CHAPTER

3

ELEMENTARY FUNCTIONS

We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable z that reduce to the elementary functions in calculus when $z = x + i0$. We start by defining the complex exponential function and then use it to develop the others.

29. THE EXPONENTIAL FUNCTION

As anticipated earlier (Sec. 14), we define here the exponential function e^z by writing

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

where Euler's formula (see Sec. 6)

$$e^{iy} = \cos y + i \sin y \tag{2}$$

is used and y is to be taken in radians. We see from this definition that e^z reduces to the usual exponential function in calculus when $y = 0$; and, following the convention used in calculus, we often write $\exp z$ for e^z.

Note that since the *positive* nth root $\sqrt[n]{e}$ of e is assigned to e^x when $x = 1/n$ $(n = 2, 3, \ldots)$, expression (1) tells us that the complex exponential function e^z is also $\sqrt[n]{e}$ when $z = 1/n$ $(n = 2, 3, \ldots)$. This is an exception to the convention (Sec. 9) that would ordinarily require us to interpret $e^{1/n}$ as the set of nth roots of e.

According to definition (1), $e^x e^{iy} = e^{x+iy}$; and, as already pointed out in Sec. 14, the definition is suggested by the additive property

$$e^{x_1} e^{x_2} = e^{x_1 + x_2}$$

of e^x in calculus. That property's extension,

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}, \tag{3}$$

to complex analysis is easy to verify. To do this, we write

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Then

$$e^{z_1} e^{z_2} = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}) = (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2}).$$

But x_1 and x_2 are both real, and we know from Sec. 7 that

$$e^{iy_1} e^{iy_2} = e^{i(y_1 + y_2)}.$$

Hence

$$e^{z_1} e^{z_2} = e^{(x_1 + x_2)} e^{i(y_1 + y_2)};$$

and, since

$$(x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2,$$

the right-hand side of this last equation becomes $e^{z_1 + z_2}$. Property (3) is now established.

Observe how property (3) enables us to write $e^{z_1 - z_2} e^{z_2} = e^{z_1}$, or

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}. \tag{4}$$

From this and the fact that $e^0 = 1$, it follows that $1/e^z = e^{-z}$.

There are a number of other important properties of e^z that are expected. According to Example 1 in Sec. 22, for instance,

$$\frac{d}{dz} e^z = e^z \tag{5}$$

everywhere in the z plane. Note that the differentiability of e^z for all z tells us that e^z *is entire* (Sec. 24). It is also true that

$$e^z \neq 0 \quad \text{for any complex number } z. \tag{6}$$

This is evident upon writing definition (1) in the form

$$e^z = \rho e^{i\phi} \quad \text{where} \quad \rho = e^x \text{ and } \phi = y,$$

which tells us that

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi \ \ (n = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

Statement (6) then follows from the observation that $|e^z|$ is always positive.

Some properties of e^z are, however, *not* expected. For example, since

$$e^{z+2\pi i} = e^z e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1,$$

we find that e^z *is periodic,* with a pure imaginary period of $2\pi i$:

$$e^{z+2\pi i} = e^z. \tag{8}$$

For another property of e^z that e^x does not have, we note that while e^x is always positive, e^z can be negative. We recall (Sec. 6), for instance, that $e^{i\pi} = -1$. In fact,

$$e^{i(2n+1)\pi} = e^{i2n\pi + i\pi} = e^{i2n\pi} e^{i\pi} = (1)(-1) = -1 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

There are, moreover, values of z such that e^z is any given nonzero complex number. This is shown in the next section, where the logarithmic function is developed, and is illustrated in the following example.

EXAMPLE. In order to find numbers $z = x + iy$ such that

$$e^z = 1 + i, \tag{9}$$

we write equation (9) as

$$e^x e^{iy} = \sqrt{2}\, e^{i\pi/4}.$$

Then, in view of the statment in italics at the beginning of Sec. 9 regarding the equality of two nonzero complex numbers in exponential form,

$$e^x = \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Because $\ln(e^x) = x$, it follows that

$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{4}\right)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots);$$

and so

$$z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{10}$$

EXERCISES

1. Show that

(a) $\exp(2 \pm 3\pi i) = -e^2$; *(b)* $\exp\left(\dfrac{2+\pi i}{4}\right) = \sqrt{\dfrac{e}{2}}\,(1+i)$;

(c) $\exp(z + \pi i) = -\exp z$.

2. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp \overline{z}$ is not analytic anywhere.

4. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative?

Ans. $f'(z) = 2z \exp(z^2)$.

5. Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

6. Show that $|\exp(z^2)| \le \exp(|z|^2)$.

7. Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

8. Find all values of z such that

(a) $e^z = -2$; *(b)* $e^z = 1 + \sqrt{3}i$; *(c)* $\exp(2z - 1) = 1$.

Ans. *(a)* $z = \ln 2 + (2n + 1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $z = \ln 2 + \left(2n + \dfrac{1}{3}\right)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $z = \dfrac{1}{2} + n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

9. Show that $\overline{\exp(iz)} = \exp(i\overline{z})$ if and only if $z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$. (Compare with Exercise 4, Sec. 28.)

10. *(a)* Show that if e^z is real, then $\operatorname{Im} z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$.

(b) If e^z is pure imaginary, what restriction is placed on z?

11. Describe the behavior of $e^z = e^x e^{iy}$ as *(a)* x tends to $-\infty$; *(b)* y tends to ∞.

12. Write $\operatorname{Re}(e^{1/z})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

13. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D. State why the functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \quad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

are harmonic in D and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.

14. Establish the identity

$$(e^z)^n = e^{nz} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

in the following way.

(a) Use mathematical induction to show that it is valid when $n = 0, 1, 2, \ldots$.
(b) Verify it for negative integers n by first recalling from Sec. 7 that

$$z^n = (z^{-1})^m \qquad (m = -n = 1, 2, \ldots)$$

when $z \neq 0$ and writing $(e^z)^n = (1/e^z)^m$. Then use the result in part *(a)*, together with the property $1/e^z = e^{-z}$ (Sec. 29) of the exponential function.

30. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z \tag{1}$$

for w, where z is any *nonzero* complex number. To do this, we note that when z and w are written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$ and $w = u + iv$, equation (1) becomes

$$e^u e^{iv} = re^{i\Theta}.$$

According to the statement in italics at the beginning of Sec. 9 about the equality of two complex numbers expressed in exponential form, this tells us that

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi$$

where n is any integer. Since the equation $e^u = r$ is the same as $u = \ln r$, it follows that equation (1) is satisfied if and only if w has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus, if we write

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{2}$$

equation (1) tells us that

$$e^{\log z} = z \qquad (z \neq 0), \tag{3}$$

which serves to motivate expression (2) as the *definition* of the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$.

EXAMPLE 1. If $z = -1 - \sqrt{3}i$, then $r = 2$ and $\Theta = -2\pi/3$. Hence

$$\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i$$
$$(n = 0, \pm 1, \pm 2, \ldots).$$

It should be emphasized that it is *not* true that the left-hand side of equation (3) with the order of the exponential and logarithmic functions reversed reduces to just z. More precisely, since expression (2) can be written

$$\log z = \ln |z| + i \arg z$$

and since (Sec. 29)

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi \ \ (n = 0, \pm 1, \pm 2, \ldots)$$

when $z = x + iy$, we know that

$$\log(e^z) = \ln |e^z| + i \arg(e^z) = \ln(e^x) + i(y + 2n\pi) = (x + iy) + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$\log(e^z) = z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{4}$$

The *principal value* of $\log z$ is the value obtained from equation (2) when $n = 0$ there and is denoted by Log z. Thus

$$\operatorname{Log} z = \ln r + i\Theta. \tag{5}$$

Note that Log z is well defined and single-valued when $z \neq 0$ and that

$$\log z = \operatorname{Log} z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

It reduces to the usual logarithm in calculus when z is a positive real number $z = r$. To see this, one need only write $z = re^{i0}$, in which case equation (5) becomes $\operatorname{Log} z = \ln r$. That is, $\operatorname{Log} r = \ln r$.

EXAMPLE 2. From expression (2), we find that

$$\log 1 = \ln 1 + i(0 + 2n\pi) = 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

As anticipated, Log $1 = 0$.

Our final example here reminds us that although we were unable to find logarithms of *negative* real numbers in calculus, we can now do so.

EXAMPLE 3. Observe that

$$\log(-1) = \ln 1 + i(\pi + 2n\pi) = (2n + 1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that $\operatorname{Log}(-1) = \pi i$.

31. BRANCHES AND DERIVATIVES OF LOGARITHMS

If $z = re^{i\theta}$ is a nonzero complex number, the argument θ has any one of the values $\theta = \Theta + 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $\Theta = \text{Arg } z$. Hence the definition

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

of the multiple-valued logarithmic function in Sec. 30 can be written

$$\log z = \ln r + i\theta. \tag{1}$$

If we let α denote any real number and restrict the value of θ in expression (1) so that $\alpha < \theta < \alpha + 2\pi$, the function

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi), \tag{2}$$

with components

$$u(r, \theta) = \ln r \quad \text{and} \quad v(r, \theta) = \theta, \tag{3}$$

is *single-valued* and continuous in the stated domain (Fig. 35). Note that if the function (2) were to be defined on the ray $\theta = \alpha$, it would not be continuous there. For if z is a point on that ray, there are points arbitrarily close to z at which the values of v are near α and also points such that the values of v are near $\alpha + 2\pi$.

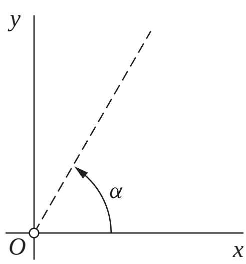

FIGURE 35

The function (2) is not only continuous but also analytic throughout the domain $r > 0$, $\alpha < \theta < \alpha + 2\pi$ since the first-order partial derivatives of u and v are continuous there and satisfy the polar form (Sec. 23)

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations. Furthermore, according to Sec. 23,

$$\frac{d}{dz} \log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}};$$

that is,

$$\frac{d}{dz}\log z = \frac{1}{z} \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi). \tag{4}$$

In particular,

$$\frac{d}{dz}\,\text{Log}\, z = \frac{1}{z} \qquad (|z| > 0, -\pi < \text{Arg}\ z < \pi). \tag{5}$$

A *branch* of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f. The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f. Observe that for each fixed α, the single-valued function (2) is a branch of the multiple-valued function (1). The function

$$\text{Log}\, z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi) \tag{6}$$

is called the *principal branch.*

A *branch cut* is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Points on the branch cut for F are singular points (Sec. 24) of F, and any point that is common to all branch cuts of f is called a *branch point.* The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic function. The branch cut for the principal branch (6) consists of the origin and the ray $\Theta = \pi$. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

Special care must be taken in using branches of the logarithmic function, especially since expected identities involving logarithms do not always carry over from calculus.

EXAMPLE. When the principal branch (6) is used, one can see that

$$\text{Log}(i^3) = \text{Log}(-i) = \ln 1 - i\frac{\pi}{2} = -\frac{\pi}{2}i$$

and

$$3\,\text{Log}\, i = 3\left(\ln 1 + i\frac{\pi}{2}\right) = \frac{3\pi}{2}i.$$

Hence

$$\text{Log}(i^3) \neq 3\,\text{Log}\, i.$$

(See also Exercises 3 and 4.)

In Sec. 32, we shall derive some identities involving logarithms that *do* carry over from calculus, sometimes with qualifications as to how they are to be interpreted. A reader who wishes to pass to Sec. 33 can simply refer to results in Sec. 32 when needed.

EXERCISES

1. Show that

(a) $\text{Log}(-ei) = 1 - \dfrac{\pi}{2}i$; *(b)* $\text{Log}(1 - i) = \dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}i$.

2. Show that

(a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\log i = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Show that

(a) $\text{Log}(1 + i)^2 = 2\,\text{Log}(1 + i)$; *(b)* $\text{Log}(-1 + i)^2 \neq 2\,\text{Log}(-1 + i)$.

4. Show that

(a) $\log(i^2) = 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{\pi}{4} < \theta < \dfrac{9\pi}{4}\right)$;

(b) $\log(i^2) \neq 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{3\pi}{4} < \theta < \dfrac{11\pi}{4}\right)$.

5. Show that

(a) the set of values of $\log(i^{1/2})$ is

$$\left(n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that the same is true of $(1/2)\log i$;

(b) the set of values of $\log(i^2)$ is *not* the same as the set of values of $2\log i$.

6. Given that the branch $\log z = \ln r + i\theta\ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 30)

$$e^{\log z} = z \qquad (z \neq 0)$$

and using the chain rule.

7. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

8. Suppose that the point $z = x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta\ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$. [Compare with equation (4), Sec. 30.]

9. Show that

(a) the function $f(z) = \text{Log}(z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;

(b) the function

$$f(z) = \frac{\text{Log}(z + 4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

10. Show in two ways that the function $\ln(x^2 + y^2)$ is harmonic in every domain that does not contain the origin.

11. Show that

$$\operatorname{Re}[\log(z-1)] = \frac{1}{2}\ln[(x-1)^2 + y^2] \qquad (z \neq 1).$$

Why must this function satisfy Laplace's equation when $z \neq 1$?

32. SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{1}$$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2}$$

was in Sec. 8. That is, if values of two of the three logarithms are specified, then there is a value of the third such that equation (1) holds.

The verification of statement (1) can be based on statement (2) in the following way. Since $|z_1 z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|.$$

So it follows from this and equation (2) that

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2). \tag{3}$$

Finally, because of the way in which equations (1) and (2) are to be interpreted, equation (3) is the same as equation (1).

EXAMPLE. To illustrate statement (1), write $z_1 = z_2 = -1$ and recall from Examples 2 and 3 in Sec. 30 that

$$\log 1 = 2n\pi i \quad \text{and} \quad \log(-1) = (2n+1)\pi i,$$

where $n = 0, \pm 1, \pm 2, \ldots$. Noting that $z_1 z_2 = 1$ and using the values

$$\log(z_1 z_2) = 0 \quad \text{and} \quad \log z_1 = \pi i,$$

we find that equations (1) is satisfied when the value $\log z_2 = -\pi i$ is chosen.

If, on the other hand, the principal values

$$\text{Log}\, 1 = 0 \quad \text{and} \quad \text{Log}(-1) = \pi i$$

are used,

$$\text{Log}(z_1 z_2) = 0 \quad \text{and} \quad \text{Log}\, z_1 + \log z_2 = 2\pi i$$

for the same numbers z_1 and z_2. Thus statement (1), which is sometimes true when *log* is replaced by *Log* (see Exercise 1), is not always true when principal values are used in all three of its terms.

Verification of the statement

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2, \tag{4}$$

which is to be interpreted in the same way as statement (1), is left to the exercises.

We include here two other properties of $\log z$ that will be of special interest in Sec. 33. If z is a nonzero complex number, then

$$z^n = e^{n \log z} \qquad (n = 0 \pm 1, \pm 2, \ldots) \tag{5}$$

for any value of $\log z$ that is taken. When $n = 1$, this reduces, of course, to relation (3), Sec. 30. Equation (5) is readily verified by writing $z = re^{i\theta}$ and noting that each side becomes $r^n e^{in\theta}$.

It is also true that when $z \neq 0$,

$$z^{1/n} = \exp\left(\frac{1}{n} \log z\right) \qquad (n = 1, 2, \ldots). \tag{6}$$

That is, the term on the right here has n distinct values, and those values are the nth roots of z. To prove this, we write $z = r \exp(i\Theta)$, where Θ is the principal value of $\arg z$. Then, in view of definition (2), Sec. 30, of $\log z$,

$$\exp\left(\frac{1}{n} \log z\right) = \exp\left[\frac{1}{n} \ln r + \frac{i(\Theta + 2k\pi)}{n}\right]$$

where $k = 0, \pm 1, \pm 2, \ldots$. Thus

$$\exp\left(\frac{1}{n} \log z\right) = \sqrt[n]{r} \exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

Because $\exp(i2k\pi/n)$ has distinct values only when $k = 0, 1, \ldots, n-1$, the right-hand side of equation (7) has only n values. That right-hand side is, in fact, an expression for the nth roots of z (Sec. 9), and so it can be written $z^{1/n}$. This establishes property (6), which is actually valid when n is a negative integer too (see Exercise 5).

EXERCISES

1. Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

Suggestion: Write $\Theta_1 = \operatorname{Arg} z_1$ and $\Theta_2 = \operatorname{Arg} z_2$. Then observe how it follows from the stated restrictions on z_1 and z_2 that $-\pi < \Theta_1 + \Theta_2 < \pi$.

2. Show that for any two nonzero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare with Exercise 1.)

3. Verify expression (4), Sec. 32, for $\log(z_1/z_2)$ by

(a) using the fact that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (Sec. 8);

(b) showing that $\log(1/z) = -\log z \ (z \neq 0)$, in the sense that $\log(1/z)$ and $-\log z$ have the same set of values, and then referring to expression (1), Sec. 32, for $\log(z_1 z_2)$.

4. By choosing specific nonzero values of z_1 and z_2, show that expression (4), Sec. 32, for $\log(z_1/z_2)$ is not always valid when *log* is replaced by *Log*.

5. Show that property (6), Sec. 32, also holds when n is a negative integer. Do this by writing $z^{1/n} = (z^{1/m})^{-1}$ $(m = -n)$, where n has any one of the negative values $n = -1, -2, \ldots$ (see Exercise 9, Sec. 10), and using the fact that the property is already known to be valid for positive integers.

6. Let z denote any nonzero complex number, written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$, and let n denote any fixed positive integer $(n = 1, 2, \ldots)$. Show that all of the values of $\log(z^{1/n})$ are given by the equation

$$\log(z^{1/n}) = \frac{1}{n}\ln r + i\frac{\Theta + 2(pn + k)\pi}{n},$$

where $p = 0, \pm 1, \pm 2, \ldots$ and $k = 0, 1, 2, \ldots, n - 1$. Then, after writing

$$\frac{1}{n}\log z = \frac{1}{n}\ln r + i\frac{\Theta + 2q\pi}{n},$$

where $q = 0, \pm 1, \pm 2, \ldots,$ show that the set of values of $\log(z^{1/n})$ is the same as the set of values of $(1/n)\log z$. Thus show that $\log(z^{1/n}) = (1/n)\log z$ where, corresponding to a value of $\log(z^{1/n})$ taken on the left, the appropriate value of $\log z$ is to be selected on the right, and conversely. [The result in Exercise 5*(a)*, Sec. 31, is a special case of this one.]

Suggestion: Use the fact that the remainder upon dividing an integer by a positive integer n is always an integer between 0 and $n - 1$, inclusive; that is, when a positive integer n is specified, any integer q can be written $q = pn + k$, where p is an integer and k has one of the values $k = 0, 1, 2, \ldots, n - 1$.

33. COMPLEX EXPONENTS

When $z \neq 0$ and *the exponent c is any complex number,* the function z^c is defined by means of the equation

$$z^c = e^{c \log z}, \tag{1}$$

where $\log z$ denotes the multiple-valued logarithmic function. Equation (1) provides a consistent definition of z^c in the sense that it is already known to be valid (see Sec. 32) when $c = n$ $(n = 0, \pm 1, \pm 2, \ldots)$ and $c = 1/n$ $(n = \pm 1, \pm 2, \ldots)$. Definition (1) is, in fact, suggested by those particular choices of c.

EXAMPLE 1. Powers of z are, in general, multiple-valued, as illustrated by writing

$$i^{-2i} = \exp(-2i \log i)$$

and then

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = \left(2n + \frac{1}{2}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

This shows that

$$i^{-2i} = \exp[(4n + 1)\pi] \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

Note that these values of i^{-2i} are all *real numbers.*

Since the exponential function has the property $1/e^z = e^{-z}$ (Sec. 29), one can see that

$$\frac{1}{z^c} = \frac{1}{\exp(c \log z)} = \exp(-c \log z) = z^{-c}$$

and, in particular, that $1/i^{2i} = i^{-2i}$. According to expression (2), then,

$$\frac{1}{i^{2i}} = \exp[(4n + 1)\pi] \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{3}$$

If $z = re^{i\theta}$ and α is any real number, the branch

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function is single-valued and analytic in the indicated domain (Sec. 31). When that branch is used, it follows that the function $z^c = \exp(c \log z)$ is single-valued and analytic in the same domain. The derivative of such a *branch of* z^c is found by first using the chain rule to write

$$\frac{d}{dz} z^c = \frac{d}{dz} \exp(c \log z) = \frac{c}{z} \exp(c \log z)$$

and then recalling (Sec. 30) the identity $z = \exp(\log z)$. That yields the result

$$\frac{d}{dz} z^c = c\frac{\exp(c \log z)}{\exp(\log z)} = c \exp[(c-1)\log z],$$

or

$$\frac{d}{dz} z^c = cz^{c-1} \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi). \tag{4}$$

The *principal value* of z^c occurs when $\log z$ is replaced by Log z in definition (1):

$$\text{P.V. } z^c = e^{c \operatorname{Log} z}. \tag{5}$$

Equation (5) also serves to define the *principal branch* of the function z^c on the domain $|z| > 0, -\pi < \operatorname{Arg} z < \pi$.

EXAMPLE 2. The principal value of $(-i)^i$ is

$$\exp[i \operatorname{Log}(-i)] = \exp\left[i\left(\ln 1 - i\frac{\pi}{2}\right)\right] = \exp\frac{\pi}{2}.$$

That is,

$$\text{P.V. } (-i)^i = \exp\frac{\pi}{2}. \tag{6}$$

EXAMPLE 3. The principal branch of $z^{2/3}$ can be written

$$\exp\left(\frac{2}{3}\operatorname{Log} z\right) = \exp\left(\frac{2}{3}\ln r + \frac{2}{3}i\Theta\right) = \sqrt[3]{r^2}\exp\left(i\frac{2\Theta}{3}\right).$$

Thus

$$\text{P.V. } z^{2/3} = \sqrt[3]{r^2}\cos\frac{2\Theta}{3} + i\sqrt[3]{r^2}\sin\frac{2\Theta}{3}. \tag{7}$$

This function is analytic in the domain $r > 0, -\pi < \Theta < \pi$, as one can see directly from the theorem in Sec. 23.

While familiar laws of exponents used in calculus often carry over to complex analysis, there are exceptions when certain numbers are involved.

EXAMPLE 4. Consider the nonzero complex numbers

$$z_1 = 1 + i, \quad z_2 = 1 - i, \quad \text{and} \quad z_3 = -1 - i.$$

When principal values of the powers are taken,

$$(z_1z_2)^i = 2^i = e^{i\,\mathrm{Log}\,2} = e^{i(\ln 2 + i0)} = e^{i\ln 2}$$

and

$$z_1^i = e^{i\,\mathrm{Log}(1+i)} = e^{i(\ln\sqrt{2} + i\pi/4)} = e^{-\pi/4}e^{i(\ln 2)/2},$$
$$z_2^i = e^{i\,\mathrm{Log}(1-i)} = e^{i(\ln\sqrt{2} - i\pi/4)} = e^{\pi/4}e^{i(\ln 2)/2}.$$

Thus

$$(z_1z_2)^i = z_1^i z_2^i, \tag{8}$$

as might be expected.

On the other hand, continuing to use principal values, we see that

$$(z_2z_3)^i = (-2)^i = e^{i\,\mathrm{Log}(-2)} = e^{i(\ln 2 + i\pi)} = e^{-\pi}\,e^{i\ln 2}$$

and

$$z_3^i = e^{i\,\mathrm{Log}(-1-i)} = e^{i(\ln\sqrt{2} - i3\pi/4)} = e^{3\pi/4}e^{i(\ln 2)/2}.$$

Hence

$$(z_2z_3)^i = \left[e^{\pi/4}e^{i(\ln 2)/2}\right]\left[e^{3\pi/4}e^{i(\ln 2)/2}\right]e^{-2\pi},$$

or

$$(z_2z_3)^i = z_2^i z_3^i\,e^{-2\pi}. \tag{9}$$

According to definition (1), *the exponential function with base* c, where c is any nonzero complex constant, is written

$$c^z = e^{z\log c}. \tag{10}$$

Note that although e^z is, in general, multiple-valued according to definition (10), the usual interpretation of e^z occurs when the principal value of the logarithm is taken. This is because the principal value of $\log e$ is unity.

When a value of $\log c$ is specified, c^z is an entire function of z. In fact,

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z\log c} = e^{z\log c}\log c\,;$$

and this shows that

$$\frac{d}{dz}c^z = c^z\log c. \tag{11}$$

EXERCISES

1. Show that

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $(-1)^{1/\pi} = e^{(2n+1)i} \quad (n = 0, \pm 1, \pm 2, \ldots)$.

2. Find the principal value of

(a) i^i; *(b)* $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; *(c)* $(1-i)^{4i}$.

Ans. *(a)* $\exp(-\pi/2)$; *(b)* $-\exp(2\pi^2)$; *(c)* $e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)]$.

3. Use definition (1), Sec. 33, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

4. Show that the result in Exercise 3 could have been obtained by writing

(a) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$ and first finding the square roots of $-1+\sqrt{3}i$;

(b) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^3]^{1/2}$ and first cubing $-1+\sqrt{3}i$.

5. Show that the *principal* nth root of a nonzero complex number z_0 that was defined in Sec. 9 is the same as the principal value of $z_0^{1/n}$ defined by equation (5), Sec. 33.

6. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(a \ln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

7. Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

Ans. c is real.

8. Let c, c_1, c_2, and z denote complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then

(a) $z^{c_1}z^{c_2} = z^{c_1+c_2}$; *(b)* $\frac{z^{c_1}}{z^{c_2}} = z^{c_1-c_2}$; *(c)* $(z^c)^n = z^{cn} \quad (n = 1, 2, \ldots)$.

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

34. TRIGONOMETRIC FUNCTIONS

Euler's formula (Sec. 6) tells us that

$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x$$

for every real number x. Hence

$$e^{ix} - e^{-ix} = 2i\sin x \quad \text{and} \quad e^{ix} + e^{-ix} = 2\cos x.$$

That is,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

It is, therefore, natural to *define* the sine and cosine functions of a complex variable z as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{1}$$

These functions are entire since they are linear combinations (Exercise 3, Sec. 25) of the entire functions e^{iz} and e^{-iz}. Knowing the derivatives

$$\frac{d}{dz} e^{iz} = ie^{iz} \quad \text{and} \quad \frac{d}{dz} e^{-iz} = -ie^{-iz}$$

of those exponential functions, we find from equations (1) that

$$\frac{d}{dz} \sin z = \cos z \quad \text{and} \quad \frac{d}{dz} \cos z = -\sin z. \tag{2}$$

It is easy to see from definitions (1) that the sine and cosine functions remain odd and even, respectively:

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z. \tag{3}$$

Also,

$$e^{iz} = \cos z + i \sin z. \tag{4}$$

This is, of course, Euler's formula (Sec. 6) when z is real.

A variety of identities carry over from trigonometry. For instance (see Exercises 2 and 3),

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \tag{5}$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2. \tag{6}$$

From these, it follows readily that

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z, \tag{7}$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z, \tag{8}$$

and [Exercise 4*(a)*]

$$\sin^2 z + \cos^2 z = 1. \tag{9}$$

The periodic character of $\sin z$ and $\cos z$ is also evident:

$$\sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z, \tag{10}$$

$$\cos(z + 2\pi) = \cos z, \quad \cos(z + \pi) = -\cos z. \tag{11}$$

When y is any real number, definitions (1) and the hyperbolic functions

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

from calculus can be used to write

$$\sin(iy) = i \sinh y \quad \text{and} \quad \cos(iy) = \cosh y. \tag{12}$$

Also, the real and imaginary components of sin z and cos z can be displayed in terms of those hyperbolic functions:

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \tag{13}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y, \tag{14}$$

where $z = x + iy$. To obtain expressions (13) and (14), we write

$$z_1 = x \quad \text{and} \quad z_2 = iy$$

in identities (5) and (6) and then refer to relations (12). Observe that once expression (13) is obtained, relation (14) also follows from the fact (Sec. 21) that if the derivative of a function

$$f(z) = u(x, y) + iv(x, y)$$

exists at a point $z = (x, y)$, then

$$f'(z) = u_x(x, y) + iv_x(x, y).$$

Expressions (13) and (14) can be used (Exercise 7) to show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \tag{15}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \tag{16}$$

Inasmuch as $\sinh y$ tends to infinity as y tends to infinity, it is clear from these two equations that $\sin z$ and $\cos z$ are *not bounded* on the complex plane, whereas the absolute values of $\sin x$ and $\cos x$ are less than or equal to unity for all values of x. (See the definition of a bounded function at the end of Sec. 18.)

A *zero* of a given function $f(z)$ is a number z_0 such that $f(z_0) = 0$. Since $\sin z$ becomes the usual sine function in calculus when z is real, we know that the real numbers $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ are all zeros of $\sin z$. To show that *there are no other zeros,* we assume that $\sin z = 0$ and note how it follows from equation (15) that

$$\sin^2 x + \sinh^2 y = 0.$$

This sum of two squares reveals that

$$\sin x = 0 \quad \text{and} \quad \sinh y = 0.$$

Evidently, then, $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ and $y = 0$; that is,

$$\sin z = 0 \quad \text{if and only if} \quad z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots). \tag{17}$$

Since

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

according to the second of identities (8),

$$\cos z = 0 \quad \text{if and only if} \quad z = \frac{\pi}{2} + n\pi \ (n = 0, \pm 1, \pm 2, \ldots). \tag{18}$$

So, as was the case with $\sin z$, the zeros of $\cos z$ are all real.

The other four trigonometric functions are defined in terms of the sine and cosine functions by the expected relations:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \tag{19}$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}. \tag{20}$$

Observe that the quotients $\tan z$ and $\sec z$ are analytic everywhere except at the singularities (Sec. 24)

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which are the zeros of $\cos z$. Likewise, $\cot z$ and $\csc z$ have singularities at the zeros of $\sin z$, namely

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

By differentiating the right-hand sides of equations (19) and (20), we obtain the anticipated differentiation formulas

$$\frac{d}{dz} \tan z = \sec^2 z, \qquad \frac{d}{dz} \cot z = -\csc^2 z, \tag{21}$$

$$\frac{d}{dz} \sec z = \sec z \tan z, \quad \frac{d}{dz} \csc z = -\csc z \cot z. \tag{22}$$

The periodicity of each of the trigonometric functions defined by equations (19) and (20) follows readily from equations (10) and (11). For example,

$$\tan(z + \pi) = \tan z. \tag{23}$$

Mapping properties of the transformation $w = \sin z$ are especially important in the applications later on. A reader who wishes at this time to learn some of those properties is sufficiently prepared to read Sec. 96 (Chap. 8), where they are discussed.

EXERCISES

1. Give details in the derivation of expressions (2), Sec. 34, for the derivatives of $\sin z$ and $\cos z$.

2. *(a)* With the aid of expression (4), Sec. 34, show that

$$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 34, to show how it follows that

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(b) Use the results in part *(a)* and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 34.

3. According to the final result in Exercise 2*(b)*,

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 34.

4. Verify identity (9) in Sec. 34 using

(a) identity (6) and relations (3) in that section;

(b) the lemma in Sec. 27 and the fact that the entire function

$$f(z) = \sin^2 z + \cos^2 z - 1$$

has zero values along the x axis.

5. Use identity (9) in Sec. 34 to show that

(a) $1 + \tan^2 z = \sec^2 z$;   *(b)* $1 + \cot^2 z = \csc^2 z$.

6. Establish differentiation formulas (21) and (22) in Sec. 34.

7. In Sec. 34, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

8. Point out how it follows from expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$ that

(a) $|\sin z| \geq |\sin x|$;   *(b)* $|\cos z| \geq |\cos x|$.

9. With the aid of expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$, show that

(a) $|\sinh y| \le |\sin z| \le \cosh y$;   *(b)* $|\sinh y| \le |\cos z| \le \cosh y$.

10. *(a)* Use definitions (1), Sec. 34, of $\sin z$ and $\cos z$ to show that

$$2\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1.$$

(b) With the aid of the identity obtained in part *(a)*, show that if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

11. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that neither $\sin \overline{z}$ nor $\cos \overline{z}$ is an analytic function of z anywhere.

12. Use the reflection principle (Sec. 28) to show that for all z,

(a) $\overline{\sin z} = \sin \overline{z}$;   *(b)* $\overline{\cos z} = \cos \overline{z}$.

13. With the aid of expressions (13) and (14) in Sec. 34, give direct verifications of the relations obtained in Exercise 12.

14. Show that

(a) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z;

(b) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

15. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and then the imaginary parts of $\sin z$ and $\cosh 4$.

Ans. $\left(\dfrac{\pi}{2} + 2n\pi\right) \pm 4i \qquad (n = 0, \pm 1, \pm 2, \ldots)$.

16. With the aid of expression (14), Sec. 34, show that the roots of the equaion $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then express them in the form

$$z = 2n\pi \pm i\ln(2 + \sqrt{3}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

35. HYPERBOLIC FUNCTIONS

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1}$$

Since e^z and e^{-z} are entire, it follows from definitions (1) that $\sinh z$ and $\cosh z$ are entire. Furthermore,

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z. \tag{2}$$

Because of the way in which the exponential function appears in definitions (1) and in the definitions (Sec. 34)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

of $\sin z$ and $\cos z$, the hyperbolic sine and cosine functions are closely related to those trigonometric functions:

$$-i \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z, \tag{3}$$

$$-i \sin(iz) = \sinh z, \quad \cos(iz) = \cosh z. \tag{4}$$

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \tag{5}$$

$$\cosh^2 z - \sinh^2 z = 1, \tag{6}$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \tag{7}$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \tag{8}$$

and

$$\sinh z = \sinh x \cos y + i \cosh x \sin y, \tag{9}$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y, \tag{10}$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y, \tag{11}$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y, \tag{12}$$

where $z = x + iy$. While these identities follow directly from definitions (1), they are often more easily obtained from related trigonometric identities, with the aid of relations (3) and (4).

EXAMPLE. To illustrate the method of proof just suggested, let us verify identity (11). According to the first of relations (4), $|\sinh z|^2 = |\sin(iz)|^2$. That is,

$$|\sinh z|^2 = |\sin(-y + ix)|^2, \tag{13}$$

where $z = x + iy$. But from equation (15), Sec. 34, we know that

$$|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y\,;$$

and this enables us to write equation (13) in the desired form (11).

In view of the periodicity of $\sin z$ and $\cos z$, it follows immediately from relaions (4) that $\sinh z$ and $\cosh z$ are periodic with period $2\pi i$. Relations (4), together with statements (17) and (18) in Sec. 34, also tell us that

$$\sinh z = 0 \quad \text{if and only if} \quad z = n\pi i \;\; (n = 0, \pm 1, \pm 2, \ldots) \tag{14}$$

and

$$\cosh z = 0 \quad \text{if and only if} \quad z = \left(\frac{\pi}{2} + n\pi\right)i \;\; (n = 0, \pm 1, \pm 2, \ldots). \tag{15}$$

The hyperbolic tangent of z is defined by means of the equation

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{16}$$

and is analytic in every domain in which $\cosh z \neq 0$. The functions $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are the reciprocals of $\tanh z$, $\cosh z$, and $\sinh z$, respectively. It is straightforward to verify the following differentiation formulas, which are the same as those established in calculus for the corresponding functions of a real variable:

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z, \tag{17}$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z. \tag{18}$$

EXERCISES

1. Verify that the derivatives of $\sinh z$ and $\cosh z$ are as stated in equations (2), Sec. 35.

2. Prove that $\sinh 2z = 2 \sinh z \cosh z$ by starting with

(a) definitions (1), Sec. 35, of $\sinh z$ and $\cosh z$;

(b) the identity $\sin 2z = 2 \sin z \cos z$ (Sec. 34) and using relations (3) in Sec. 35.

3. Show how identities (6) and (8) in Sec. 35 follow from identities (9) and (6), respectively, in Sec. 34.

4. Write $\sinh z = \sinh(x + iy)$ and $\cosh z = \cosh(x + iy)$, and then show how expressions (9) and (10) in Sec. 35 follow from identities (7) and (8), respectively, in that section.

5. Verify expression (12), Sec. 35, for $|\cosh z|^2$.

6. Show that $|\sinh x| \le |\cosh z| \le \cosh x$ by using

(a) identity (12), Sec. 35;

(b) the inequalities $|\sinh y| \le |\cos z| \le \cosh y$, obtained in Exercise 9*(b)*, Sec. 34.

7. Show that

(a) $\sinh(z + \pi i) = -\sinh z$; *(b)* $\cosh(z + \pi i) = \cosh z$;
(c) $\tanh(z + \pi i) = \tanh z$.

8. Give details showing that the zeros of $\sinh z$ and $\cosh z$ are as in statements (14) and (l5), Sec. 35.

9. Using the results proved in Exercise 8, locate all zeros and singularities of the hyperbolic tangent function.

10. Derive differentiation formulas (17), Sec. 35.

11. Use the reflection principle (Sec. 28) to show that for all z,

(a) $\overline{\sinh z} = \sinh \overline{z}$; *(b)* $\overline{\cosh z} = \cosh \overline{z}$.

12. Use the results in Exercise 11 to show that $\overline{\tanh z} = \tanh \overline{z}$ at points where $\cosh z \neq 0$.

13. By accepting that the stated identity is valid when z is replaced by the real variable x and using the lemma in Sec. 27, verify that

(a) $\cosh^2 z - \sinh^2 z = 1$; *(b)* $\sinh z + \cosh z = e^z$.

[Compare with Exercise 4*(b)*, Sec. 34.]

14. Why is the function $\sinh(e^z)$ entire? Write its real component as a function of x and y, and state why that function must be harmonic everywhere.

15. By using one of the identities (9) and (10) in Sec. 35 and then proceeding as in Exercise 15, Sec. 34, find all roots of the equation

(a) $\sinh z = i$; *(b)* $\cosh z = \dfrac{1}{2}$.

Ans. (a) $z = \left(2n + \dfrac{1}{2}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $z = \left(2n \pm \dfrac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$.

16. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 34.)

Ans. $z = \pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

36. INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms.

In order to define the inverse sine function $\sin^{-1} z$, we write

$$w = \sin^{-1} z \quad \text{when} \quad z = \sin w.$$

That is, $w = \sin^{-1} z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

If we put this equation in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

which is quadratic in e^{iw}, and solve for e^{iw} [see Exercise 8*(a)*, Sec. 10], we find that

$$e^{iw} = iz + (1 - z^2)^{1/2} \tag{1}$$

where $(1 - z^2)^{1/2}$ is, of course, a double-valued function of z. Taking logarithms of each side of equation (1) and recalling that $w = \sin^{-1} z$, we arrive at the expression

$$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]. \tag{2}$$

The following example emphasizes the fact that $\sin^{-1} z$ is a multiple-valued function, with infinitely many values at each point z.

EXAMPLE. Expression (2) tells us that

$$\sin^{-1}(-i) = -i \log(1 \pm \sqrt{2}).$$

But

$$\log(1 + \sqrt{2}) = \ln(1 + \sqrt{2}) + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(1 - \sqrt{2}) = \ln(\sqrt{2} - 1) + (2n + 1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since

$$\ln(\sqrt{2} - 1) = \ln \frac{1}{1 + \sqrt{2}} = -\ln(1 + \sqrt{2}),$$

then, the numbers

$$(-1)^n \ln(1 + \sqrt{2}) + n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

constitute the set of values of $\log(1 \pm \sqrt{2})$. Thus, in rectangular form,

$$\sin^{-1}(-i) = n\pi + i(-1)^{n+1} \ln(1 + \sqrt{2}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

One can apply the technique used to derive expression (2) for $\sin^{-1} z$ to show that

$$\cos^{-1} z = -i \log\left[z + i(1 - z^2)^{1/2}\right] \tag{3}$$

and that

$$\tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}. \tag{4}$$

The functions $\cos^{-1} z$ and $\tan^{-1} z$ are also multiple-valued. When specific branches of the square root and logarithmic functions are used, all three inverse functions become single-valued and analytic because they are then compositions of analytic functions.

The derivatives of these three functions are readily obtained from their logarithmic expressions. The derivatives of the first two depend on the values chosen for the square roots:

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \tag{5}$$

$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}. \tag{6}$$

The derivative of the last one,

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}, \tag{7}$$

does not, however, depend on the manner in which the function is made single-valued.

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

$$\sinh^{-1} z = \log\left[z + (z^2+1)^{1/2}\right], \tag{8}$$

$$\cosh^{-1} z = \log\left[z + (z^2-1)^{1/2}\right], \tag{9}$$

and

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}. \tag{10}$$

Finally, we remark that common alternative notation for all of these inverse functions is arcsin z, etc.

EXERCISES

1. Find all the values of

(a) $\tan^{-1}(2i)$; *(b)* $\tan^{-1}(1+i)$; *(c)* $\cosh^{-1}(-1)$; *(d)* $\tanh^{-1} 0$.

Ans. *(a)* $\left(n+\frac{1}{2}\right)\pi + \frac{i}{2}\ln 3 \ (n = 0, \pm 1, \pm 2, \ldots)$;

(d) $n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

2. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and then imaginary parts in that equation;

(b) using expression (2), Sec. 36, for $\sin^{-1} z$.

Ans. $z = \left(2n+\frac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Solve the equation $\cos z = \sqrt{2}$ for z.

4. Derive formula (5), Sec. 36, for the derivative of $\sin^{-1} z$.

5. Derive expression (4), Sec. 36, for $\tan^{-1} z$.

6. Derive formula (7), Sec. 36, for the derivative of $\tan^{-1} z$.

7. Derive expression (9), Sec. 36, for $\cosh^{-1} z$.