CHAPTER

2

ANALYTIC FUNCTIONS

We now consider functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

12. FUNCTIONS OF A COMPLEX VARIABLE

Let S be a set of complex numbers. A *function* f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the *domain of definition* of f.*

It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Also, it is not always convenient to use notation that distinguishes between a given function and its values.

EXAMPLE 1. If f is defined on the set $z \neq 0$ by means of the equation $w = 1/z$, it may be referred to only as the function $w = 1/z$, or simply the function $1/z$.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy).$$

*Although the domain of definition is often a domain as defined in Sec. 11, it need not be.

Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x, y) + iv(x, y). \tag{1}$$

If the polar coordinates r and θ, instead of x and y, are used, then

$$u + iv = f(re^{i\theta})$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta). \tag{2}$$

EXAMPLE 2. If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

When polar coordinates are used,

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Consequently,

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

If, in either of equations (1) and (2), the function v always has value zero, then the value of f is always real. That is, f is a *real-valued function* of a complex variable.

EXAMPLE 3. A real-valued function that is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0.$$

If n is zero or a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n \neq 0$, the function

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$$

is a *polynomial* of degree n. Note that the sum here has a finite number of terms and that the domain of definition is the entire z plane. Quotients $P(z)/Q(z)$ of polynomials are called *rational functions* and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition. These *multiple-valued functions* occur in the theory of functions of a complex variable, just as they do in the case of a real variable. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function.

EXAMPLE 4. Let z denote any nonzero complex number. We know from Sec. 9 that $z^{1/2}$ has the two values

$$z^{1/2} = \pm\sqrt{r}\exp\left(i\frac{\Theta}{2}\right),$$

where $r = |z|$ and Θ $(-\pi < \Theta \leq \pi)$ is the *principal value* of $\arg z$. But, if we choose only the positive value of $\pm\sqrt{r}$ and write

$$f(z) = \sqrt{r}\exp\left(i\frac{\Theta}{2}\right) \qquad (r > 0, -\pi < \Theta \leq \pi), \tag{3}$$

the (single-valued) function (3) is well defined on the set of nonzero numbers in the z plane. Since zero is the only square root of zero, we also write $f(0) = 0$. The function f is then well defined on the entire plane.

EXERCISES

1. For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2+1}$; *(b)* $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z+\overline{z}}$; *(d)* $f(z) = \dfrac{1}{1-|z|^2}$.

Ans. (a) $z \neq \pm i$; *(c)* $\operatorname{Re} z \neq 0$.

2. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

Ans. $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$.

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 5)

$$x = \frac{z+\overline{z}}{2} \quad \text{and} \quad y = \frac{z-\overline{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

Ans. $f(z) = \overline{z}^2 + 2iz$.

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r,\theta) + iv(r,\theta)$.

Ans. $f(z) = \left(r + \dfrac{1}{r}\right)\cos\theta + i\left(r - \dfrac{1}{r}\right)\sin\theta.$

13. MAPPINGS

Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when $w = f(z)$, where z and w are complex, no such convenient graphical representation of the function f is available because each of the numbers z and w is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $w = (u, v)$. To do this, it is generally simpler to draw the z and w planes separately.

When a function f is thought of in this way, it is often referred to as a *mapping*, or transformation. The *image* of a point z in the domain of definition S is the point $w = f(z)$, and the set of images of all points in a set T that is contained in S is called the image of T. The image of the entire domain of definition S is called the *range* of f. The *inverse image* of a point w is the set of all points z in the domain of definition of f that have w as their image. The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when w is not in the range of f.

Terms such as *translation, rotation,* and *reflection* are used to convey dominant geometric characteristics of certain mappings. In such cases, it is sometimes convenient to consider the z and w planes to be the same. For example, the mapping

$$w = z + 1 = (x + 1) + iy,$$

where $z = x + iy$, can be thought of as a translation of each point z one unit to the right. Since $i = e^{i\pi/2}$, the mapping

$$w = iz = r\exp\left[i\left(\theta + \frac{\pi}{2}\right)\right],$$

where $z = re^{i\theta}$, rotates the radius vector for each nonzero point z through a right angle about the origin in the counterclockwise direction; and the mapping

$$w = \overline{z} = x - iy$$

transforms each point $z = x + iy$ into its reflection in the real axis.

More information is usually exhibited by sketching images of curves and regions than by simply indicating images of individual points. In the following three examples, we illustrate this with the transformation $w = z^2$. We begin by finding the images of some *curves* in the z plane.

EXAMPLE 1. According to Example 2 in Sec. 12, the mapping $w = z^2$ can be thought of as the transformation

$$u = x^2 - y^2, \quad v = 2xy \tag{1}$$

from the xy plane into the uv plane. This form of the mapping is especially useful in finding the images of certain hyperbolas.

It is easy to show, for instance, that each branch of a hyperbola

$$x^2 - y^2 = c_1 \qquad (c_1 > 0) \tag{2}$$

is mapped in a one to one manner onto the vertical line $u = c_1$. We start by noting from the first of equations (1) that $u = c_1$ when (x, y) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the second of equations (1) tells us that $v = 2y\sqrt{y^2 + c_1}$. Thus the image of the right-hand branch can be expressed parametrically as

$$u = c_1, \quad v = 2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty);$$

and it is evident that the image of a point (x, y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction (Fig. 17). Likewise, since the pair of equations

$$u = c_1, \quad v = -2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty)$$

furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going *downward* along the entire left-hand branch is seen to move up the entire line $u = c_1$.

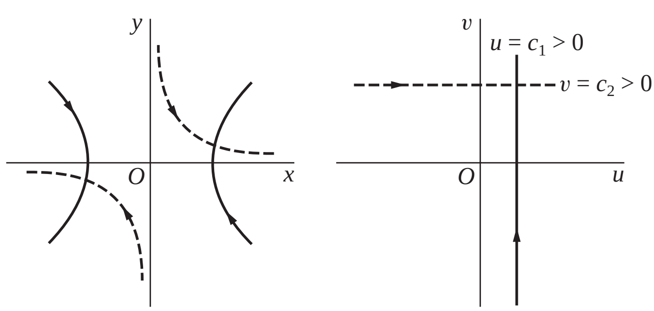

FIGURE 17
$w = z^2$.

On the other hand, each branch of a hyperbola

$$2xy = c_2 \qquad (c_2 > 0) \tag{3}$$

is transformed into the line $v = c_2$, as indicated in Fig. 17. To verify this, we note from the second of equations (1) that $v = c_2$ when (x, y) is a point on either branch. Suppose that (x, y) is on the branch lying in the first quadrant. Then, since $y = c_2/(2x)$, the first of equations (1) reveals that the branch's image has parametric representation

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2 \qquad (0 < x < \infty).$$

Observe that

$$\lim_{\substack{x \to 0 \\ x > 0}} u = -\infty \quad \text{and} \quad \lim_{x \to \infty} u = \infty.$$

Since u depends continuously on x, then, it is clear that as (x, y) travels down the entire upper branch of hyperbola (3), its image moves to the right along the entire horizontal line $v = c_2$. Inasmuch as the image of the lower branch has parametric representation

$$u = \frac{c_2^2}{4y^2} - y^2, \quad v = c_2 \qquad (-\infty < y < 0)$$

and since

$$\lim_{y \to -\infty} u = -\infty \quad \text{and} \quad \lim_{\substack{y \to 0 \\ y < 0}} u = \infty,$$

it follows that the image of a point moving *upward* along the entire lower branch also travels to the right along the entire line $v = c_2$ (see Fig. 17).

We shall now use Example 1 to find the image of a certain *region.*

EXAMPLE 2. The domain $x > 0, y > 0, xy < 1$ consists of all points lying on the upper branches of hyperbolas from the family $2xy = c$, where $0 < c < 2$ (Fig. 18). We know from Example 1 that as a point travels downward along the entirety of such a branch, its image under the transformation $w = z^2$ moves to the right along the entire line $v = c$. Since, for all values of c between 0 and 2, these upper branches fill out the domain $x > 0, y > 0, xy < 1$, that domain is mapped onto the horizontal strip $0 < v < 2$.

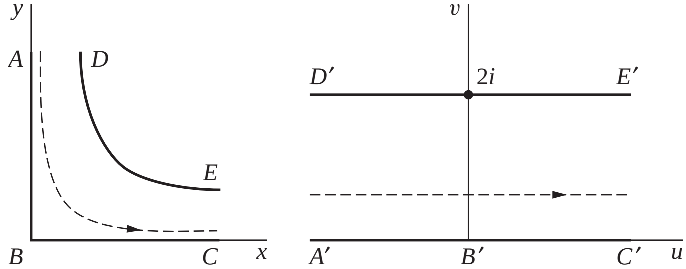

FIGURE 18
$w = z^2$.

In view of equations (1), the image of a point $(0, y)$ in the z plane is $(-y^2, 0)$. Hence as $(0, y)$ travels downward to the origin along the y axis, its image moves to the right along the negative u axis and reaches the origin in the w plane. Then, since the image of a point $(x, 0)$ is $(x^2, 0)$, that image moves to the right from the origin along the u axis as $(x, 0)$ moves to the right from the origin along the x axis. The image of the upper branch of the hyperbola $xy = 1$ is, of course, the horizontal line $v = 2$. Evidently, then, the closed region $x \geq 0$, $y \geq 0$, $xy \leq 1$ is mapped onto the closed strip $0 \leq v \leq 2$, as indicated in Fig. 18.

Our last example here illustrates how polar coordinates can be useful in analyzing certain mappings.

EXAMPLE 3. The mapping $w = z^2$ becomes

$$w = r^2 e^{i2\theta} \tag{4}$$

when $z = re^{i\theta}$. Evidently, then, the image $w = \rho e^{i\phi}$ of any nonzero point z is found by squaring the modulus $r = |z|$ and doubling the value θ of arg z that is used:

$$\rho = r^2 \quad \text{and} \quad \phi = 2\theta. \tag{5}$$

Observe that points $z = r_0 e^{i\theta}$ on a circle $r = r_0$ are transformed into points $w = r_0^2 e^{i2\theta}$ on the circle $\rho = r_0^2$. As a point on the first circle moves counterclockwise from the positive real axis to the positive imaginary axis, its image on the second circle moves counterclockwise from the positive real axis to the negative real axis (see Fig. 19). So, as all possible positive values of r_0 are chosen, the corresponding arcs in the z and w planes fill out the first quadrant and the upper half plane, respectively. The transformation $w = z^2$ is, then, a one to one mapping of the first quadrant $r \geq 0$, $0 \leq \theta \leq \pi/2$ in the z plane onto the upper half $\rho \geq 0, 0 \leq \phi \leq \pi$ of the w plane, as indicated in Fig. 19. The point $z = 0$ is, of course, mapped onto the point $w = 0$.

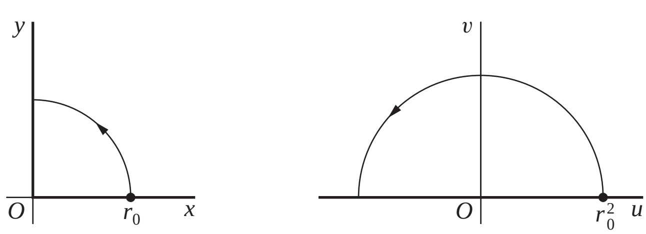

FIGURE 19
$w = z^2$.

The transformation $w = z^2$ also maps the upper half plane $r \geq 0, 0 \leq \theta \leq \pi$ onto the entire w plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z plane are mapped onto the positive real axis in the w plane.

When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $w = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \le r_0, 0 \le \theta \le 2\pi/n$ is mapped onto the disk $\rho \le r_0^n$, but not in a one to one manner.

Other, but somewhat more involved, mappings by $w = z^2$ appear in Example 1, Sec. 97, and Exercises 1 through 4 of that section.

14. MAPPINGS BY THE EXPONENTIAL FUNCTION

In Chap. 3 we shall introduce and develop properties of a number of elementary functions which do not involve polynomials. That chapter will start with the exponential function

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

the two factors e^x and e^{iy} being well defined at this time (see Sec. 6). Note that definition (1), which can also be written

$$e^{x+iy} = e^x e^{iy},$$

is suggested by the familiar additive property

$$e^{x_1+x_2} = e^{x_1} e^{x_2}$$

of the exponential function in calculus.

The object of this section is to use the function e^z to provide the reader with additional examples of mappings that continue to be reasonably simple. We begin by examining the images of vertical and horizontal lines.

EXAMPLE 1. The transformation

$$w = e^z \tag{2}$$

can be written $w = e^x e^{iy}$, where $z = x + iy$, according to equation (1). Thus, if $w = \rho e^{i\phi}$, transformation (2) can be expressed in the form

$$\rho = e^x, \quad \phi = y. \tag{3}$$

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = \exp c_1$ and $\phi = y$ in the w plane. That image moves counterclockwise around the circle shown in Fig. 20 as z moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

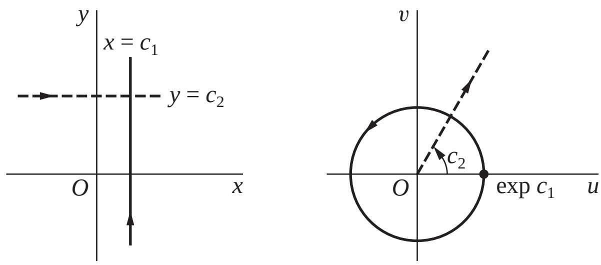

FIGURE 20
$w = \exp z$.

A horizontal line $y = c_2$ is mapped in a one to one manner onto the ray $\phi = c_2$. To see that this is so, we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$ and $\phi = c_2$. Consequently, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\phi = c_2$, as indicated in Fig. 20.

Vertical and horizontal line *segments* are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in Example 1. This is illustrated in the following example.

EXAMPLE 2. Let us show that the transformation $w = e^z$ maps the rectangular region $a \le x \le b, c \le y \le d$ onto the region $e^a \le \rho \le e^b, c \le \phi \le d$. The two regions and corresponding parts of their boundaries are indicated in Fig. 21. The vertical line segment AD is mapped onto the arc $\rho = e^a, c \le \phi \le d$, which is labeled $A'D'$. The images of vertical line segments to the right of AD and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment BC is the arc $\rho = e^b, c \le \phi \le d$, labeled $B'C'$. The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \le \phi \le \pi$; and the rectangular region is mapped onto half of a circular ring, as shown in Fig. 8, Appendix 2.

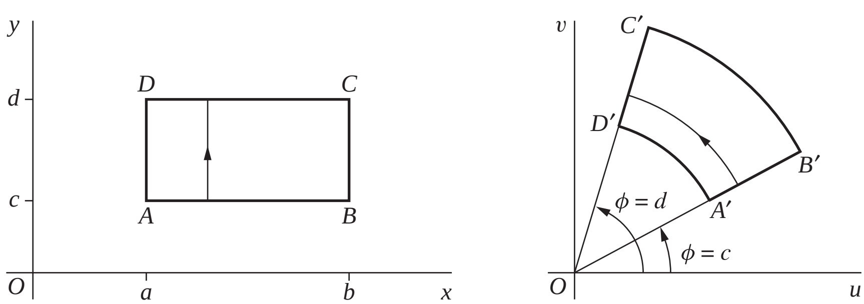

FIGURE 21
$w = \exp z$.

Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

EXAMPLE 3. When $w = e^z$, the image of the infinite strip $0 \leq y \leq \pi$ is the upper half $v \geq 0$ of the w plane (Fig. 22). This is seen by recalling from Example 1 how a horizontal line $y = c$ is transformed into a ray $\phi = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$. This mapping is also shown in Fig. 6 of Appendix 2, where corresponding points on the boundaries of the two regions are indicated.

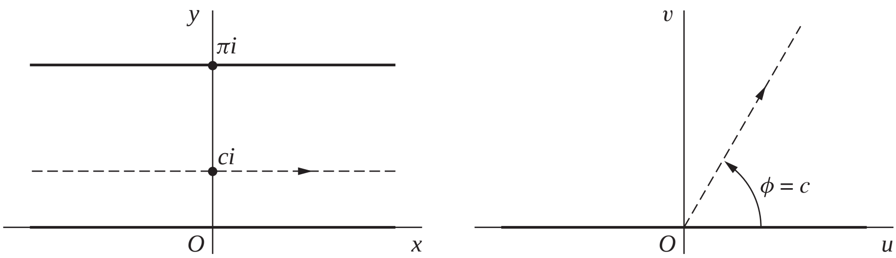

FIGURE 22
$w = \exp z$.

EXERCISES

1. By referring to Example 1 in Sec. 13, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1, u = 2, v = 1$, and $v = 2$. (See Fig. 2, Appendix 2.)

2. Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$.

3. Sketch the region onto which the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ is mapped by the transformation *(a)* $w = z^2$; *(b)* $w = z^3$; *(c)* $w = z^4$.

4. Show that the lines $ay = x$ $(a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$.

5. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \leq x \leq b, c \leq y \leq d$ under the transformation $w = \exp z$ is the region $e^a \leq \rho \leq e^b, c \leq \phi \leq d$, as shown in Fig. 21 (Sec. 14).

6. Verify the mapping of the region and boundary shown in Fig. 7 of Appendix 2, where the transformation is $w = \exp z$.

7. Find the image of the semi-infinite strip $x \geq 0, 0 \leq y \leq \pi$ under the transformation $w = \exp z$, and label corresponding portions of the boundaries.

8. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a *vector field* in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by *(a)* $w = iz$; *(b)* $w = z/|z|$.

15. LIMITS

Let a function f be defined at all points z in some deleted neighborhood (Sec. 11) of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 23). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.

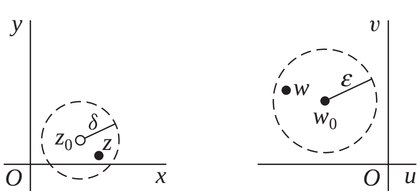

FIGURE 23

It is easy to show that *when a limit of a function* $f(z)$ *exists at a point* z_0, *it is unique.* To do this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for each positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0$$