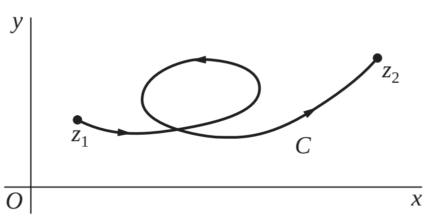

FIGURE 43

Expression (5) is also valid when C is a contour that is not necessarily smooth since a contour consists of a finite number of smooth arcs C_k $(k = 1, 2, \ldots, n)$, joined end to end. More precisely, suppose that each C_k extends from z_k to z_{k+1}. Then

$$\int_C z\,dz = \sum_{k=1}^{n}\int_{C_k} z\,dz = \sum_{k=1}^{n}\int_{z_k}^{z_{k+1}} z\,dz = \sum_{k=1}^{n}\frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}, \tag{6}$$

where this last summation has telescoped and z_1 is the initial point of C and z_{n+1} is its final point.

It follows from expression (6) that the integral of the function $f(z) = z$ around each closed contour in the plane has value zero. (Once again, compare with Example 2, where the value of the integral of a given function around a closed contour was *not* zero.) The question of predicting when an integral around a closed contour has value zero will be discussed in Secs. 44, 46, and 48.

42. EXAMPLES WITH BRANCH CUTS

The path in a contour integral can contain a point on a branch cut of the integrand involved. The next two examples illustrate this.

EXAMPLE 1. Let C denote the semicircular path

$$z = 3\,e^{i\theta} \quad (0 \le \theta \le \pi)$$

from the point $z = 3$ to the point $z = -3$ (Fig. 44). Although the branch

$$f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) \quad (|z| > 0,\, 0 < \arg z < 2\pi)$$

of the multiple-valued function $z^{1/2}$ is not defined at the initial point $z = 3$ of the contour C, the integral

$$I = \int_C z^{1/2}\,dz \tag{1}$$

nevertheless exists. For the integrand is piecewise continuous on C. To see that this is so, we first observe that when $z(\theta) = 3\,e^{i\theta}$,

$$f[z(\theta)] = \exp\left[\frac{1}{2}(\ln 3 + i\theta)\right] = \sqrt{3}\,e^{i\theta/2}.$$

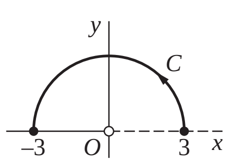

FIGURE 44

Hence the right-hand limits of the real and imaginary components of the function

$$f[z(\theta)]z'(\theta) = \sqrt{3}\,e^{i\theta/2}3ie^{i\theta} = 3\sqrt{3}ie^{i3\theta/2} = -3\sqrt{3}\sin\frac{3\theta}{2} + i3\sqrt{3}\cos\frac{3\theta}{2}$$
$$(0 < \theta \le \pi)$$

at $\theta = 0$ exist, those limits being 0 and $i3\sqrt{3}$, respectively. This means that $f[z(\theta)]z'(\theta)$ is continuous on the closed interval $0 \le \theta \le \pi$ when its value at $\theta = 0$ is defined as $i3\sqrt{3}$. Consequently,

$$I = 3\sqrt{3}i\int_0^{\pi} e^{i3\theta/2}\,d\theta.$$

Since

$$\int_0^{\pi} e^{i3\theta/2}\,d\theta = \frac{2}{3i}e^{i3\theta/2}\Big]_0^{\pi} = -\frac{2}{3i}(1+i),$$

we now have the value

(2) $$I = -2\sqrt{3}(1+i)$$

of integral (1).

EXAMPLE 2. Suppose that C is the positively oriented circle (Fig. 45)

$$z = Re^{i\theta} \quad (-\pi \le \theta \le \pi)$$

about the origin, and left a denote any nonzero real number. Using the principal branch

$$f(z) = z^{a-1} = \exp[(a-1)\text{Log } z] \quad (|z| > 0, -\pi < \text{Arg } z < \pi)$$

of the power function z^{a-1}, let us evaluate the integral

(3) $$I = \int_C z^{a-1}\,dz.$$

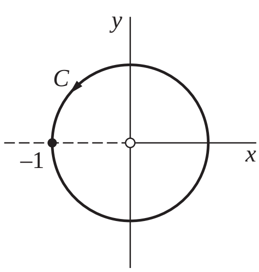

FIGURE 45

When $z(\theta) = Re^{i\theta}$, it is easy to see that

$$f[z(\theta)]z'(\theta) = iR^a e^{ia\theta} = -R^a \sin a\theta + iR^a \cos a\theta,$$

where the positive value of R^a is to be taken. Inasmuch as this function is piecewise continuous on $-\pi < \theta < \pi$, integral (3) exists. In fact,

(4) $$I = iR^a \int_{-\pi}^{\pi} e^{ia\theta}\,d\theta = iR^a\left[\frac{e^{ia\theta}}{ia}\right]_{-\pi}^{\pi} = i\frac{2R^a}{a}\cdot\frac{e^{ia\pi}-e^{-ia\pi}}{2i} = i\frac{2R^a}{a}\sin a\pi.$$

Note that if a is a nonzero integer n, this result tells us that

$$\int_C z^{n-1}\,dz = 0 \quad (n = \pm 1, \pm 2, \ldots). \tag{5}$$

If a is allowed to be zero, we have

$$\int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} d\theta = 2\pi i. \tag{6}$$

EXERCISES

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\,dz.$$

1. $f(z) = (z+2)/z$ and C is

(a) the semicircle $z = 2\,e^{i\theta}$ $(0 \le \theta \le \pi)$;
(b) the semicircle $z = 2\,e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
(c) the circle $z = 2\,e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. *(a)* $-4 + 2\pi i$; *(b)* $4 + 2\pi i$; *(c)* $4\pi i$.

2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

(a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
(b) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

Ans. *(a)* 0; *(b)* 0.

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points $0, 1, 1 + i$, and i, the orientation of C being in the counterclockwise direction.

Ans. $4(e^{\pi} - 1)$.

4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans. $2 + 3i$.

5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.

Ans. $z_2 - z_1$.

6. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. $i(1 - e^{-2\pi})$.

7. $f(z)$ is the principal branch

$$z^i = \exp(i\operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Ans. $-\dfrac{1 + e^{-\pi}}{2}(1 - i)$.

8. With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \,\overline{z}^{\,n} dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

9. Evaluate the integral I in Example 1, Sec. 41, using this representation for C:

$$z = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

(See Exercise 2, Sec. 39.)

10. Let C_0 and C denote the circles

$$z = z_0 + Re^{i\theta} \ (-\pi \le \theta \le \pi) \quad \text{and} \quad z = Re^{i\theta} \ (-\pi \le \theta \le \pi),$$

respectively.

(a) Use these parametric representations to show that

$$\int_{C_0} f(z - z_0)\, dz = \int_C f(z)\, dz$$

when f is piecewise continuous on C.

(b) Apply the result in part *(a)* to integrals (5) and (6) in Sec. 42 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \ (n = \pm 1, \pm 2, \ldots) \quad \text{and} \quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i.$$

11. *(a)* Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)$ $(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)$ $(\alpha \le \tau \le \beta)$ is the function described in Sec. 39, then

$$\int_a^b f[z(t)]z'(t)\, dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\, d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

(b) Point out how it follows that the identity obtained in part *(a)* remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one.

Suggestion: In part *(a)*, use the result in Exercise 1*(b)*, Sec. 39, and then refer to expression (14) in that section.

43. UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

We turn now to an inequality involving contour integrals that is extremely important in various applications. We present the result as a theorem but preface it with a needed lemma involving functions $w(t)$ of the type encountered in Secs. 37 and 38.

Lemma. *If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \le t \le b$, then*

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt. \tag{1}$$

This inequality clearly holds when the value of the integral on the left is zero. Thus, in the verification we may assume that its value is a *nonzero* complex number and write

$$\int_a^b w(t)\, dt = r_0\, e^{i\theta_0}.$$

Solving for r_0, we have

$$r_0 = \int_a^b e^{-i\theta_0} w(t)\, dt. \tag{2}$$

Now the left-hand side of this equation is a real number, and so the right-hand side is too. Thus, using the fact that the real part of a real number is the number itself, we find that

$$r_0 = \text{Re} \int_a^b e^{-i\theta_0} w(t)\, dt,$$

or

$$r_0 = \int_a^b \text{Re}[e^{-i\theta_0} w(t)]\, dt. \tag{3}$$

But

$$\text{Re}[e^{-i\theta_0} w(t)] \le |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}|\, |w(t)| = |w(t)|,$$

and it follows from equation (3) that

$$r_0 \le \int_a^b |w(t)|\, dt.$$

Because r_0 is, in fact, the left-hand side of inequality (1), the verification of the lemma is complete.

Theorem. *Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that*

$$|f(z)| \le M \tag{4}$$

for all points z on C at which $f(z)$ is defined, then

$$\left|\int_C f(z)\,dz\right| \le ML. \tag{5}$$

To prove this, let $z = z(t)$ $(a \le t \le b)$ be a parametric representation of C. According to the above lemma,

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f[z(t)]z'(t)\,dt\right| \le \int_a^b |f[z(t)]z'(t)|\,dt.$$

Inasmuch as

$$|f[z(t)]z'(t)| = |f[z(t)]|\,|z'(t)| \le M\,|z'(t)|$$

when $a \le t \le b$, it follows that

$$\left|\int_C f(z)\,dz\right| \le M\int_a^b |z'(t)|\,dt.$$

Since the integral on the right here represents the length L of C (see Sec. 39), inequality (5) is established. It is, of course, a strict inequality if inequality (4) is strict.

Note that since C is a contour and f is piecewise continuous on C, a number M such as the one appearing in inequality (4) will always exist. This is because the real-valued function $|f[z(t)]|$ is continuous on the closed bounded interval $a \le t \le b$ when f is continuous on C; and such a function always reaches a maximum value M on that interval.* Hence $|f(z)|$ has a maximum value on C when f is continuous on it. The same is, then, true when f is *piecewise* continuous on C.

EXAMPLE 1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant (Fig. 46). Inequality (5) can be used to show that

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}. \tag{6}$$

This is done by noting first that if z is a point on C, so that $|z| = 2$, then

$$|z+4| \le |z| + 4 = 6$$

*See, for instance A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 86–90, 1983.

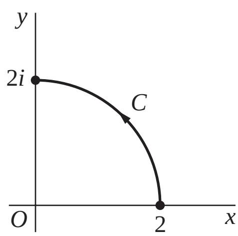

FIGURE 46

and

$$|z^3 - 1| \geq ||z|^3 - 1| = 7.$$

Thus, when z lies on C,

$$\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \leq \frac{6}{7}.$$

Writing $M = 6/7$ and observing that $L = \pi$ is the length of C, we may now use inequality (5) to obtain inequality (6).

EXAMPLE 2. Here C_R is the semicircular path

$$z = Re^{i\theta} \qquad (0 \leq \theta \leq \pi),$$

and $z^{1/2}$ denotes the branch

$$z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the square root function. (See Fig. 47.) Without actually finding the value of the integral, one can easily show that

$$\lim_{R\to\infty}\int_{C_R}\frac{z^{1/2}}{z^2+1}\,dz = 0. \tag{7}$$

For, when $|z| = R > 1$,

$$|z^{1/2}| = |\sqrt{R}e^{i\theta/2}| = \sqrt{R}$$

and

$$|z^2 + 1| \geq ||z^2| - 1| = R^2 - 1.$$

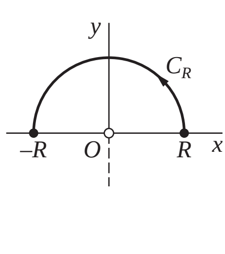

FIGURE 47

Consequently, at points on C_R,

$$\left|\frac{z^{1/2}}{z^2+1}\right| \le M_R \quad \text{where} \quad M_R = \frac{\sqrt{R}}{R^2-1}.$$

Since the length of C_R is the number $L = \pi R$, it follows from inequality (5) that

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz\right| \le M_R L.$$

But

$$M_R L = \frac{\pi R\sqrt{R}}{R^2-1}\cdot\frac{1/R^2}{1/R^2} = \frac{\pi/\sqrt{R}}{1-(1/R^2)},$$

and it is clear that the term on the far right here tends to zero as R tends to infinity. Limit (7) is, therefore, established.

EXERCISES

1. Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$$

when C is the same arc as the one in Example 1, Sec. 43.

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.

3. Show that if C is the boundary of the triangle with vertices at the points $0, 3i$, and -4, oriented in the counterclockwise direction (see Fig. 48), then

$$\left|\int_C (e^z - \overline{z})\,dz\right| \le 60.$$

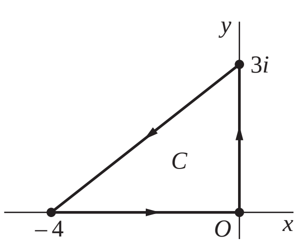

FIGURE 48

4. Let C_R denote the upper half of the circle $|z| = R\ (R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

5. Let C_R be the circle $|z| = R\ (R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho\ (0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).

7. Apply inequality (1), Sec. 43, to show that for all values of x in the interval $-1 \le x \le 1$, the functions*

$$P_n(x) = \frac{1}{\pi} \int_0^\pi \left(x + i\sqrt{1 - x^2} \cos\theta\right)^n d\theta \qquad (n = 0, 1, 2, \ldots)$$

satisfy the inequality $|P_n(x)| \le 1$.

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \ge |\sin x| \quad \text{and} \quad |\sin z| \ge |\sinh y|,$$

obtained in Exercises 8*(a)* and 9*(a)* of Sec. 34, show that $|\sin z| \ge 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \ge A$ for all points z lying on the contour C_N.

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*These functions are actually polynomials in x. They are known as *Legendre polynomials* and are important in applied mathematics. See, for example, Chap. 4 of the book by Lebedev that is listed in Appendix 1.

(b) Using the final result in part *(a)*, show that

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \le \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

44. ANTIDERIVATIVES

Although the value of a contour integral of a function $f(z)$ from a fixed point z_1 to a fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals from z_1 to z_2 have values that are *independent of path*. (Recall Examples 2 and 3 in Sec. 41.) The examples just cited also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

The theorem contains an extension of the fundamental theorem of calculus that simplifies the evaluation of many contour integrals. The extension involves the concept on an antiderivative of a continuous function $f(z)$ on a domain D, or a function $F(z)$ such that $F'(z) = f(z)$ for all z in D. Note that an antiderivative is, of necessity, an analytic function. Note, too, that *an antiderivative of a given function $f(z)$ is unique except for an additive constant*. This is because the derivative of the difference $F(z) - G(z)$ of any two such antiderivatives is zero; and, according to the theorem in Sec. 24, an analytic function is constant in a domain D when its derivative is zero throughout D.

Theorem. *Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:*

(a) *$f(z)$ has an antiderivative $F(z)$ throughout D;*

(b) *the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely*

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) *the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.*

It should be emphasized that the theorem does *not* claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true. The next section is devoted to the proof of the theorem and can be easily skipped by a reader who wishes to get on with other important aspects of integration theory. But we include here a number of examples illustrating how the theorem can be used.

EXAMPLE 1. The continuous function $f(z) = z^2$ has an antiderivative $F(z) = z^3/3$ throughout the plane. Hence

$$\int_0^{1+i} z^2\,dz = \frac{z^3}{3}\bigg]_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3}(-1+i)$$

for every contour from $z = 0$ to $z = 1 + i$.

EXAMPLE 2. The function $f(z) = 1/z^2$, which is continuous everywhere except at the origin, has an antiderivative $F(z) = -1/z$ in the domain $|z| > 0$, consisting of the entire plane with the origin deleted. Consequently,

$$\int_C \frac{dz}{z^2} = 0$$

when C is the positively oriented circle (Fig. 49)

$$z = 2\,e^{i\theta} \qquad (-\pi \le \theta \le \pi) \tag{1}$$

about the origin.

Note that the integral of the function $f(z) = 1/z$ around the same circle *cannot* be evaluated in a similar way. For, although the derivative of any branch $F(z)$ of $\log z$ is $1/z$ (Sec. 31), $F(z)$ is not differentiable, or even defined, along its branch cut. In particular, if a ray $\theta = \alpha$ from the origin is used to form the branch cut, $F'(z)$ fails to exist at the point where that ray intersects the circle C (see Fig. 49). So C does not lie in any domain throughout which $F'(z) = 1/z$, and one cannot make direct use of an antiderivative. Example 3, just below, illustrates how a combination of *two* different antiderivatives can be used to evaluate $f(z) = 1/z$ around C.

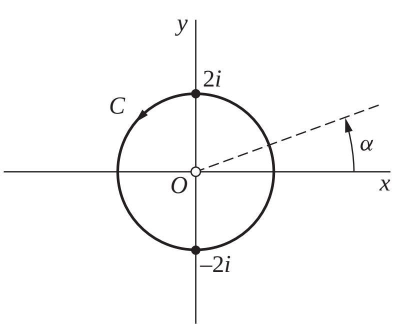

FIGURE 49

EXAMPLE 3. Let C_1 denote the right half

$$z = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right) \tag{2}$$

of the circle C in Example 2. The principal branch

$$\operatorname{Log} z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi)$$

of the logarithmic function serves as an antiderivative of the function $1/z$ in the evaluation of the integral of $1/z$ along C_1 (Fig. 50):

$$\begin{aligned}\int_{C_1} \frac{dz}{z} &= \int_{-2i}^{2i} \frac{dz}{z} = \operatorname{Log} z \Big]_{-2i}^{2i} = \operatorname{Log}(2i) - \operatorname{Log}(-2i)\\ &= \left(\ln 2 + i\frac{\pi}{2}\right) - \left(\ln 2 - i\frac{\pi}{2}\right) = \pi i.\end{aligned}$$

This integral was evaluated in another way in Example 1, Sec. 41, where representation (2) for the semicircle was used.

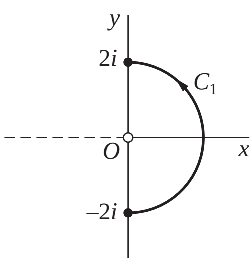

FIGURE 50

Next, let C_2 denote the *left* half

(3) $$z = 2\,e^{i\theta} \qquad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$$

of the same circle C and consider the branch

$$\log z = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

of the logarithmic function (Fig. 51). One can write

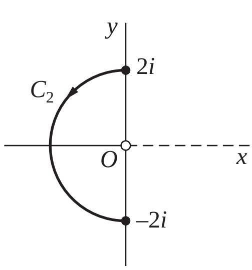

FIGURE 51

$$\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} = \log z \Big]_{2i}^{-2i} = \log(-2i) - \log(2i)$$
$$= \left(\ln 2 + i\frac{3\pi}{2}\right) - \left(\ln 2 + i\frac{\pi}{2}\right) = \pi i.$$

The value of the integral of $1/z$ around the entire circle $C = C_1 + C_2$ is thus obtained:

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

EXAMPLE 4. Let us use an antiderivative to evaluate the integral

$$\int_{C_1} z^{1/2}\, dz, \tag{4}$$

where the integrand is the branch

$$f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, 0 < \theta < 2\pi) \tag{5}$$

of the square root function and where C_1 is any contour from $z = -3$ to $z = 3$ that, except for its end points, lies above the x axis (Fig. 52). Although the integrand is piecewise continuous on C_1, and the integral therefore exists, the branch (5) of $z^{1/2}$ is not defined on the ray $\theta = 0$, in particular at the point $z = 3$. But another branch,

$$f_1(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right),$$

is defined and continuous everywhere on C_1. The values of $f_1(z)$ at all points on C_1 except $z = 3$ coincide with those of our integrand (5); so the integrand can be replaced by $f_1(z)$. Since an antiderivative of $f_1(z)$ is the function

$$F_1(z) = \frac{2}{3}z^{3/2} = \frac{2}{3}r\sqrt{r}e^{i3\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right),$$

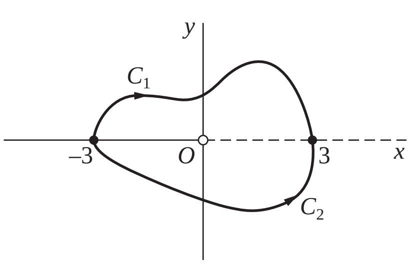

FIGURE 52

we can now write

$$\int_{C_1} z^{1/2}\, dz = \int_{-3}^{3} f_1(z)\, dz = F_1(z)\Big]_{-3}^{3} = 2\sqrt{3}(e^{i0} - e^{i3\pi/2}) = 2\sqrt{3}(1+i).$$

(Compare with Example 1 in Sec. 42.)

The integral

$$\int_{C_2} z^{1/2}\, dz \tag{6}$$

of the function (5) over any contour C_2 that extends from $z = -3$ to $z = 3$ *below* the real axis can be evaluated in a similar way. In this case, we can replace the integrand by the branch

$$f_2(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}\right),$$

whose values coincide with those of the integrand at $z = -3$ and at all points on C_2 below the real axis. This enables us to use an antiderivative of $f_2(z)$ to evaluate integral (6). Details are left to the exercises.

45. PROOF OF THE THEOREM

To prove the theorem in the previous section, it is sufficient to show that statement *(a)* implies statement *(b)*, that statement *(b)* implies statement *(c)*, and finally that statement *(c)* implies statement *(a)*.

Let us assume that statement *(a)* is true, or that $f(z)$ has an antiderivative $F(z)$ on the domain D being considered. To show how statement *(b)* follows, we need to show that integration in independent of path in D and that the fundamental theorem of calculus can be extended using $F(z)$. If a contour C from z_1 to z_2 is a *smooth* arc lying in D, with parametric representation $z = z(t)$ $(a \le t \le b)$, we know from Exercise 5, Sec. 39, that

$$\frac{d}{dt}F[z(t)] = F'[z(t)]z'(t) = f[z(t)]z'(t) \quad (a \le t \le b).$$

Because the fundamental theorem of calculus can be extended so as to apply to complex-valued functions of a real variable (Sec. 38), it follows that

$$\int_C f(z)\, dz = \int_a^b f\big[z(t)\big]z'(t)\, dt = F[z(t)]\Big]_a^b = F[z(b)] - F[z(a)].$$

Since $z(b) = z_2$ and $z(a) = z_1$, the value of this contour integral is then

$$F(z_2) - F(z_1);$$

and that value is evidently independent of the contour C as long as C extends from z_1 to z_2 and lies entirely in D. That is,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1) = F(z)\Big]_{z_1}^{z_2} \tag{1}$$

when C is smooth. Expression (1) is also valid when C is *any* contour, not necessarily a smooth one, that lies in D. For, if C consists of a finite number of smooth arcs C_k $(k = 1, 2, \ldots, n)$, each C_k extending from a point z_k to a point z_{k+1}, then

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} f(z)\,dz = \sum_{k=1}^{n} [F(z_{k+1}) - F(z_k)].$$

Because the last sum here telescopes to $F(z_{n+1}) - F(z_1)$, we arrive at the expression

$$\int_C f(z)\,dz = F(z_{n+1}) - F(z_1).$$

(Compare with Example 3, Sec. 41.) The fact that statement *(b)* follows from statement *(a)* is now established.

To see that statement *(b)* implies statement *(c)*, we now show that the value of any integral around a closed contour in D is zero when integration is independent of path there. To do this, we let z_1 and z_2 denote two points on any closed contour C lying in D and form two paths C_1 and C_2, each with initial point z_1 and final point z_2, such that $C = C_1 - C_2$ (Fig. 53). Assuming that integration is independent of path in D, one can write

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz, \tag{2}$$

or

$$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0. \tag{3}$$

That is, the integral of $f(z)$ around the closed contour $C = C_1 - C_2$ has value zero.

It remains to show statement *(c)* implies statement *(a)*. That is, we need to show that if integrals of $f(z)$ around closed contours in D always have value zero,

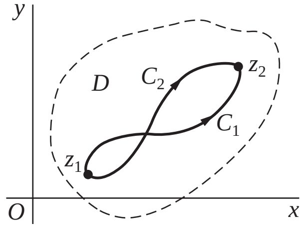

FIGURE 53

then $f(z)$ has an antiderivative on D. Assuming that the values of such integrals are in fact zero, we start by showing that integration is independent of path in D. We let C_1 and C_2 denote any two contours, lying in D, from a point z_1 to a point z_2 and observe that since integrals around closed paths lying in D have value zero, equation (3) holds (see Fig. 53). Thus equation (2) holds. Integration is, therefore, independent of path in D; and we can define the function

$$F(z) = \int_{z_0}^{z} f(s)\,ds$$

on D. The proof of the theorem is complete once we show that $F'(z) = f(z)$ everywhere in D. We do this by letting $z + \Delta z$ be any point distinct from z and lying in some neighborhood of z that is small enough to be contained in D. Then

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds = \int_{z}^{z+\Delta z} f(s)\,ds,$$

where the path of integration may be selected as a line segment (Fig. 54). Since

$$\int_{z}^{z+\Delta z} ds = \Delta z$$

(see Exercise 5, Sec. 42), one can write

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,ds;$$

and it follows that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(s) - f(z)]\,ds.$$

But f is continuous at the point z. Hence, for each positive number ε, a positive number δ exists such that

$$|f(s) - f(z)| < \varepsilon \quad \text{whenever} \quad |s - z| < \delta.$$

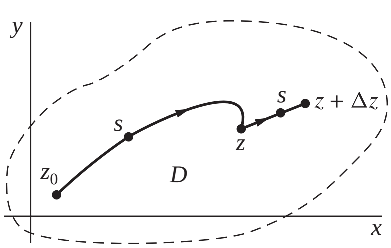

FIGURE 54

Consequently, if the point $z + \Delta z$ is close enough to z so that $|\Delta z| < \delta$, then

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \varepsilon |\Delta z| = \varepsilon;$$

that is,

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z),$$

or $F'(z) = f(z)$.

EXERCISES

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n \, dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}) \qquad (n = 0, 1, 2, \ldots).$$

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

(a) $\displaystyle\int_i^{i/2} e^{\pi z} \, dz$; *(b)* $\displaystyle\int_0^{\pi + 2i} \cos\left(\frac{z}{2}\right) dz$; *(c)* $\displaystyle\int_1^3 (z-2)^3 \, dz$.

Ans. *(a)* $(1+i)/\pi$; *(b)* $e + (1/e)$; *(c)* 0.

3. Use the theorem in Sec. 44 to show that

$$\int_{C_0} (z - z_0)^{n-1} \, dz = 0 \qquad (n = \pm 1, \pm 2, \ldots)$$

when C_0 is any closed contour which does not pass through the point z_0. [Compare with Exercise 10*(b)*, Sec. 42.]

4. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of $z^{1/2}$ in Example 4, Sec. 44, to show that integral (6) there has value $2\sqrt{3}(-1 + i)$. Note that the value of the integral of the function (5) around the closed contour $C_2 - C_1$ in that example is, therefore, $-4\sqrt{3}$.

5. Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2}(1 - i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 7, Sec. 42.)

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.