8. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a *vector field* in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by *(a)* $w = iz$; *(b)* $w = z/|z|$.

15. LIMITS

Let a function f be defined at all points z in some deleted neighborhood (Sec. 11) of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 23). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.

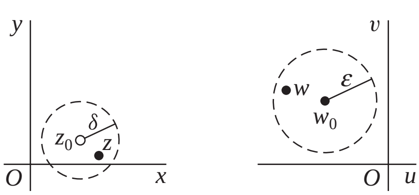

FIGURE 23

It is easy to show that *when a limit of a function $f(z)$ exists at a point z_0, it is unique.* To do this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for each positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1.$$

So if $0 < |z - z_0| < \delta$, where δ is any positive number that is smaller than δ_0 and δ_1, we find that

$$|w_1 - w_0| = |[f(z) - w_0] - [f(z) - w_1]| \le |f(z) - w_0| + |f(z) - w_1| < \varepsilon + \varepsilon = 2\varepsilon.$$

But $|w_1 - w_0|$ is a nonnegative constant, and ε can be chosen arbitrarily small. Hence

$$w_1 - w_0 = 0, \quad \text{or} \quad w_1 = w_0.$$

Definition (2) requires that f be defined at all points in some deleted neighborhood of z_0. Such a deleted neighborhood, of course, always exists when z_0 is an interior point of a region on which f is defined. We can extend the definition of limit to the case in which z_0 is a boundary point of the region by agreeing that the first of inequalities (2) need be satisfied by only those points z that lie in both the region *and* the deleted neighborhood.

EXAMPLE 1. Let us show that if $f(z) = i\bar{z}/2$ in the open disk $|z| < 1$, then

(3) $$\lim_{z \to 1} f(z) = \frac{i}{2},$$

the point 1 being on the boundary of the domain of definition of f. Observe that when z is in the disk $|z| < 1$,

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2}.$$

Hence, for any such z and each positive number ε (see Fig. 24),

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 1| < 2\varepsilon.$$

Thus condition (2) is satisfied by points in the region $|z| < 1$ when δ is equal to 2ε or any smaller positive number.

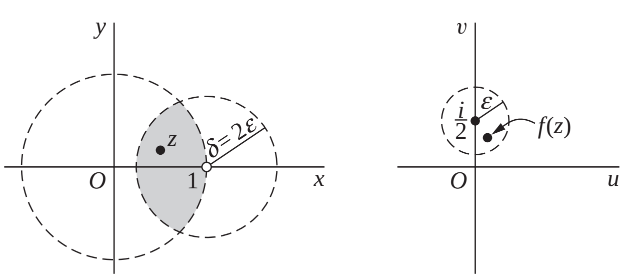

FIGURE 24

If limit (1) exists, the symbol $z \to z_0$ implies that z is allowed to approach z_0 in an arbitrary manner, not just from some particular direction. The next example emphasizes this.

EXAMPLE 2. If

$$f(z) = \frac{z}{\overline{z}}, \tag{4}$$

the limit

$$\lim_{z \to 0} f(z) \tag{5}$$

does not exist. For, if it did exist, it could be found by letting the point $z = (x, y)$ approach the origin in any manner. But when $z = (x, 0)$ is a nonzero point on the real axis (Fig. 25),

$$f(z) = \frac{x + i0}{x - i0} = 1;$$

and when $z = (0, y)$ is a nonzero point on the imaginary axis,

$$f(z) = \frac{0 + iy}{0 - iy} = -1.$$

Thus, by letting z approach the origin along the real axis, we would find that the desired limit is 1. An approach along the imaginary axis would, on the other hand, yield the limit -1. Since a limit is unique, we must conclude that limit (5) does not exist.

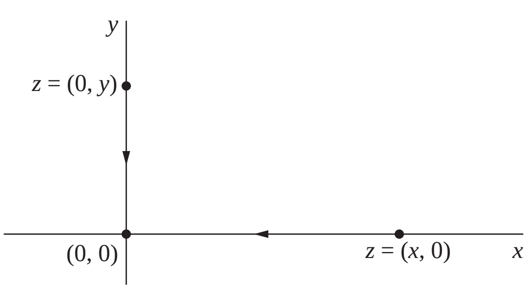

FIGURE 25

While definition (2) provides a means of testing whether a given point w_0 is a limit, it does not directly provide a method for determining that limit. Theorems on limits, presented in the next section, will enable us to actually find many limits.

16. THEOREMS ON LIMITS

We can expedite our treatment of limits by establishing a connection between limits of functions of a complex variable and limits of real-valued functions of two real variables. Since limits of the latter type are studied in calculus, we use their definition and properties freely.

Theorem 1. *Suppose that*

$$f(z) = u(x, y) + iv(x, y) \quad (z = x + iy)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0.$$

Then

$$\lim_{z \to z_0} f(z) = w_0 \tag{1}$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0. \tag{2}$$

To prove the theorem, we first assume that limits (2) hold and obtain limit (1). Limits (2) tell us that for each positive number ε, there exist positive numbers δ_1 and δ_2 such that

$$|u - u_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 \tag{3}$$

and

$$|v - v_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2. \tag{4}$$

Let δ be any positive number smaller than δ_1 and δ_2. Since

$$|(u + iv) - (u_0 + iv_0)| = |(u - u_0) + i(v - v_0)| \le |u - u_0| + |v - v_0|$$

and

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)| = |(x + iy) - (x_0 + iy_0)|,$$

it follows from statements (3) and (4) that

$$|(u + iv) - (u_0 + iv_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta.$$

That is, limit (1) holds.

Let us now start with the assumption that limit (1) holds. With that assumption, we know that for each positive number ε, there is a positive number δ such that

$$|(u + iv) - (u_0 + iv_0)| < \varepsilon \tag{5}$$

whenever

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta. \tag{6}$$

But

$$|u - u_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|,$$
$$|v - v_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|,$$

and

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Hence it follows from inequalities (5) and (6) that

$$|u - u_0| < \varepsilon \quad \text{and} \quad |v - v_0| < \varepsilon$$

whenever

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

This establishes limits (2), and the proof of the theorem is complete.

Theorem 2. *Suppose that*

$$\lim_{z \to z_0} f(z) = w_0 \quad \textit{and} \quad \lim_{z \to z_0} F(z) = W_0. \tag{7}$$

Then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0, \tag{8}$$

$$\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0\,; \tag{9}$$

and, if $W_0 \neq 0$,

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}. \tag{10}$$

This important theorem can be proved directly by using the definition of the limit of a function of a complex variable. But, with the aid of Theorem 1, it follows almost immediately from theorems on limits of real-valued functions of two real variables.

To verify property (9), for example, we write

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y),$$

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0, \quad W_0 = U_0 + iV_0.$$

Then, according to hypotheses (7) and Theorem 1, the limits as (x, y) approaches (x_0, y_0) of the functions u, v, U, and V exist and have the values u_0, v_0, U_0, and V_0, respectively. So the real and imaginary components of the product

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$

have the limits $u_0U_0 - v_0V_0$ and $v_0U_0 + u_0V_0$, respectively, as (x, y) approaches (x_0, y_0). Hence, by Theorem 1 again, $f(z)F(z)$ has the limit

$$(u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0)$$

as z approaches z_0; and this is equal to w_0W_0. Property (9) is thus established. Corresponding verifications of properties (8) and (10) can be given.

It is easy to see from definition (2), Sec. 15, of limit that

$$\lim_{z\to z_0} c = c \quad \text{and} \quad \lim_{z\to z_0} z = z_0,$$

where z_0 and c are any complex numbers; and, by property (9) and mathematical induction, it follows that

$$\lim_{z\to z_0} z^n = z_0^n \qquad (n = 1, 2, \ldots).$$

So, in view of properties (8) and (9), the limit of a polynomial

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$$

as z approaches a point z_0 is the value of the polynomial at that point:

$$\lim_{z\to z_0} P(z) = P(z_0). \tag{11}$$

17. LIMITS INVOLVING THE POINT AT INFINITY

It is sometimes convenient to include with the complex plane the *point at infinity,* denoted by ∞, and to use limits involving it. The complex plane together with this point is called the *extended* complex plane. To visualize the point at infinity, one can think of the complex plane as passing through the equator of a unit sphere centered at the origin (Fig. 26). To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is the point where the line through z and the north pole N intersects the sphere. In like manner, to each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one

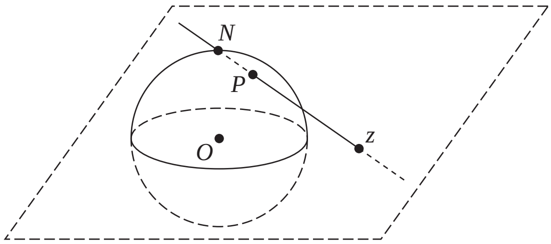

FIGURE 26

point z in the plane. By letting the point N of the sphere correspond to the point at infinity, we obtain a one to one correspondence between the points of the sphere and the points of the extended complex plane. The sphere is known as the *Riemann sphere*, and the correspondence is called a *stereographic projection.*

Observe that the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and the point N deleted. Moreover, for each small positive number ε, those points in the complex plane exterior to the circle $|z| = 1/\varepsilon$ correspond to points on the sphere close to N. We thus call the set $|z| > 1/\varepsilon$ an ε *neighborhood*, or neighborhood, of ∞.

Let us agree that in referring to a point z, we mean a point in the *finite* plane. Hereafter, when the point at infinity is to be considered, it will be specifically mentioned.

A meaning is now readily given to the statement

$$\lim_{z \to z_0} f(z) = w_0$$

when either z_0 or w_0, or possibly each of these numbers, is replaced by the point at infinity. In the definition of limit in Sec. 15, we simply replace the appropriate neighborhoods of z_0 and w_0 by neighborhoods of ∞. The proof of the following theorem illustrates how this is done.

Theorem. *If z_0 and w_0 are points in the z and w planes, respectively, then*

(1) $$\lim_{z \to z_0} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

and

(2) $$\lim_{z \to \infty} f(z) = w_0 \quad \textit{if and only if} \quad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0.$$

Moreover,

(3) $$\lim_{z \to \infty} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z \to 0} \frac{1}{f(1/z)} = 0.$$

We start the proof by noting that the first of limits (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{4}$$

That is, the point $w = f(z)$ lies in the ε neighborhood $|w| > 1/\varepsilon$ of ∞ whenever z lies in the deleted neighborhood $0 < |z - z_0| < \delta$ of z_0. Since statement (4) can be written

$$\left|\frac{1}{f(z)} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta,$$

the second of limits (1) follows.

The first of limits (2) means that for each positive number ε, a positive number δ exists such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad |z| > \frac{1}{\delta}. \tag{5}$$

Replacing z by $1/z$ in statement (5) and then writing the result as

$$\left|f\left(\frac{1}{z}\right) - w_0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta,$$

we arrive at the second of limits (2).

Finally, the first of limits (3) is to be interpreted as saying that for each positive number ε, there is a positive number δ such that

$$|f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad |z| > \frac{1}{\delta}. \tag{6}$$

When z is replaced by $1/z$, this statement can be put in the form

$$\left|\frac{1}{f(1/z)} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta;$$

and this gives us the second of limits (3).

EXAMPLES. Observe that

$$\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty \quad \text{since} \quad \lim_{z \to -1} \frac{z + 1}{iz + 3} = 0$$

and

$$\lim_{z \to \infty} \frac{2z + i}{z + 1} = 2 \quad \text{since} \quad \lim_{z \to 0} \frac{(2/z) + i}{(1/z) + 1} = \lim_{z \to 0} \frac{2 + iz}{1 + z} = 2.$$

Furthermore,

$$\lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty \quad \text{since} \quad \lim_{z \to 0} \frac{(1/z^2) + 1}{(2/z^3) - 1} = \lim_{z \to 0} \frac{z + z^3}{2 - z^3} = 0.$$

18. CONTINUITY

A function f is *continuous* at a point z_0 if all three of the following conditions are satisfied:

(1) $$\lim_{z\to z_0} f(z) \text{ exists},$$

(2) $$f(z_0) \text{ exists},$$

(3) $$\lim_{z\to z_0} f(z) = f(z_0).$$

Observe that statement (3) actually contains statements (1) and (2), since the existence of the quantity on each side of the equation there is needed. Statement (3) says, of course, that for each positive number ε, there is a positive number δ such that

(4) $$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

A function of a complex variable is said to be continuous in a region R if it is continuous at each point in R.

If two functions are continuous at a point, their sum and product are also continuous at that point; their quotient is continuous at any such point if the denominator is not zero there. These observations are direct consequences of Theorem 2, Sec. 16. Note, too, that a polynomial is continuous in the entire plane because of limit (11) in Sec. 16.

We turn now to two expected properties of continuous functions whose verifications are not so immediate. Our proofs depend on definition (4) of continuity, and we present the results as theorems.

Theorem 1. *A composition of continuous functions is itself continuous.*

A precise statement of this theorem is contained in the proof to follow. We let $w = f(z)$ be a function that is defined for all z in a neighborhood $|z - z_0| < \delta$ of a point z_0, and we let $W = g(w)$ be a function whose domain of definition contains the image (Sec. 13) of that neighborhood under f. The composition $W = g[f(z)]$ is, then, defined for all z in the neighborhood $|z - z_0| < \delta$. Suppose now that f is continuous at z_0 and that g is continuous at the point $f(z_0)$ in the w plane. In view of the continuity of g at $f(z_0)$, there is, for each positive number ε, a positive number γ such that

$$|g[f(z)] - g[f(z_0)]| < \varepsilon \quad \text{whenever} \quad |f(z) - f(z_0)| < \gamma.$$

(See Fig. 27.) But the continuity of f at z_0 ensures that the neighborhood $|z - z_0| < \delta$ can be made small enough that the second of these inequalities holds. The continuity of the composition $g[f(z)]$ is, therefore, established.

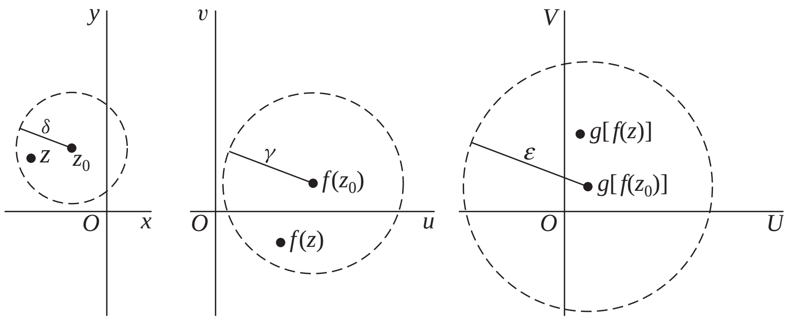

FIGURE 27

Theorem 2. *If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.*

Assuming that $f(z)$ is, in fact, continuous and nonzero at z_0, we can prove Theorem 2 by assigning the positive value $|f(z_0)|/2$ to the number ε in statement (4). This tells us that there is a positive number δ such that

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever} \quad |z - z_0| < \delta.$$

So if there is a point z in the neighborhood $|z - z_0| < \delta$ at which $f(z) = 0$, we have the contradiction

$$|f(z_0)| < \frac{|f(z_0)|}{2};$$

and the theorem is proved.

The continuity of a function

$$f(z) = u(x, y) + iv(x, y) \tag{5}$$

is closely related to the continuity of its component functions $u(x, y)$ and $v(x, y)$. We note, for instance, how it follows from Theorem 1 in Sec. 16 that *the function* (5) *is continuous at a point $z_0 = (x_0, y_0)$ if and only if its component functions are continuous there.* Our proof of the next theorem illustrates the use of this statement. The theorem is extremely important and will be used often in later chapters, especially in applications. Before stating the theorem, we recall from Sec. 11 that a region R is *closed* if it contains all of its boundary points and that it is *bounded* if it lies inside some circle centered at the origin.

Theorem 3. *If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that*

$$|f(z)| \leq M \qquad \text{for all points } z \text{ in } R, \tag{6}$$

where equality holds for at least one such z.

To prove this, we assume that the function f in equation (5) is continuous and note how it follows that the function

$$\sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is continuous throughout R and thus reaches a maximum value M somewhere in R.* Inequality (6) thus holds, and we say that *f is bounded on R.*

EXERCISES

1. Use definition (2), Sec. 15, of limit to prove that

(a) $\lim_{z \to z_0} \operatorname{Re} z = \operatorname{Re} z_0$; *(b)* $\lim_{z \to z_0} \overline{z} = \overline{z_0}$; *(c)* $\lim_{z \to 0} \dfrac{\overline{z}^2}{z} = 0$.

2. Let a, b, and c denote complex constants. Then use definition (2), Sec. 15, of limit to show that

(a) $\lim_{z \to z_0} (az + b) = az_0 + b$; *(b)* $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$;

(c) $\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i \quad (z = x + iy)$.

3. Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

(a) $\lim_{z \to z_0} \dfrac{1}{z^n} \quad (z_0 \neq 0)$; *(b)* $\lim_{z \to i} \dfrac{iz^3 - 1}{z + i}$; *(c)* $\lim_{z \to z_0} \dfrac{P(z)}{Q(z)}$.

Ans. (a) $1/z_0^n$; *(b)* 0; *(c)* $P(z_0)/Q(z_0)$.

4. Use mathematical induction and property (9), Sec. 16, of limits to show that

$$\lim_{z \to z_0} z^n = z_0^n$$

when n is a positive integer $(n = 1, 2, \ldots)$.

5. Show that the limit of the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. [Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

6. Prove statement (8) in Theorem 2 of Sec. 16 using

(a) Theorem 1 in Sec. 16 and properties of limits of real-valued functions of two real variables;

(b) definition (2), Sec. 15, of limit.

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*See, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 125–126 and p. 529, 1983.

7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \quad \lim_{z \to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z \to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how the first of inequalities (9), Sec. 4, enables one to write

$$||f(z)| - |w_0|| \le |f(z) - w_0|.$$

8. Write $\Delta z = z - z_0$ and show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0.$$

9. Show that

$$\lim_{z \to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0$$

and if there exists a positive number M such that $|g(z)| \le M$ for all z in some neighborhood of z_0.

10. Use the theorem in Sec. 17 to show that

(a) $\displaystyle\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4$; *(b)* $\displaystyle\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$; *(c)* $\displaystyle\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$.

11. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \ne 0),$$

(a) $\displaystyle\lim_{z \to \infty} T(z) = \infty$ if $c = 0$;

(b) $\displaystyle\lim_{z \to \infty} T(z) = \frac{a}{c}$ and $\displaystyle\lim_{z \to -d/c} T(z) = \infty$ if $c \ne 0$.

12. State why limits involving the point at infinity are unique.

13. Show that a set S is unbounded (Sec. 11) if and only if every neighborhood of the point at infinity contains at least one point in S.

19. DERIVATIVES

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The *derivative* of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and the function f is said to be *differentiable* at z_0 when $f'(z_0)$ exists.

By expressing the variable z in definition (1) in terms of the new complex variable

$$\Delta z = z - z_0 \quad (z \ne z_0),$$