7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \quad \lim_{z \to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z \to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how the first of inequalities (9), Sec. 4, enables one to write

$$||f(z)| - |w_0|| \le |f(z) - w_0|.$$

8. Write $\Delta z = z - z_0$ and show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0.$$

9. Show that

$$\lim_{z \to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0$$

and if there exists a positive number M such that $|g(z)| \le M$ for all z in some neighborhood of z_0.

10. Use the theorem in Sec. 17 to show that

(a) $\displaystyle\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4$; *(b)* $\displaystyle\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$; *(c)* $\displaystyle\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$.

11. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \ne 0),$$

(a) $\displaystyle\lim_{z \to \infty} T(z) = \infty$ if $c = 0$;

(b) $\displaystyle\lim_{z \to \infty} T(z) = \frac{a}{c}$ and $\displaystyle\lim_{z \to -d/c} T(z) = \infty$ if $c \ne 0$.

12. State why limits involving the point at infinity are unique.

13. Show that a set S is unbounded (Sec. 11) if and only if every neighborhood of the point at infinity contains at least one point in S.

19. DERIVATIVES

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The *derivative* of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and the function f is said to be *differentiable* at z_0 when $f'(z_0)$ exists.

By expressing the variable z in definition (1) in terms of the new complex variable

$$\Delta z = z - z_0 \quad (z \ne z_0),$$

one can write that definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2}$$

Because f is defined throughout a neighborhood of z_0, the number $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small (Fig. 28).

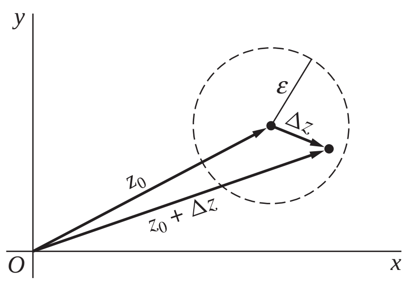

FIGURE 28

When taking form (2) of the definition of derivative, we often drop the subscript on z_0 and introduce the number

$$\Delta w = f(z + \Delta z) - f(z),$$

which denotes the change in the value $w = f(z)$ of f corresponding to a change Δz in the point at which f is evaluated. Then, if we write dw/dz for $f'(z)$, equation (2) becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{3}$$

EXAMPLE 1. Suppose that $f(z) = z^2$. At any point z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

since $2z + \Delta z$ is a polynomial in Δz. Hence $dw/dz = 2z$, or $f'(z) = 2z$.

EXAMPLE 2. If $f(z) = \overline{z}$, then

$$\frac{\Delta w}{\Delta z} = \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}. \tag{4}$$

If the limit of $\Delta w/\Delta z$ exists, it can be found by letting the point $\Delta z = (\Delta x, \Delta y)$ approach the origin $(0, 0)$ in the Δz plane in any manner. In particular, as Δz approaches $(0, 0)$ horizontally through the points $(\Delta x, 0)$ on the real axis (Fig. 29),

$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z.$$

In that case, expression (4) tells us that

$$\frac{\Delta w}{\Delta z} = \frac{\Delta z}{\Delta z} = 1.$$

Hence if the limit of $\Delta w/\Delta z$ exists, its value must be unity. However, when Δz approaches $(0, 0)$ vertically through the points $(0, \Delta y)$ on the imaginary axis, so that

$$\overline{\Delta z} = \overline{0 + i\,\Delta y} = 0 - i\,\Delta y = -(0 + i\,\Delta y) = -\Delta z,$$

we find from expression (4) that

$$\frac{\Delta w}{\Delta z} = \frac{-\Delta z}{\Delta z} = -1.$$

Hence the limit must be -1 if it exists. Since limits are unique (Sec. 15), it follows that dw/dz does not exist anywhere.

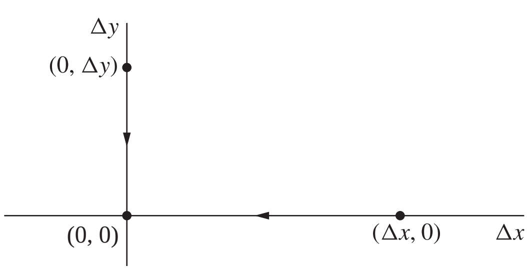

FIGURE 29

EXAMPLE 3. Consider the real-valued function $f(z) = |z|^2$. Here

$$(5) \qquad \frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} = \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}.$$

Proceeding as in Example 2, where horizontal and vertical approaches of Δz toward the origin gave us

$$\overline{\Delta z} = \Delta z \quad \text{and} \quad \overline{\Delta z} = -\Delta z,$$

respectively, we have the expressions

$$\frac{\Delta w}{\Delta z} = \overline{z} + \Delta z + z \quad \text{when} \quad \Delta z = (\Delta x, 0)$$

and

$$\frac{\Delta w}{\Delta z} = \overline{z} - \Delta z - z \quad \text{when} \quad \Delta z = (0, \Delta y).$$

Hence if the limit of $\Delta w/\Delta z$ exists as Δz tends to zero, the uniqueness of limits, used in Example 2, tells us that

$$\overline{z} + z = \overline{z} - z,$$

or $z = 0$. Evidently, then dw/dz cannot exist when $z \neq 0$.

To show that dw/dz does, in fact, exist at $z = 0$, we need only observe that expression (5) reduces to

$$\frac{\Delta w}{\Delta z} = \overline{\Delta z}$$

when $z = 0$. We conclude, therefore, that dw/dz exists *only* at $z = 0$, its value there being 0.

Example 3 shows that a function $f(z) = u(x, y) + iv(x, y)$ can be differentiable at a point $z = (x, y)$ but nowhere else in any neighborhood of that point. Since

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0 \tag{6}$$

when $f(z) = |z|^2$, it also shows that the real and imaginary components of a function of a complex variable can have continuous partial derivatives of all orders at a point $z = (x, y)$ and yet the function may not be differentiable there.

The function $f(z) = |z|^2$ is continuous at each point in the plane since its components (6) are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there. It is, however, true that *the existence of the derivative of a function at a point implies the continuity of the function at that point.* To see this, we assume that $f'(z_0)$ exists and write

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0,$$

from which it follows that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This is the statement of continuity of f at z_0 (Sec. 18).

Geometric interpretations of derivatives of functions of a complex variable are not as immediate as they are for derivatives of functions of a real variable. We defer the development of such interpretations until Chap. 9.

20. DIFFERENTIATION FORMULAS

The definition of derivative in Sec. 19 is identical in form to that of the derivative of a real-valued function of a real variable. In fact, the basic differentiation formulas given below can be derived from the definition in Sec. 19 by essentially the same steps as the ones used in calculus. In these formulas, the derivative of a function f at a point z is denoted by either

$$\frac{d}{dz}f(z) \quad \text{or} \quad f'(z),$$

depending on which notation is more convenient.

Let c be a complex constant, and let f be a function whose derivative exists at a point z. It is easy to show that

$$\frac{d}{dz}c = 0, \quad \frac{d}{dz}z = 1, \quad \frac{d}{dz}[cf(z)] = cf'(z). \tag{1}$$

Also, if n is a positive integer,

$$\frac{d}{dz}z^n = nz^{n-1}. \tag{2}$$

This formula remains valid when n is a negative integer, provided that $z \neq 0$.

If the derivatives of two functions f and g exist at a point z, then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z), \tag{3}$$

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)\,; \tag{4}$$

and, when $g(z) \neq 0$,

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}. \tag{5}$$

Let us derive formula (4). To do this, we write the following expression for the change in the product $w = f(z)g(z)$:

$$\begin{aligned}\Delta w &= f(z + \Delta z)g(z + \Delta z) - f(z)g(z)\\ &= f(z)[g(z + \Delta z) - g(z)] + [f(z + \Delta z) - f(z)]g(z + \Delta z).\end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = f(z)\,\frac{g(z + \Delta z) - g(z)}{\Delta z} + \frac{f(z + \Delta z) - f(z)}{\Delta z}\,g(z + \Delta z)\,;$$

and, letting Δz tend to zero, we arrive at the desired formula for the derivative of $f(z)g(z)$. Here we have used the fact that g is continuous at the point z, since

$g'(z)$ exists; thus $g(z+\Delta z)$ tends to $g(z)$ as Δz tends to zero (see Exercise 8, Sec. 18).

There is also a chain rule for differentiating composite functions. Suppose that f has a derivative at z_0 and that g has a derivative at the point $f(z_0)$. Then the function $F(z) = g[f(z)]$ has a derivative at z_0, and

$$F'(z_0) = g'[f(z_0)]f'(z_0). \tag{6}$$

If we write $w = f(z)$ and $W = g(w)$, so that $W = F(z)$, the chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}.$$

EXAMPLE. To find the derivative of $(2z^2+i)^5$, write $w = 2z^2 + i$ and $W = w^5$. Then

$$\frac{d}{dz}(2z^2+i)^5 = 5w^4 4z = 20z(2z^2+i)^4.$$

To start the derivation of formula (6), choose a specific point z_0 at which $f'(z_0)$ exists. Write $w_0 = f(z_0)$ and also assume that $g'(w_0)$ exists. There is, then, some ε neighborhood $|w - w_0| < \varepsilon$ of w_0 such that for all points w in that neighborhood, we can define a function Φ having the values $\Phi(w_0) = 0$ and

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \quad \text{when} \quad w \neq w_0. \tag{7}$$

Note that in view of the definition of derivative,

$$\lim_{w \to w_0} \Phi(w) = 0. \tag{8}$$

Hence Φ is continuous at w_0.

Now expression (7) can be put in the form

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \qquad (|w - w_0| < \varepsilon), \tag{9}$$

which is valid even when $w = w_0$; and since $f'(z_0)$ exists and f is therefore continuous at z_0, we can choose a positive number δ such that the point $f(z)$ lies in the ε neighborhood $|w - w_0| < \varepsilon$ of w_0 if z lies in the δ neighborhood $|z - z_0| < \delta$ of z_0. Thus it is legitimate to replace the variable w in equation (9) by $f(z)$ when z is any point in the neighborhood $|z - z_0| < \delta$. With that substitution, and with $w_0 = f(z_0)$, equation (9) becomes

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\}\frac{f(z) - f(z_0)}{z - z_0} \tag{10}$$

$$(0 < |z - z_0| < \delta),$$

where we must stipulate that $z \neq z_0$ so that we are not dividing by zero. As already noted, f is continuous at z_0 and Φ is continuous at the point $w_0 = f(z_0)$. Hence the composition $\Phi[f(z)]$ is continuous at z_0; and since $\Phi(w_0) = 0$,

$$\lim_{z \to z_0} \Phi[f(z)] = 0.$$

So equation (10) becomes equation (6) in the limit as z approaches z_0.

EXERCISES

1. Use results in Sec. 20 to find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$; *(b)* $f(z) = (1 - 4z^2)^3$;

(c) $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq -1/2)$; *(d)* $f(z) = \dfrac{(1+z^2)^4}{z^2}$ $(z \neq 0)$.

2. Using results in Sec. 20, show that

(a) a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$ is differentiable everywhere, with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1};$$

(b) the coefficients in the polynomial $P(z)$ in part *(a)* can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

3. Apply definition (3), Sec. 19, of derivative to give a direct proof that

$$\frac{dw}{dz} = -\frac{1}{z^2} \quad \text{when} \quad w = \frac{1}{z} \qquad (z \neq 0).$$

4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

5. Derive formula (3), Sec. 20, for the derivative of the sum of two functions.

6. Derive expression (2), Sec. 20, for the derivative of z^n when n is a positive integer by using

(a) mathematical induction and formula (4), Sec. 20, for the derivative of the product of two functions;

(b) definition (3), Sec. 19, of derivative and the binomial formula (Sec. 3).

7. Prove that expression (2), Sec. 20, for the derivative of z^n remains valid when n is a negative integer $(n = -1, -2, \ldots)$, provided that $z \neq 0$.

Suggestion: Write $m = -n$ and use the formula for the derivative of a quotient of two functions.

8. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when
(a) $f(z) = \operatorname{Re} z$; *(b)* $f(z) = \operatorname{Im} z$.

9. Let f denote the function whose values are

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $\Delta x\ \Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane. (Compare with Example 2, Sec. 19.)

21. CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function

$$f(z) = u(x, y) + iv(x, y) \tag{1}$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also show how to express $f'(z_0)$ in terms of those partial derivatives.

We start by writing

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta x + i\Delta y,$$

and

$$\begin{aligned} \Delta w &= f(z_0 + \Delta z) - f(z_0) \\ &= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]. \end{aligned}$$

Assuming that the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{2}$$

exists, we know from Theorem 1 in Sec. 16 that

$$f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Re} \frac{\Delta w}{\Delta z} \right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Im} \frac{\Delta w}{\Delta z} \right). \tag{3}$$

Now it is important to keep in mind that expression (3) is valid as $(\Delta x, \Delta y)$ tends to $(0, 0)$ in any manner that we may choose. In particular, we let $(\Delta x, \Delta y)$ tend to $(0, 0)$ horizontally through the points $(\Delta x, 0)$, as indicated in Fig. 29 (Sec. 19). Inasmuch as $\Delta y = 0$, the quotient $\Delta w/\Delta z$ becomes

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}.$$

Thus

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Re}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Im}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = v_x(x_0, y_0),$$

where $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ denote the first-order partial derivatives with respect to x of the functions u and v, respectively, at (x_0, y_0). Substitution of these limits into expression (3) tells us that

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0). \tag{4}$$

We might have let Δz tend to zero vertically through the points $(0, \Delta y)$. In that case, $\Delta x = 0$ and

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\ &= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}.\end{aligned}$$

Evidently, then,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Re}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} = v_y(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Im}\frac{\Delta w}{\Delta z}\right) = -\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} = -u_y(x_0, y_0).$$

Hence it follows from expression (3) that

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0), \tag{5}$$

where the partial derivatives of u and v are, this time, with respect to y. Note that equation (5) can also be written in the form

$$f'(z_0) = -i[u_y(x_0, y_0) + iv_y(x_0, y_0)].$$

Equations (4) and (5) not only give $f'(z_0)$ in terms of partial derivatives of the component functions u and v, but they also provide necessary conditions for the existence of $f'(z_0)$. To obtain those conditions, we need only equate the real parts and then the imaginary parts on the right-hand sides of equations (4) and (5) to see that the existence of $f'(z_0)$ requires that

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{6}$$

Equations (6) are the *Cauchy–Riemann equations*, so named in honor of the French mathematician A. L. Cauchy (1789–1857), who discovered and used them, and in honor of the German mathematician G. F. B. Riemann (1826–1866), who made them fundamental in his development of the theory of functions of a complex variable.

We summarize the above results as follows.

Theorem. *Suppose that*

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{7}$$

there. Also, $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x, \tag{8}$$

where these partial derivatives are to be evaluated at (x_0, y_0).

EXAMPLE 1. In Example 1, Sec. 19, we showed that the function

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

is differentiable everywhere and that $f'(z) = 2z$. To verify that the Cauchy–Riemann equations are satisfied everywhere, write

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Thus

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x.$$

Moreover, according to equation (8),

$$f'(z) = 2x + i2y = 2(x + iy) = 2z.$$

Since the Cauchy–Riemann equations are necessary conditions for the existence of the derivative of a function f at a point z_0, they can often be used to locate points at which f does *not* have a derivative.

EXAMPLE 2. When $f(z) = |z|^2$, we have

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0.$$

If the Cauchy–Riemann equations are to hold at a point (x, y), it follows that $2x = 0$ and $2y = 0$, or that $x = y = 0$. Consequently, $f'(z)$ does not exist at any nonzero point, as we already know from Example 3 in Sec. 19. Note that the theorem just proved does not ensure the existence of $f'(0)$. The theorem in the next section will, however, do this.

22. SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

Satisfaction of the Cauchy–Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. (See Exercise 6, Sec. 23.) But, with certain continuity conditions, we have the following useful theorem.

Theorem. *Let the function*

$$f(z) = u(x, y) + iv(x, y)$$

be defined throughout some ε *neighborhood of a point* $z_0 = x_0 + iy_0$, *and suppose that*

(a) the first-order partial derivatives of the functions u *and* v *with respect to* x *and* y *exist everywhere in the neighborhood;*

(b) those partial derivatives are continuous at (x_0, y_0) *and satisfy the Cauchy–Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0).

Then $f'(z_0)$ *exists, its value being*

$$f'(z_0) = u_x + iv_x$$

where the right-hand side is to be evaluated at (x_0, y_0).

To prove the theorem, we assume that conditions *(a)* and *(b)* in its hypothesis are satisfied and write $\Delta z = \Delta x + i\Delta y$, where $0 < |\Delta z| < \varepsilon$, as well as

$$\Delta w = f(z_0 + \Delta z) - f(z_0).$$

Thus

(1) $$\Delta w = \Delta u + i\,\Delta v,$$

where

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

and

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

The assumption that the first-order partial derivatives of u and v are continuous at the point (x_0, y_0) enables us to write*

(2) $$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

and

(3) $$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 tend to zero as $(\Delta x, \Delta y)$ approaches (0, 0) in the Δz plane. Substitution of expressions (2) and (3) into equation (1) now tells us that

(4) $$\begin{aligned}\Delta w = {} & u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ & + i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y].\end{aligned}$$

Because the Cauchy–Riemann equations are assumed to be satisfied at (x_0, y_0), one can replace $u_y(x_0, y_0)$ by $-v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ by $u_x(x_0, y_0)$ in equation (4) and then divide through by the quantity $\Delta z = \Delta x + i\,\Delta y$ to get

(5) $$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + i v_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}.$$

But $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$, according to inequalities (3) in Sec. 4, and so

$$\left|\frac{\Delta x}{\Delta z}\right| \le 1 \quad \text{and} \quad \left|\frac{\Delta y}{\Delta z}\right| \le 1.$$

Consequently,

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_1 + i\varepsilon_3| \le |\varepsilon_1| + |\varepsilon_3|$$

and

$$\left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}\right| \le |\varepsilon_2 + i\varepsilon_4| \le |\varepsilon_2| + |\varepsilon_4|;$$

*See, for instance, W. Kaplan, “Advanced Calculus,” 5th ed., pp. 86ff, 2003.

and this means that the last two terms on the right in equation (5) tend to zero as the variable $\Delta z = \Delta x + i\Delta y$ approaches zero. The expression for $f'(z_0)$ in the statement of the theorem is now established.

EXAMPLE 1. Consider the exponential function

$$f(z) = e^z = e^x e^{iy} \quad (z = x + iy),$$

some of whose mapping properties were discussed in Sec. 14. In view of Euler's formula (Sec. 6), this function can, of course, be written

$$f(z) = e^x \cos y + ie^x \sin y,$$

where y is to be taken in radians when $\cos y$ and $\sin y$ are evaluated. Then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere and since these derivatives are everywhere continuous, the conditions in the above theorem are satisfied at all points in the complex plane. Thus $f'(z)$ exists everywhere, and

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y.$$

Note that $f'(z) = f(z)$ for all z.

EXAMPLE 2. It also follows from our theorem that the function $f(z) = |z|^2$, whose components are

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0,$$

has a derivative at $z = 0$. In fact, $f'(0) = 0 + i0 = 0$. We saw in Example 2, Sec. 21, that this function *cannot* have a derivative at any nonzero point since the Cauchy–Riemann equations are not satisfied at such points. (See also Example 3, Sec. 19.)

23. POLAR COORDINATES

Assuming that $z_0 \neq 0$, we shall in this section use the coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

to restate the theorem in Sec. 22 in polar coordinates.

Depending on whether we write

$$z = x + iy \quad \text{or} \quad z = re^{i\theta} \qquad (z \neq 0)$$

when $w = f(z)$, the real and imaginary components of $w = u + iv$ are expressed in terms of either the variables x and y or r and θ. Suppose that the first-order partial derivatives of u and v with respect to x and y exist everywhere in some neighborhood of a given nonzero point z_0 and are continuous at z_0. The first-order partial derivatives of u and v with respect to r and θ also have those properties, and the chain rule for differentiating real-valued functions of two real variables can be used to write them in terms of the ones with respect to x and y. More precisely, since

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta},$$

one can write

$$u_r = u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x\, r \sin\theta + u_y\, r\cos\theta. \tag{2}$$

Likewise,

$$v_r = v_x \cos\theta + v_y \sin\theta, \quad v_\theta = -v_x\, r \sin\theta + v_y\, r\cos\theta. \tag{3}$$

If the partial derivatives of u and v with respect to x and y also satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{4}$$

at z_0, equations (3) become

$$v_r = -u_y \cos\theta + u_x \sin\theta, \quad v_\theta = u_y\, r \sin\theta + u_x\, r\cos\theta \tag{5}$$

at that point. It is then clear from equations (2) and (5) that

$$r u_r = v_\theta, \quad u_\theta = -r v_r \tag{6}$$

at z_0.

If, on the other hand, equations (6) are known to hold at z_0, it is straightforward to show (Exercise 7) that equations (4) must hold there. Equations (6) are, therefore, an alternative form of the Cauchy–Riemann equations (4).

In view of equations (6) and the expression for $f'(z_0)$ that is found in Exercise 8, we are now able to restate the theorem in Sec. 22 using r and θ.

Theorem. *Let the function*

$$f(z) = u(r,\theta) + iv(r,\theta)$$

be defined throughout some ε neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood;

(b) those partial derivatives are continuous at (r_0, θ_0) *and satisfy the polar form*

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations at (r_0, θ_0).

Then $f'(z_0)$ *exists, its value being*

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where the right-hand side is to be evaluated at (r_0, θ_0).

EXAMPLE 1. Consider the function

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) \quad (z \neq 0).$$

Since

$$u(r,\theta) = \frac{\cos\theta}{r} \quad \text{and} \quad v(r,\theta) = -\frac{\sin\theta}{r},$$

the conditions in this theorem are satisfied at every nonzero point $z = re^{i\theta}$ in the plane. In particular, the Cauchy–Riemann equations

$$ru_r = -\frac{\cos\theta}{r} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sin\theta}{r} = -rv_r$$

are satisfied. Hence the derivative of f exists when $z \neq 0$; and, according to the theorem,

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -e^{-i\theta}\,\frac{e^{-i\theta}}{r^2} = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}.$$

EXAMPLE 2. The theorem can be used to show that when α is a fixed real number, the function

$$f(z) = \sqrt[3]{r}e^{i\theta/3} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

has a derivative everywhere in its domain of definition. Here

$$u(r,\theta) = \sqrt[3]{r}\cos\frac{\theta}{3} \quad \text{and} \quad v(r,\theta) = \sqrt[3]{r}\sin\frac{\theta}{3}.$$

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$$ru_r = \frac{\sqrt[3]{r}}{3}\cos\frac{\theta}{3} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt[3]{r}}{3}\sin\frac{\theta}{3} = -rv_r$$

and since the other conditions in the theorem are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. The theorem tells us, moreover, that

$$f'(z) = e^{-i\theta}\left[\frac{1}{3(\sqrt[3]{r})^2}\cos\frac{\theta}{3} + i\frac{1}{3(\sqrt[3]{r})^2}\sin\frac{\theta}{3}\right],$$

or

$$f'(z) = \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2}e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r}e^{i\theta/3})^2} = \frac{1}{3[f(z)]^2}.$$

Note that when a specific point z is taken in the domain of definition of f, the value $f(z)$ is one value of $z^{1/3}$ (see Sec. 9). Hence this last expression for $f'(z)$ can be put in the form

$$\frac{d}{dz}z^{1/3} = \frac{1}{3(z^{1/3})^2}$$

when that value is taken. Derivatives of such power functions will be elaborated on in Chap. 3 (Sec. 33).

EXERCISES

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$; *(b)* $f(z) = z - \overline{z}$;
(c) $f(z) = 2x + ixy^2$; *(d)* $f(z) = e^x e^{-iy}$.

2. Use the theorem in Sec. 22 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$; *(b)* $f(z) = e^{-x}e^{-iy}$;
(c) $f(z) = z^3$; *(d)* $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Ans. *(b)* $f''(z) = f(z)$; *(d)* $f''(z) = -f(z)$.

3. From results obtained in Secs. 21 and 22, determine where $f'(z)$ exists and find its value when

(a) $f(z) = 1/z$; *(b)* $f(z) = x^2 + iy^2$; *(c)* $f(z) = z \operatorname{Im} z$.

Ans. *(a)* $f'(z) = -1/z^2$ $(z \neq 0)$; *(b)* $f'(x + ix) = 2x$; *(c)* $f'(0) = 0$.

4. Use the theorem in Sec. 23 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(a) $f(z) = 1/z^4$ $\quad (z \neq 0)$;

(b) $f(z) = \sqrt{r}e^{i\theta/2}$ $\quad (r > 0, \alpha < \theta < \alpha + 2\pi)$;

(c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ $\quad (r > 0, 0 < \theta < 2\pi)$.

Ans. *(b)* $f'(z) = \dfrac{1}{2f(z)}$; *(c)* $f'(z) = i\dfrac{f(z)}{z}$.

5. Show that when $f(z) = x^3 + i(1-y)^3$, it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

6. Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$. [Compare with Exercise 9, Sec. 20, where it is shown that $f'(0)$ nevertheless fails to exist.]

7. Solve equations (2), Sec. 23 for u_x and u_y to show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \quad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that in Sec. 23 equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 23, are the Cauchy–Riemann equations in polar form.

8. Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 7, together with the polar form (6), Sec. 23, of the Cauchy–Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 22 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

9. *(a)* With the aid of the polar form (6), Sec. 23, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

of the expression for $f'(z_0)$ found in Exercise 8.

(b) Use the expression for $f'(z_0)$ in part *(a)* to show that the derivative of the function $f(z) = 1/z$ $(z \neq 0)$ in Example 1, Sec. 23, is $f'(z) = -1/z^2$.

10. *(a)* Recall (Sec. 5) that if $z = x + iy$, then

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(b) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part *(a)*, to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the *complex form* $\partial f/\partial \overline{z} = 0$ *of the Cauchy–Riemann equations.*

24. ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is *analytic at a point* z_0 if it has a derivative at each point in some neighborhood of z_0.* It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is *analytic in an open set* if it has a derivative everywhere in that set. If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

Note that the function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 3, Sec. 19.)

An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a *singular point*, or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhere analytic.

A necessary, but by no means sufficient, condition for a function f to be analytic in a domain D is clearly the continuity of f throughout D. Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems in Secs. 22 and 23.

Other useful sufficient conditions are obtained from the differentiation formulas in Sec. 20. The derivatives of the sum and product of two functions exist wherever

*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.