By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(b) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part *(a)*, to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the *complex form* $\partial f/\partial \overline{z} = 0$ *of the Cauchy–Riemann equations.*

24. ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is *analytic at a point* z_0 if it has a derivative at each point in some neighborhood of z_0.* It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is *analytic in an open set* if it has a derivative everywhere in that set. If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

Note that the function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 3, Sec. 19.)

An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a *singular point*, or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhere analytic.

A necessary, but by no means sufficient, condition for a function f to be analytic in a domain D is clearly the continuity of f throughout D. Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems in Secs. 22 and 23.

Other useful sufficient conditions are obtained from the differentiation formulas in Sec. 20. The derivatives of the sum and product of two functions exist wherever

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*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.

the functions themselves have derivatives. Thus, *if two functions are analytic in a domain D, their sum and their product are both analytic in D*. Similarly, *their quotient is analytic in D provided the function in the denominator does not vanish at any point in D*. In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

From the chain rule for the derivative of a composite function, we find that *a composition of two analytic functions is analytic*. More precisely, suppose that a function $f(z)$ is analytic in a domain D and that the image (Sec. 13) of D under the transformation $w = f(z)$ is contained in the domain of definition of a function $g(w)$. Then the composition $g[f(z)]$ is analytic in D, with derivative

$$\frac{d}{dz}g[f(z)] = g'[f(z)]f'(z).$$

The following property of analytic functions is especially useful, in addition to being expected.

Theorem. *If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.*

We start the proof by writing $f(z) = u(x, y) + iv(x, y)$. Assuming that $f'(z) = 0$ in D, we note that $u_x + iv_x = 0$; and, in view of the Cauchy–Riemann equations, $v_y - iu_y = 0$. Consequently,

$$u_x = u_y = 0 \quad \text{and} \quad v_x = v_y = 0$$

at each point in D.

Next, we show that $u(x, y)$ is constant along any line segment L extending from a point P to a point P' and lying entirely in D. We let s denote the distance along L from the point P and let $\mathbf{U}$ denote the unit vector along L in the direction of increasing s (see Fig. 30). We know from calculus that the directional derivative du/ds can be written as the dot product

$$\frac{du}{ds} = (\text{grad } u) \cdot \mathbf{U}, \tag{1}$$

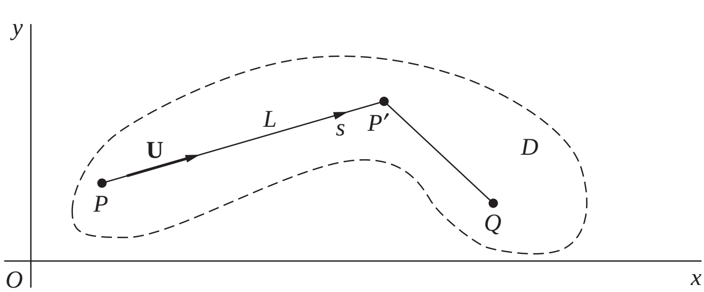

FIGURE 30

where grad u is the gradient vector

$$\text{grad}\, u = u_x \mathbf{i} + u_y \mathbf{j}. \tag{2}$$

Because u_x and u_y are zero everywhere in D, grad u is evidently the zero vector at all points on L. Hence it follows from equation (1) that the derivative du/ds is zero along L; and this means that u is constant on L.

Finally, since there is always a finite number of such line segments, joined end to end, connecting any two points P and Q in D (Sec. 11), the values of u at P and Q must be the same. We may conclude, then, that there is a real constant a such that $u(x, y) = a$ throughout D. Similarly, $v(x, y) = b$; and we find that $f(z) = a + bi$ at each point in D.

25. EXAMPLES

As pointed out in Sec. 24, it is often possible to determine where a given function is analytic by simply recalling various differentiation formulas in Sec. 20.

EXAMPLE 1. The quotient

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is evidently analytic throughout the z plane except for the singular points $z = \pm\sqrt{3}$ and $z = \pm i$. The analyticity is due to the existence of familiar differentiation formulas, which need to be applied only if the expression for $f'(z)$ is wanted.

When a function is given in terms of its component functions $u(x, y)$ and $v(x, y)$, its analyticity can be demonstrated by direct application of the Cauchy–Riemann equations.

EXAMPLE 2. If

$$f(z) = \cosh x \cos y + i \sinh x \sin y,$$

the component functions are

$$u(x, y) = \cosh x \cos y \quad \text{and} \quad v(x, y) = \sinh x \sin y.$$

Because

$$u_x = \sinh x \cos y = v_y \quad \text{and} \quad u_y = -\cosh x \sin y = -v_x$$

everywhere, it is clear from the theorem in Sec. 22 that f is entire.

Finally, we illustrate how the theorem in Sec. 24 can be used to obtain other properties of analytic functions.

EXAMPLE 3. Suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

and its conjugate

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

are *both* analytic in a given domain D. It is now easy to show that $f(z)$ must be constant throughout D.

To do this, we write $\overline{f(z)}$ as

$$\overline{f(z)} = U(x, y) + iV(x, y)$$

where

$$U(x, y) = u(x, y) \quad \text{and} \quad V(x, y) = -v(x, y). \tag{1}$$

Because of the analyticity of $f(z)$, the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{2}$$

hold in D; and the analyticity of $\overline{f(z)}$ in D tells us that

$$U_x = V_y, \quad U_y = -V_x. \tag{3}$$

In view of relations (1), equations (3) can also be written

$$u_x = -v_y, \quad u_y = v_x. \tag{4}$$

By adding corresponding sides of the first of equations (2) and (4), we find that $u_x = 0$ in D. Similarly, subtraction involving corresponding sides of the second of equations (2) and (4) reveals that $v_x = 0$. According to expression (8) in Sec. 21, then,

$$f'(z) = u_x + iv_x = 0 + i0 = 0;$$

and it follows from the theorem in Sec. 24 that $f(z)$ is constant throughout D.

EXAMPLE 4. As in Example 3, we consider a function f that is analytic throughout a given domain D. Assuming further that the modulus $|f(z)|$ is constant throughout D, one can prove that $f(z)$ must be constant there too. This result is needed to obtain an important result later on in Chap. 4 (Sec. 54).

The proof is accomplished by writing

$$|f(z)| = c \quad \text{for all } z \text{ in } D, \tag{5}$$

where c is a real constant. If $c = 0$, it follows that $f(z) = 0$ everywhere in D. If $c \neq 0$, the fact that (see Sec. 5)

$$f(z)\overline{f(z)} = c^2$$

tells us that $f(z)$ is never zero in D. Hence

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{for all } z \text{ in } D,$$

and it follows from this that $\overline{f(z)}$ is analytic everywhere in D. The main result in Example 3 just above thus ensures that $f(z)$ is constant throughout D.

EXERCISES

1. Apply the theorem in Sec. 22 to verify that each of these functions is entire:

(a) $f(z) = 3x + y + i(3y - x)$; *(b)* $f(z) = \sin x \cosh y + i \cos x \sinh y$;

(c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; *(d)* $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

(a) $f(z) = xy + iy$; *(b)* $f(z) = 2xy + i(x^2 - y^2)$; *(c)* $f(z) = e^y e^{ix}$.

3. State why a composition of two entire functions is entire. Also, state why any *linear combination* $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

(a) $f(z) = \dfrac{2z + 1}{z(z^2 + 1)}$; *(b)* $f(z) = \dfrac{z^3 + i}{z^2 - 3z + 2}$; *(c)* $f(z) = \dfrac{z^2 + 1}{(z + 2)(z^2 + 2z + 2)}$.

Ans. (a) $z = 0, \pm i$; *(b)* $z = 1, 2$; *(c)* $z = -2, -1 \pm i$.

5. According to Exercise 4*(b)*, Sec. 23, the function

$$g(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is analytic in its domain of definition, with derivative

$$g'(z) = \frac{1}{2\,g(z)}.$$

Show that the composite function $G(z) = g(2z - 2 + i)$ is analytic in the half plane $x > 1$, with derivative

$$G'(z) = \frac{1}{g(2z - 2 + i)}.$$

Suggestion: Observe that $\text{Re}(2z - 2 + i) > 0$ when $x > 1$.

6. Use results in Sec. 23 to verify that the function

$$g(z) = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = 1/z$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0, y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0, y > 0$.

7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughtout D.

26. HARMONIC FUNCTIONS

A real-valued function H of two real variables x and y is said to be *harmonic* in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0, \tag{1}$$

known as *Laplace's equation.*

Harmonic functions play an important role in applied mathematics. For example, the temperatures $T(x, y)$ in thin plates lying in the xy plane are often harmonic. A function $V(x, y)$ is harmonic when it denotes an electrostatic potential that varies only with x and y in the interior of a region of three-dimensional space that is free of charges.

EXAMPLE 1. It is easy to verify that the function $T(x, y) = e^{-y} \sin x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip $0 < x < \pi$, $y > 0$. It also assumes the values on the edges of the strip that are indicated in Fig. 31. More precisely, it satisfies all of the conditions

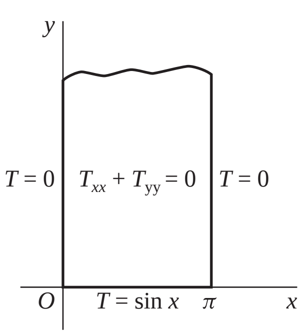

FIGURE 31

$$T_{xx}(x, y) + T_{yy}(x, y) = 0,$$

$$T(0, y) = 0, \quad T(\pi, y) = 0,$$

$$T(x, 0) = \sin x, \quad \lim_{y \to \infty} T(x, y) = 0,$$

which describe steady temperatures $T(x, y)$ in a thin homogeneous plate in the xy plane that has no heat sources or sinks and is insulated except for the stated conditions along the edges.

The use of the theory of functions of a complex variable in discovering solutions, such as the one in Example 1, of temperature and other problems is described in considerable detail later on in Chap. 10 and in parts of chapters following it.* That theory is based on the theorem below, which provides a source of harmonic functions.

Theorem 1. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

To show this, we need a result that is to be proved in Chap. 4 (Sec. 52). Namely, if a function of a complex variable is analytic at a point, then its real and imaginary components have continuous partial derivatives of all orders at that point.

Assuming that f is analytic in D, we start with the observation that the first-order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations throughout D:

$$u_x = v_y, \quad u_y = -v_x. \tag{2}$$

Differentiating both sides of these equations with respect to x, we have

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}. \tag{3}$$

Likewise, differentiation with respect to y yields

$$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}. \tag{4}$$

Now, by a theorem in advanced calculus,† the continuity of the partial derivatives of u and v ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. It then follows from equations (3) and (4) that

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.$$

That is, u and v are harmonic in D.

*Another important method is developed in the authors' "Fourier Series and Boundary Value Problems," 7th ed., 2008.

†See, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 199–201, 1983.

EXAMPLE 2. The function $f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire, as is shown in Exercise 1 *(c)*, Sec. 25. Hence its real component, which is the temperature function $T(x, y) = e^{-y}\sin x$ in Example 1, must be harmonic in every domain of the xy plane.

EXAMPLE 3. Since the function $f(z) = i/z^2$ is analytic whenever $z \neq 0$ and since

$$\frac{i}{z^2} = \frac{i}{z^2}\cdot\frac{\overline{z}^2}{\overline{z}^2} = \frac{i\overline{z}^2}{(z\overline{z})^2} = \frac{i\overline{z}^2}{|z|^4} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2},$$

the two functions

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2} \quad \text{and} \quad v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations (2) throughout D, then v is said to be a *harmonic conjugate* of u. The meaning of the word conjugate here is, of course, different from that in Sec. 5, where $\overline{z}$ is defined.

Theorem 2. *A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.*

The proof is easy. If v is a harmonic conjugate of u in D, the theorem in Sec. 22 tells us that f is analytic in D. Conversely, if f is analytic in D, we know from Theorem 1 that u and v are harmonic in D; furthermore, in view of the theorem in Sec. 21, the Cauchy–Riemann equations are satisfied.

The following example shows that if v is a harmonic conjugate of u in some domain, it is *not*, in general, true that u is a harmonic conjugate of v there. (See also Exercises 3 and 4.)

EXAMPLE 4. Suppose that

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Since these are the real and imaginary components, respectively, of the entire function $f(z) = z^2$, we know that v is a harmonic conjugate of u throughout the plane. But u cannot be a harmonic conjugate of v since, as verified in Exercise 2*(b)*, Sec. 25, the function $2xy + i(x^2 - y^2)$ is not analytic anywhere.

In Chap. 9 (Sec. 104) we shall show that a function u which is harmonic in a domain of a certain type always has a harmonic conjugate. Thus, in such domains, every harmonic function is the real part of an analytic function. It is also true (Exercise 2) that a harmonic conjugate, when it exists, is unique except for an additive constant.

EXAMPLE 5. We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function

$$u(x, y) = y^3 - 3x^2y \tag{5}$$

is readily seen to be harmonic throughout the entire xy plane. Since a harmonic conjugate $v(x, y)$ is related to $u(x, y)$ by means of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x, \tag{6}$$

the first of these equations tells us that

$$v_y(x, y) = -6xy.$$

Holding x fixed and integrating each side here with respect to y, we find that

$$v(x, y) = -3xy^2 + \phi(x) \tag{7}$$

where ϕ is, at present, an arbitrary function of x. Using the second of equations (6), we have

$$3y^2 - 3x^2 = 3y^2 - \phi'(x),$$

or $\phi'(x) = 3x^2$. Thus $\phi(x) = x^3 + C$, where C is an arbitrary real number. According to equation (7), then, the function

$$v(x, y) = -3xy^2 + x^3 + C \tag{8}$$

is a harmonic conjugate of $u(x, y)$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + C). \tag{9}$$

The form $f(z) = i(z^3 + C)$ of this function is easily verified and is suggested by noting that when $y = 0$, expression (9) becomes $f(x) = i(x^3 + C)$.

EXERCISES

1. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

(a) $u(x, y) = 2x(1 - y)$; *(b)* $u(x, y) = 2x - x^3 + 3xy^2$;
(c) $u(x, y) = \sinh x \sin y$; *(d)* $u(x, y) = y/(x^2 + y^2)$.

Ans. (a) $v(x, y) = x^2 - y^2 + 2y$; *(b)* $v(x, y) = 2y - 3x^2y + y^3$;
(c) $v(x, y) = -\cosh x \cos y$; *(d)* $v(x, y) = x/(x^2 + y^2)$.

2. Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.