(b) Point out why

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\text{Re}(z_1\overline{z_2}) \le 2|z_1||z_2|.$$

(c) Use the results in parts *(a)* and *(b)* to obtain the inequality

$$|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2,$$

and note how the triangle inequality follows.

6. EXPONENTIAL FORM

Let r and θ be polar coordinates of the point (x, y) that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in *polar form* as

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

If $z = 0$, the coordinate θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an *argument* of z, and the set of all such values is denoted by $\arg z$. The *principal value* of $\arg z$, denoted by $\text{Arg } z$, is that unique value Θ such that $-\pi < \Theta \le \pi$. Evidently, then,

$$\arg z = \text{Arg } z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

Also, when z is a negative real number, $\text{Arg } z$ has value π, not $-\pi$.

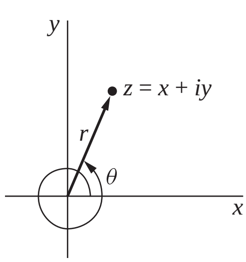

FIGURE 6

EXAMPLE 1. The complex number $-1 - i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$\text{Arg}(-1 - i) = -\frac{3\pi}{4}.$$

It must be emphasized that because of the restriction $-\pi < \Theta \le \pi$ of the principal argument Θ, it is *not* true that $\text{Arg}(-1-i) = 5\pi/4$.

According to equation (2),

$$\arg(-1-i) = -\frac{3\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Note that the term Arg z on the right-hand side of equation (2) can be replaced by any particular value of arg z and that one can write, for instance,

$$\arg(-1-i) = \frac{5\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The symbol $e^{i\theta}$, or $\exp(i\theta)$, is defined by means of *Euler's formula* as

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3}$$

where θ is to be measured in radians. It enables one to write the polar form (1) more compactly in *exponential form* as

$$z = re^{i\theta}. \tag{4}$$

The choice of the symbol $e^{i\theta}$ will be fully motivated later on in Sec. 29. Its use in Sec. 7 will, however, suggest that it is a natural choice.

EXAMPLE 2. The number $-1-i$ in Example 1 has exponential form

$$-1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}\right)\right]. \tag{5}$$

With the agreement that $e^{-i\theta} = e^{i(-\theta)}$, this can also be written $-1-i = \sqrt{2}\,e^{-i3\pi/4}$. Expression (5) is, of course, only one of an infinite number of possibilities for the exponential form of $-1-i$:

$$-1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4} + 2n\pi\right)\right] \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

Note how expression (4) with $r = 1$ tells us that the numbers $e^{i\theta}$ lie on the circle centered at the origin with radius unity, as shown in Fig. 7. Values of $e^{i\theta}$ are, then, immediate from that figure, without reference to Euler's formula. It is, for instance, geometrically obvious that

$$e^{i\pi} = -1, \quad e^{-i\pi/2} = -i, \quad \text{and} \quad e^{-i4\pi} = 1.$$

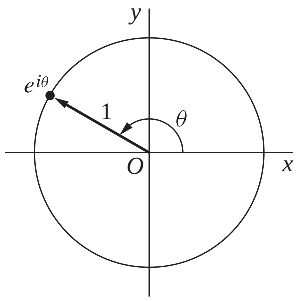

FIGURE 7

Note, too, that the equation

$$z = Re^{i\theta} \qquad (0 \le \theta \le 2\pi) \tag{7}$$

is a parametric representation of the circle $|z| = R$, centered at the origin with radius R. As the parameter θ increases from $\theta = 0$ to $\theta = 2\pi$, the point z starts from the positive real axis and traverses the circle once in the counterclockwise direction. More generally, the circle $|z - z_0| = R$, whose center is z_0 and whose radius is R, has the parametric representation

$$z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi). \tag{8}$$

This can be seen vectorially (Fig. 8) by noting that a point z traversing the circle $|z - z_0| = R$ once in the counterclockwise direction corresponds to the sum of the fixed vector z_0 and a vector of length R whose angle of inclination θ varies from $\theta = 0$ to $\theta = 2\pi$.

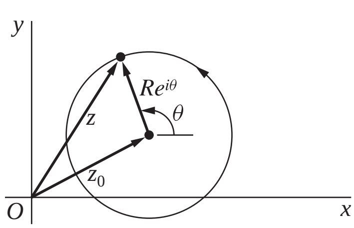

FIGURE 8

7. PRODUCTS AND POWERS IN EXPONENTIAL FORM

Simple trigonometry tells us that $e^{i\theta}$ has the familiar additive property of the exponential function in calculus:

$$\begin{aligned}
e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\
&= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\\
&= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}.
\end{aligned}$$

Thus, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, the product z_1z_2 has exponential form

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i\theta_1}e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)}. \tag{1}$$

Furthermore,

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}\cdot\frac{e^{i\theta_1}e^{-i\theta_2}}{e^{i\theta_2}e^{-i\theta_2}} = \frac{r_1}{r_2}\cdot\frac{e^{i(\theta_1-\theta_2)}}{e^{i0}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}. \tag{2}$$

Note how it follows from expression (2) that the inverse of any nonzero complex number $z = re^{i\theta}$ is

$$z^{-1} = \frac{1}{z} = \frac{1e^{i0}}{re^{i\theta}} = \frac{1}{r}e^{i(0-\theta)} = \frac{1}{r}e^{-i\theta}. \tag{3}$$

Expressions (1), (2), and (3) are, of course, easily remembered by applying the usual algebraic rules for real numbers and e^x.

Another important result that can be obtained formally by applying rules for real numbers to $z = re^{i\theta}$ is

$$z^n = r^ne^{in\theta} \qquad (n = 0, \pm1, \pm2, \ldots). \tag{4}$$

It is easily verified for positive values of n by mathematical induction. To be specific, we first note that it becomes $z = re^{i\theta}$ when $n = 1$. Next, we assume that it is valid when $n = m$, where m is any positive integer. In view of expression (1) for the product of two nonzero complex numbers in exponential form, it is then valid for $n = m + 1$:

$$z^{m+1} = z^mz = r^me^{im\theta}re^{i\theta} = (r^mr)e^{i(m\theta+\theta)} = r^{m+1}e^{i(m+1)\theta}.$$

Expression (4) is thus verified when n is a positive integer. It also holds when $n = 0$, with the convention that $z^0 = 1$. If $n = -1, -2, \ldots,$ on the other hand, we define z^n in terms of the multiplicative inverse of z by writing

$$z^n = (z^{-1})^m \quad \text{where} \quad m = -n = 1, 2, \ldots .$$

Then, since equation (4) is valid for positive integers, it follows from the exponential form (3) of z^{-1} that

$$z^n = \left[\frac{1}{r}e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^ne^{in\theta}$$
$$(n = -1, -2, \ldots).$$

Expression (4) is now established for all integral powers.

Expression (4) can be useful in finding powers of complex numbers even when they are given in rectangular form and the result is desired in that form.

EXAMPLE 1. In order to put $(\sqrt{3}+1)^7$ in rectangular form, one need only write

$$(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3}+i).$$

Finally, we observe that if $r = 1$, equation (4) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{6}$$

this is known as *de Moivre's formula*. The following example uses a special case of it.

EXAMPLE 2. Formula (6) with $n = 2$ tells us that

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta,$$

or

$$\cos^2\theta - \sin^2\theta + i2\sin\theta \quad \cos\theta = \cos 2\theta + i\sin 2\theta.$$

By equating real parts and then imaginary parts here, we have the familiar trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

(See also Exercises 10 and 11, Sec. 8.)

8. ARGUMENTS OF PRODUCTS AND QUOTIENTS

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the expression

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1+\theta_2)} \tag{1}$$

in Sec. 7 can be used to obtain an important identity involving arguments:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{2}$$

This result is to be interpreted as saying that if values of two of the three (multiple-valued) arguments are specified, then there is a value of the third such that the equation holds.

We start the verification of statement (2) by letting θ_1 and θ_2 denote any values of $\arg z_1$ and $\arg z_2$, respectively. Expression (1) then tells us that $\theta_1 + \theta_2$ is a value of $\arg(z_1 z_2)$. (See Fig. 9.) If, on the other hand, values of $\arg(z_1 z_2)$ and

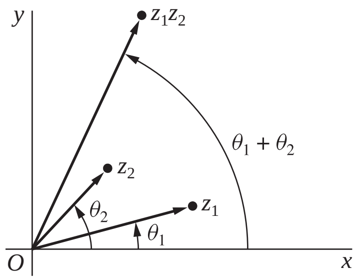

FIGURE 9

$\arg z_1$ are specified, those values correspond to particular choices of n and n_1 in the expressions

$$\arg(z_1z_2) = (\theta_1 + \theta_2) + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\arg z_1 = \theta_1 + 2n_1\pi \qquad (n_1 = 0, \pm 1, \pm 2, \ldots).$$

Since

$$(\theta_1 + \theta_2) + 2n\pi = (\theta_1 + 2n_1\pi) + [\theta_2 + 2(n - n_1)\pi],$$

equation (2) is evidently satisfied when the value

$$\arg z_2 = \theta_2 + 2(n - n_1)\pi$$

is chosen. Verification when values of $\arg(z_1z_2)$ and $\arg z_2$ are specified follows by symmetry.

Statement (2) is sometimes valid when *arg* is replaced everywhere by *Arg* (see Exercise 6). But, as the following example illustrates, that is *not always* the case.

EXAMPLE 1. When $z_1 = -1$ and $z_2 = i$,

$$\text{Arg}(z_1z_2) = \text{Arg}(-i) = -\frac{\pi}{2} \quad \text{but} \quad \text{Arg}\, z_1 + \text{Arg}\, z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

If, however, we take the values of $\arg z_1$ and $\arg z_2$ just used and select the value

$$\text{Arg}(z_1z_2) + 2\pi = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$$

of $\arg(z_1z_2)$, we find that equation (2) *is* satisfied.

Statement (2) tells us that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1z_2^{-1}) = \arg z_1 + \arg(z_2^{-1});$$

and, since (Sec. 7)

$$z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2},$$

one can see that

$$\arg(z_2^{-1}) = -\arg z_2. \tag{3}$$

Hence

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{4}$$

Statement (3) is, of course, to be interpreted as saying that the set of all values on the left-hand side is the same as the set of all values on the right-hand side. Statement (4) is, then, to be interpreted in the same way that statement (2) is.

EXAMPLE 2. In order to find the principal argument Arg z when

$$z = \frac{-2}{1 + \sqrt{3}i},$$

observe that

$$\arg z = \arg(-2) - \arg(1 + \sqrt{3}i).$$

Since

$$\text{Arg}(-2) = \pi \quad \text{and} \quad \text{Arg}(1 + \sqrt{3}i) = \frac{\pi}{3},$$

one value of $\arg z$ is $2\pi/3$; and, because $2\pi/3$ is between $-\pi$ and π, we find that $\text{Arg}\, z = 2\pi/3$.

EXERCISES

1. Find the principal argument $\text{Arg}\, z$ when

(a) $z = \dfrac{i}{-2 - 2i}$; *(b)* $z = (\sqrt{3} - i)^6$.

Ans. (a) $-3\pi/4$; *(b)* π.

2. Show that *(a)* $|e^{i\theta}| = 1$; *(b)* $\overline{e^{i\theta}} = e^{-i\theta}$.

3. Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)} \qquad (n = 2, 3, \ldots).$$

4. Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1 (see Sec. 4), give a geometric argument to find a value of θ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

Ans. π.

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; *(b)* $5i/(2+i) = 1+2i$;

(c) $(-1+i)^7 = -8(1+i)$; *(d)* $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

6. Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

7. Let z be a nonzero complex number and n a negative integer $(n = -1, -2, \ldots)$. Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$. Using the expressions

$$z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \left(\frac{1}{r}\right) e^{i(-\theta)},$$

verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec. 7 could have been written alternatively as $z^n = (z^m)^{-1}$.

8. Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \overline{c_2}$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = \exp(i\theta_1)$$

and [see Exercise 2*(b)*]

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)} = \exp(i\theta_2).$$

9. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; *(b)* $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

11. *(a)* Use the binomial formula (Sec. 3) and de Moivre's formula (Sec. 7) to write

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n = 0, 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{is odd} \end{cases}$$

and use the above summation to show that [compare with Exercise 10*(a)*]

$$\cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta\,\sin^{2k}\theta \qquad (n = 0, 1, 2, \ldots).$$

(b) Write $x = \cos\theta$ in the final summation in part *(a)* to show that it becomes a polynomial

$$T_n(x) = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k}(1-x^2)^k$$

of degree n $(n = 0, 1, 2, \ldots)$ in the variable x.*

9. ROOTS OF COMPLEX NUMBERS

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r (Fig. 10). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is true when θ is decreased by 2π. It is, therefore, evident from Fig. 10 that *two nonzero complex numbers*

$$z_1 = r_1 e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2 e^{i\theta_2}$$

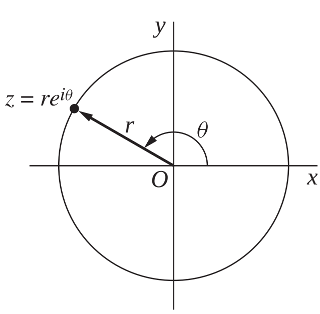

FIGURE 10

*These are called Chebyshev polynomials and are prominent in approximation theory.

are equal if and only if

$$r_1 = r_2 \quad \textit{and} \quad \theta_1 = \theta_2 + 2k\pi,$$

where k is some integer $(k = 0, \pm 1, \pm 2, \ldots)$.

This observation, together with the expression $z^n = r^n e^{in\theta}$ in Sec. 7 for integral powers of complex numbers $z = re^{i\theta}$, is useful in finding the nth roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$, where n has one of the values $n = 2, 3, \ldots$. The method starts with the fact that an nth root of z_0 is a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

According to the statement in italics just above, then,

$$r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi,$$

where k is any integer $(k = 0, \pm 1, \pm 2, \ldots)$. So $r = \sqrt[n]{r_0}$, where this radical denotes the unique *positive* nth root of the positive real number r_0, and

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Consequently, the complex numbers

$$z = \sqrt[n]{r_0} \exp\left[i \left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

are the nth roots of z_0. We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced every $2\pi/n$ radians, starting with argument θ_0/n. Evidently, then, all of the *distinct* roots are obtained when $k = 0, 1, 2, \ldots, n-1$, and no further roots arise with other values of k. We let c_k $(k = 0, 1, 2, \ldots, n-1)$ denote these distinct roots and write

(1) $$c_k = \sqrt[n]{r_0} \exp\left[i \left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] \qquad (k = 0, 1, 2, \ldots, n-1).$$

(See Fig. 11.)

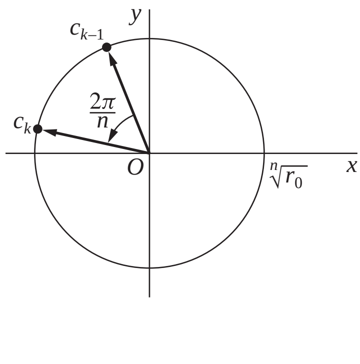

FIGURE 11

The number $\sqrt[n]{r_0}$ is the length of each of the radius vectors representing the n roots. The first root c_0 has argument θ_0/n; and the two roots when $n = 2$ lie at the opposite ends of a diameter of the circle $|z| = \sqrt[n]{r_0}$, the second root being $-c_0$. When $n \geq 3$, the roots lie at the vertices of a regular polygon of n sides inscribed in that circle.

We shall let $z_0^{1/n}$ denote the *set* of nth roots of z_0. If, in particular, z_0 is a positive real number r_0, the symbol $r_0^{1/n}$ denotes the entire set of roots; and the symbol $\sqrt[n]{r_0}$ in expression (1) is reserved for the one positive root. When the value of θ_0 that is used in expression (1) is the principal value of $\arg z_0$ $(-\pi < \theta_0 \leq \pi)$, the number c_0 is referred to as the *principal root.* Thus when z_0 is a positive real number r_0, its principal root is $\sqrt[n]{r_0}$.

Observe that if we write expression (1) for the roots of z_0 as

$$c_k = \sqrt[n]{r_0}\exp\left(i\frac{\theta_0}{n}\right)\exp\left(i\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1),$$

and also write

$$\omega_n = \exp\left(i\frac{2\pi}{n}\right), \tag{2}$$

it follows from property (5), Sec. 7. of $e^{i\theta}$ that

$$\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1) \tag{3}$$

and hence that

$$c_k = c_0\omega_n^k \quad (k = 0, 1, 2, \ldots, n-1). \tag{4}$$

The number c_0 here can, of course, be replaced by any particular nth root of z_0, since ω_n represents a counterclockwise rotation through $2\pi/n$ radians.

Finally, a convenient way to remember expression (1) is to write z_0 in its most general exponential form (compare with Example 2 in Sec. 6)

$$z_0 = r_0\, e^{i(\theta_0 + 2k\pi)} \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{5}$$

and to *formally* apply laws of fractional exponents involving real numbers, keeping in mind that there are precisely n roots:

$$z_0^{1/n} = \left[r_0\, e^{i(\theta_0+2k\pi)}\right]^{1/n} = \sqrt[n]{r_0}\exp\left[\frac{i(\theta_0 + 2k\pi)}{n}\right] = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]$$
$$(k = 0, 1, 2, \ldots, n-1).$$

The examples in the next section serve to illustrate this method for finding roots of complex numbers.

10. EXAMPLES

In each of the examples here, we start with expression (5), Sec. 9, and proceed in the manner described just after it.

EXAMPLE 1. Let us find all values of $(-8i)^{1/3}$, or the three cube roots of the number $-8i$. One need only write

$$-8i = 8\exp\left[i\left(-\frac{\pi}{2} + 2k\pi\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

to see that the desired roots are

$$c_k = 2\exp\left[i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0, 1, 2). \tag{1}$$

They lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$, and are equally spaced around that circle every $2\pi/3$ radians, starting with the principal root (Fig. 12)

$$c_0 = 2\exp\left[i\left(-\frac{\pi}{6}\right)\right] = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{3} - i.$$

Without any further calculations, it is then evident that $c_1 = 2i$; and, since c_2 is symmetric to c_0 with respect to the imaginary axis, we know that $c_2 = -\sqrt{3} - i$.

Note how it follows from expressions (2) and (4) in Sec. 9 that these roots can be written

$$c_0,\ c_0\omega_3,\ c_0\omega_3^2 \quad \text{where} \quad \omega_3 = \exp\left(i\frac{2\pi}{3}\right).$$

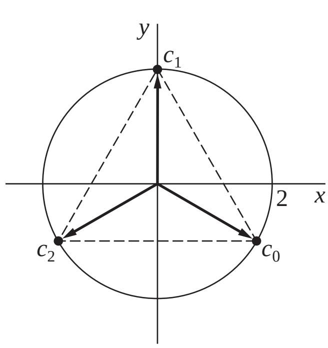

FIGURE 12

EXAMPLE 2. In order to determine the nth roots of unity, we start with

$$1 = 1\exp[i(0 + 2k\pi)] \qquad (k = 0, \pm 1, \pm 2 \ldots)$$

and find that

$$1^{1/n} = \sqrt[n]{1}\exp\left[i\left(\frac{0}{n} + \frac{2k\pi}{n}\right)\right] = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1). \tag{2}$$

When $n = 2$, these roots are, of course, ± 1. When $n \geq 3$, the regular polygon at whose vertices the roots lie is inscribed in the unit circle $|z| = 1$, with one vertex corresponding to the principal root $z = 1$ $(k = 0)$. In view of expression (3), Sec. 9, these roots are simply

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1} \quad \text{where} \quad \omega_n = \exp\left(i\frac{2\pi}{n}\right).$$

See Fig. 13, where the cases $n = 3, 4$, and 6 are illustrated. Note that $\omega_n^n = 1$.

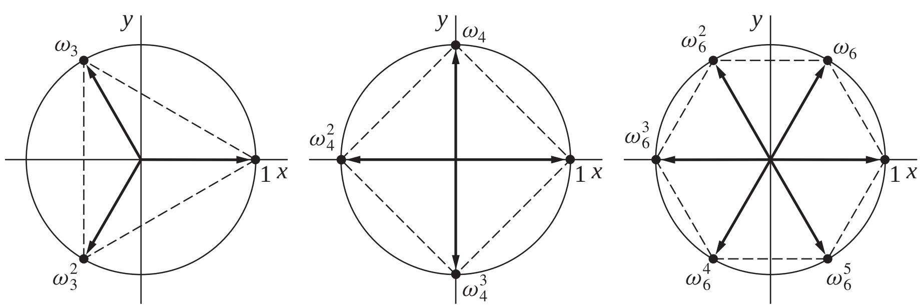

FIGURE 13

EXAMPLE 3. The two values c_k $(k = 0, 1)$ of $(\sqrt{3} + i)^{1/2}$, which are the square roots of $\sqrt{3} + i$, are found by writing

$$\sqrt{3} + i = 2\exp\left[i\left(\frac{\pi}{6} + 2k\pi\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

and (see Fig. 14)

$$c_k = \sqrt{2}\exp\left[i\left(\frac{\pi}{12} + k\pi\right)\right] \qquad (k = 0, 1). \tag{3}$$

Euler's formula tells us that

$$c_0 = \sqrt{2}\exp\left(i\frac{\pi}{12}\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right),$$

and the trigonometric identities

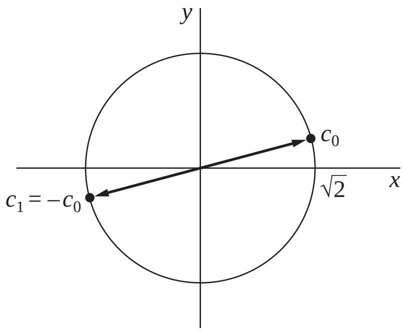

FIGURE 14

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}, \quad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \tag{4}$$

enable us to write

$$\cos^2\frac{\pi}{12} = \frac{1}{2}\left(1+\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4},$$
$$\sin^2\frac{\pi}{12} = \frac{1}{2}\left(1-\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}.$$

Consequently,

$$c_0 = \sqrt{2}\left(\sqrt{\frac{2+\sqrt{3}}{4}} + i\sqrt{\frac{2-\sqrt{3}}{4}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right).$$

Since $c_1 = -c_0$, the two square roots of $\sqrt{3}+i$ are, then,

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right). \tag{5}$$

EXERCISES

1. Find the square roots of *(a)* $2i$; *(b)* $1-\sqrt{3}i$ and express them in rectangular coordinates.

Ans. *(a)* $\pm(1+i)$; *(b)* $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

2. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

(a) $(-16)^{1/4}$; *(b)* $(-8-8\sqrt{3}i)^{1/4}$.

Ans. *(a)* $\pm\sqrt{2}(1+i)$, $\pm\sqrt{2}(1-i)$; *(b)* $\pm(\sqrt{3}-i)$, $\pm(1+\sqrt{3}i)$.

3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; *(b)* $8^{1/6}$.

Ans. (b) $\pm\sqrt{2},\ \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}},\ \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

4. According to Sec. 9, the three cube roots of a nonzero complex number z_0 can be written $c_0, c_0\omega_3, c_0\omega_3^2$ where c_0 is the principal cube root of z_0 and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2}+4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \quad c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

5. *(a)* Let a denote any fixed real number and show that the two square roots of $a+i$ are

$$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)$$

where $A = \sqrt{a^2+1}$ and $\alpha = \operatorname{Arg}(a+i)$.

(b) With the aid of the trigonometric identities (4) in Example 3 of Sec. 10, show that the square roots obtained in part *(a)* can be written

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a}+i\sqrt{A-a}\right).$$

(Note that this becomes the final result in Example 3, Sec. 10, when $a = \sqrt{3}$.)

6. Find the four zeros of the polynomial z^4+4, one of them being

$$z_0 = \sqrt{2}\,e^{i\pi/4} = 1+i.$$

Then use those zeros to factor z^2+4 into quadratic factors with real coefficients.

Ans. $(z^2+2z+2)(z^2-2z+2)$.

7. Show that if c is any nth root of unity other than unity itself, then

$$1+c+c^2+\cdots+c^{n-1}=0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

8. *(a)* Prove that the usual formula solves the quadratic equation

$$az^2+bz+c=0 \qquad (a \neq 0)$$

when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the *quadratic formula*

$$z = \frac{-b+(b^2-4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2-4ac \neq 0$,