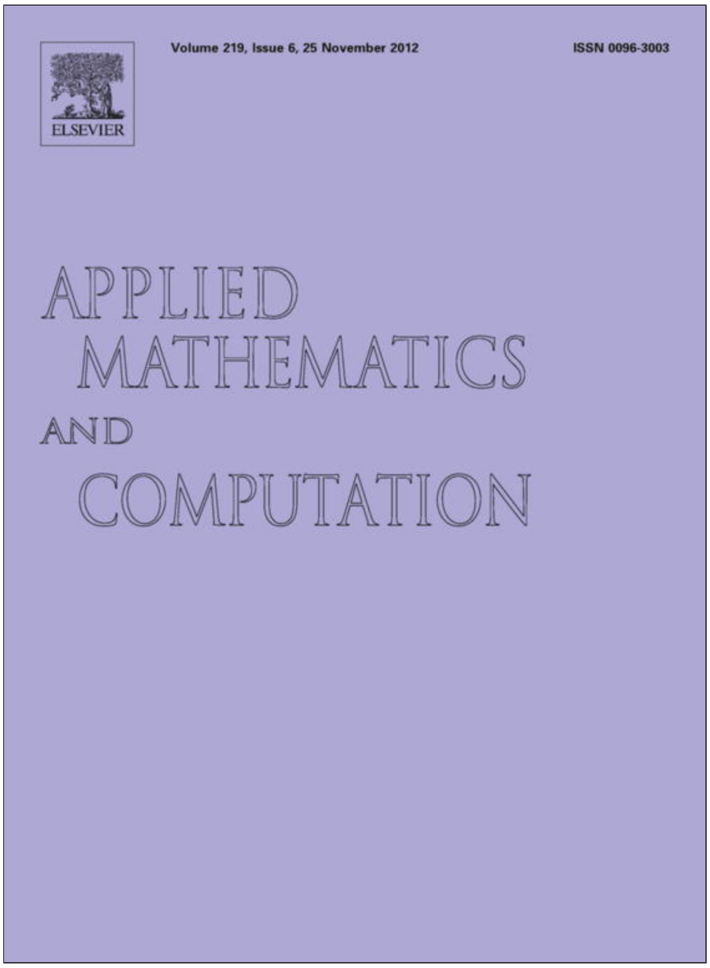
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Monotonicity-preserving C^2 rational cubic spline for monotone data

Muhammad Abbas *, Ahmad Abd. Majid, Jamaludin Md. Ali

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia

A R T I C L E   I N F O

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A B S T R A C T

Designers in industries need to generate splines which can interpolate the data points in such a way that they preserve the inherited shape characteristics (positivity, monotonicity, convexity) of data. Among the properties that the spline for curves and surfaces need to satisfy, smoothness and shape preservation of given data are mostly needed by all the designers. In this paper, a rational cubic function with three shape parameters has been developed. Data dependent sufficient constraints are derived for one of these shape parameters to preserve the inherited shape feature like monotonicity of data. Remaining two shape parameters are left free for designer to refine the shape of the monotone curve as desired. Numerical examples and interpolation error analysis show that the interpolant is not only C^2, local, computationally economical and visually pleasant but also smooth. The error of rational cubic function is also calculated when the arbitrary function being interpolated is C^3 in an interpolating interval. The order of approximation of interpolant is $O(h_i^3)$.

1. Introduction

Spline interpolation is a significant tool in Computer Graphics, Computer Aided Geometric Design and Engineering as well. The problem of shape preserving of given data plays imperative role in the field of data visualization. Data visualization is the study of visual display of data. The main purpose of data visualization is a graphical representation of information in pretty effective and clear way. These graphical representations of data have great significance in many fields including engineering, military, education, art, medicine, advertising, transport, etc. Therefore, in these fields, it is often needed to generate a monotonicity-preserving interpolating curve and surface through given monotone data. The objective of this paper is to preserve the hereditary attribute that is the monotonicity of data.

Monotonicity is a prevailing shape property of curve. There are many physical situations that arise from different sciences and art where entities only have a meaning when their values are monotone. Examples include approximations of couple and quasi couples in statistics, approximation of potential functions in physical and chemical systems and dose response curves in biochemistry and pharmacology. The specification of certain devices like digital-to-analog (DAC-used in audio/video devices) and analog-to-digital (ADC-used in music recordings, digital signal processing) requires monotonicity, which seems to be a sort of confusion. In these devices the output direction is supposed to be the same as that of input direction. In terms of monotonicity, as the input to the device increases (decreases) the output must also increase (decrease) accordingly. Monotonicity of monotone data is also involved in some other areas like, the level of blood uric acid in gout patients, data generated from stress and strain of a material, graphical display of Newton's law of cooling, medical diagnosis and economic forecasting.

* Corresponding author. Address: Department of Mathematics, University of Sargodha, Sargodha, Pakistan.
E-mail addresses: m.abbas@uos.edu.pk (M. Abbas), jamaluma@cs.usm.my (J.Md. Ali).

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The ordinary spline interpolating curve schemes are not appropriate to preserve the inherent shape feature (monotonicity) of shaped data as shown in the Figs. 1 and 4. Since the traditional interpolating schemes merely depend on the data points, so just a change in data points can cause a modification or an alteration in the shape of curves. For this reason the need for some proficient shape preserving interpolating schemes arises which not only preserve the shape of the input data but also pay heed to the underlying smoothness of curves. This motivates to come up with the scheme which can preserve the inherent shape feature of data. In this paper an interpolating scheme is developed which not only provide the smoothness in the shape preserving curves but also control the shape of data everywhere. The scheme preserves the monotonicity of monotone data only.

Some work [1–11] on shape preservation has been published in recent years. Abbas et al. [1] developed a C^1 piecewise rational cubic function with three shape parameters. Data dependent conditions were derived for shape parameter to maintain the shape of monotone data. Butt [2] produced the flexibility of cubic Hermite interpolant by insertion of extra knots rather than by particular choice of slopes. Convexity, monotonicity and positivity were considered in turn and the author derived the conditions on the positions of the knots and the slopes of the cubic Hermite interpolant at these knots to preserve the shape. Cripps and Hussain [3] developed a monotonicity preserving curve interpolant based on rational cubic Bézier basis functions. Sufficient conditions, expressed in terms of the weights, were derived for the C^1 cubic Hermite interpolation to preserve the monotonicity. Duan et al. [4] developed rational interpolation based on function values and also discussed constrained control of the interpolanting curves. They obtained conditions on function values for constraining the interpolating curves to lie above, below or between the given straight lines. The authors assumed suitable values of parameters to obtain C^2 continuous curve and the method worked for equally spaced data only.

Fiorot and Tabka [5] used C^2 cubic polynomial spline to preserve the shape of convex or monotone data. The authors obtained the values of derivative parameters by solving three systems of linear equations. Lamberti and Manni [7] presented and investigated the approximation order of a global C^2 shape preserving interpolating function using parametric cubic curves. The tension parameters were used to control the shape of curve. The authors derived the necessary and sufficient conditions for convexity whereas only sufficient conditions for positivity and monotonocity of data. Sarfraz et al. [9] developed a C^2 rational cubic spline with two families of free parameters for positive, monotone and convex curve. Sufficient data dependent constraints were made for free parameters to maintain the shape of data. The scheme did not provide a liberty to designer for the refinement of positivity, monotonicity and convexity preserving curves.

Wang and Tan [11] constructed a C^2 piecewise rational quartic spline function (quartic/linear) with two shape parameters. Sufficient conditions were derived for the derivative parameters to produce a monotonicity preserving curve for given monotone data. Piah and Unsworth [8] constructed a Bernstein-Bézier quartic rational (quartic/ linear) interpolant with single shape parameter to preserve the monotonicity of the data, without any error estimation of the interpolant. They improved the sufficient conditions for the derivative parameters and monotonicity region proposed by [11].

In this paper a rational interpolating scheme is developed which not only provides smoothness in the shape preserving curves but also control the shape of data everywhere. The problem of constructing a monotonicity-shape preserving curve through monotone data using C^2 rational cubic spline with three shape parameters is discussed. One of the substantial features of this spline which distinguish it from the ordinary splines is shape parameters. These shape parameters provide the opportunity to the designer to refine the shape of curves without changing the data. Simple data dependent sufficient constraints are derived for these shape parameters which guarantee to preserve the shape of data. This paper is a

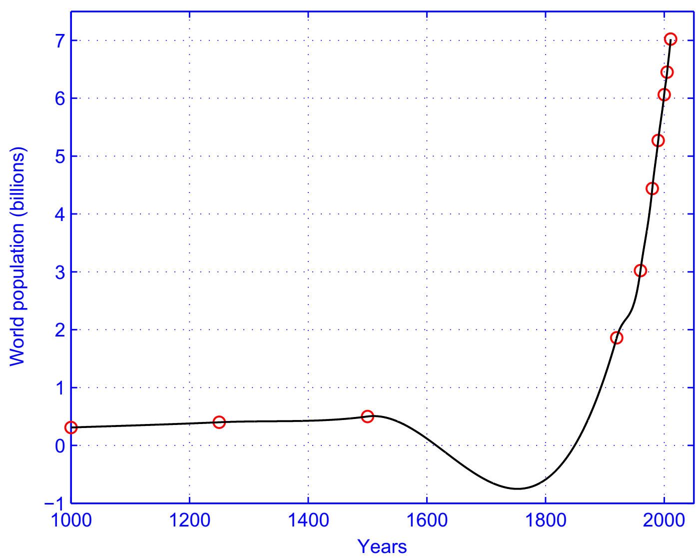

Fig. 1. Cubic Hermite curve.

contribution towards the advancement of such results that have been carried out by many authors. The technique used in this paper has many outstanding features.

- In [1,2], the smoothness of interpolant is C^1 while in this work the degree of smoothness is C^2.
- Experimental and interpolation error analysis suggest that the scheme is not only computationally economical but also produce a visually pleasant curve as compared to existing schemes [1,8,9,11].
- In [2], the author developed the schemes to achieve the desired shape of data by inserting extra knots between any two knots in the interval while we preserve the monotonicity by only imposing constraints on free parameters without any extra knots.
- In [4], the authors developed the scheme that works for equally spaced data only while the scheme developed in this paper is applicable both for equally and unequally spaced data.
- The authors [5] achieved the values of derivative parameters by solving the three systems of linear equations, which is computationally expensive as compared to methods developed in this paper where there exists only one tri-diagonal system of linear equations for finding the values of derivative parameters.
- In [9], the scheme does not allow the designer to refine the shape of curves as desired. Whereas, in this paper this job is done by introducing free parameters used in the description of C^2 rational cubic function.
- The scheme gives a noteworthy error bound $O(h^3)$ and optimal error $c_i = 0.0640$ when compared with the existing scheme.
- The proposed curve scheme is unique in its representation and it is equally applicable for the data with derivative or without derivatives.
- The rational quartic function is used by Wang and Tan [11] and improved sufficient conditions for the derivative parameters by Piah and Unsworth [8] to preserve the shape of monotone data while in this paper we have used rational cubic function (cubic/quadratic) to maintain the monotone curve through given monotone data.

This paper is organized as follows: In Section 2, a C^2 piecewise rational cubic function with three shape parameters is developed. Monotonicity-preserving C^2 interpolating rational cubic scheme is discussed in Section 3. Sufficient numerical examples are given in Section 4 and error estimation of interpolation is calculated in Section 5 to prove the worth of the scheme. The concluding remarks are presented to end the paper.

2. Rational cubic spline function

Let $\{(x_i, f_i) : i = 0, 1, 2, \ldots, n\}$ be the given set of data points such that $x_0 < x_1 < x_2 < \ldots < x_n$. A piecewise rational cubic function with three shape parameters, in each subinterval $I_i = [x_i, x_{i+1}],\ i = 0, 1, 2, \ldots, n-1$ is defined as:

$$S(x) = S_i(x) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i\eta_i}{q_i(\theta)} \tag{1}$$

Let $S'(x)$ and $S''(x)$ denote the first and second ordered derivatives with respect to x. The following interpolatory conditions are imposed for the C^2 continuity of the piecewise rational cubic function (1),

$$\begin{cases} S(x_i) = f_i, & S(x_{i+1}) = f_{i+1} \\ S'(x_i) = d_i, & S'(x_{i+1}) = d_{i+1} \\ S''(x_{i+}) = S''(x_{i-}), & i = 1, 2, \ldots, n-1 \end{cases} \tag{2}$$

From Eq. (2), the interpolating conditions produce the following system of linear equations and unknown coefficients (η_i) of equation (1).

$$\mu_i d_{i-1} + v_i d_i + \omega_i d_{i+1} = \lambda_i \tag{3}$$

with,

$$\begin{cases} \mu_i = \alpha_i\alpha_{i-1}h_i, \\ v_i = h_i\alpha_i(\alpha_{i-1} + \beta_{i-1} + \gamma_{i-1}) + h_{i-1}\beta_{i-1}(\alpha_i + \beta_i + \gamma_i), \\ \omega_i = \beta_i\beta_{i-1}h_{i-1}, \\ \lambda_i = \beta_{i-1}h_{i-1}(\alpha_i + 2\beta_i + \gamma_i)\Delta_i + \alpha_i h_i(2\alpha_{i-1} + \beta_{i-1} + \gamma_{i-1})\Delta_{i-1} \end{cases}$$

and

$$\begin{aligned} \eta_0 &= \alpha_i f_i, \\ \eta_1 &= f_i(2\alpha_i + \beta_i + \gamma_i) + \alpha_i h_i d_i, \\ \eta_2 &= f_{i+1}(\alpha_i + 2\beta_i + \gamma_i) - \beta_i h_i d_{i+1}, \\ \eta_3 &= \beta_i f_{i+1}, \end{aligned} \tag{4}$$

where $h_i = x_{i+1} - x_i$, $\theta = \frac{x-x_i}{h_i}$, $\Delta_i = (f_{i+1} - f_i)/h_i$, $\theta \epsilon [0,1]$ and α_i, $\beta_i > 0$, $\gamma_i \geq 0$ are shape parameters that are used to control the shape of interpolating curve and provide the designer liberty to refine the curve as desired. Let d_i denotes the derivative value at knots x_i that is used for the smoothness of curve.

The C^2 piecewise rational cubic function (1) is reformulated after using equation (4) as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} \tag{5}$$

with,

$$p_i(\theta) = \begin{cases} \alpha_i f_i (1-\theta)^3 + (f_i(2\alpha_i + \beta_i + \gamma_i) + \alpha_i h_i d_i)\theta(1-\theta)^2, \\ +(f_{i+1}(\alpha_i + 2\beta_i + \gamma_i) - \beta_i h_i d_{i+1})\theta^2(1-\theta) + \beta_i f_{i+1}\theta^3, \end{cases}$$

$$q_i(\theta) = (1-\theta)^2\alpha_i + \theta(1-\theta)(\gamma_i + \alpha_i + \beta_i) + \theta^2\beta_i.$$

Remark 1. The system of linear equations defined in Eq. (3) is a strictly tri-diagonal and has a unique solution for the derivatives parameters d_i, $i = 1, 2, \ldots, n-1$ for all α_i, $\beta_i > 0$ and $\gamma_i \geqslant 0$. Moreover, it is efficient to apply LU decomposition method to solve the system for the values of derivatives parameters $d_i' s$.

Remark 2. To make the rational cubic function smoother, C^2 continuity is applied at each knot. The system (3) involves $n-1$ linear equations while unknown derivative values are $n+1$. So, two more equations are required for unique solution. For this, we impose end conditions at end knots as:

$$S'(x_0) = d_0, \quad S'(x_n) = d_n. \tag{6}$$

Remark 3. For the values of shape parameters $\alpha_i = 1$, $\beta_i = 1$ and $\gamma_i = 0$ in each subinterval $I_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$, the rational cubic function reduces to standard cubic Hermite spline.

3. Monotonicity-preserving C^2 rational cubic spline interpolation

This section deals with the problem of shape preserving C^2 rational cubic function for monotone data. For this purpose we use rational cubic function (5) and impose conditions on shape parameters which assure to preserve the shape of data. This requires some further mathematical treatment so that the desired shape feature of the curve is attained.

Let $\{(x_i, f_i) : i = 0, 1, 2, \ldots, n\}$ be the given monotonically increasing data set i.e.

$$f_i \leqslant f_{i+1}, \quad i = 0, 1, 2, \ldots, n-1$$

or equivalently

$$\Delta_i \geqslant 0, \quad i = 0, 1, 2, \ldots, n-1.$$

For monotonicity, the necessary conditions are imposed on the derivative parameters as:

$$\begin{cases} \Delta_i \geqslant 0, & i = 0, 1, 2, \ldots, n-1, \\ d_i \geqslant 0, & i = 0, 1, 2, \ldots, n. \end{cases} \tag{7}$$

There arise two cases for monotonically increasing data.

Case 1. $\Delta_i = 0$.

In this case, $d_i = d_{i+1} = 0$, the interpolant $S(x)$ reduces to:

$$S_i(x) = f_i, \quad \forall x \in [x_i, x_{i+1}], \quad i = 0, 1, 2, \ldots, n-1,$$

which shows that the interpolant is monotone.

Case 2. $\Delta_i > 0$.

The C^2 rational cubic function (5) is monotonically increasing if,

$$S_i'(x) > 0, \quad \forall x \in [x_i, x_{i+1}].$$

The $S_i'(x)$ can be presented in the simpler form as:

$$S_i'(x) = \frac{\sum_{k=0}^{4}(1-\theta)^{4-k}\theta^k M_{k,i}}{(q_i(\theta))^2} \tag{8}$$

with,

$$\begin{aligned}
M_{0,i} &= \alpha_i^2 d_i,\\
M_{1,i} &= 2\alpha_i\{(\alpha_i+2\beta_i+\gamma_i)\Delta_i-\beta_i d_{i+1}\},\\
M_{2,i} &= M_{1,i}+M_{3,i}-(M_{0,i}+M_{4,i})+\gamma_i(\alpha_i+\beta_i+\gamma_i)\Delta_i-2\alpha_i\beta_i(d_i+d_{i+1}),\\
M_{3,i} &= 2\beta_i\{(2\alpha_i+\beta_i+\gamma_i)\Delta_i-\alpha_i d_i\},\\
M_{4,i} &= \beta_i^2 d_{i+1}.
\end{aligned}$$

Necessary conditions for monotonicity of rational cubic function are given as:

$$\begin{cases} d_i \geqslant 0,\\ d_{i+1} \geqslant 0,\\ M_{k,i} \geqslant 0, \quad k=0,1,2,3,4. \end{cases} \tag{9}$$

From Eq. (9) it is obvious that both $M_{0,i} > 0$ and $M_{4,i} > 0$.
$M_{1,i} > 0$ if

$$\gamma_i > \frac{d_{i+1}\beta_i - \Delta_i(\alpha_i+2\beta_i)}{\Delta_i}, \tag{10}$$

$M_{3,i} > 0$ if

$$\gamma_i > \frac{d_i\alpha_i - \Delta_i(2\alpha_i+\beta_i)}{\Delta_i}. \tag{11}$$

These choices in Eqs. (10) and (11) satisfy $M_{2,i} > 0$.

To control the shape of the curve and preserve the shape feature of data according to the demand the constraints for shape parameters in Eqs. (10) and (11) can be rewritten as:

$$\begin{cases} \alpha_i > 0, \quad \beta_i > 0,\\ \gamma_i > max\left\{0, \frac{d_{i+1}\beta_i-\Delta_i(\alpha_i+2\beta_i)}{\Delta_i}, \frac{d_i\alpha_i-\Delta_i(2\alpha_i+\beta_i)}{\Delta_i}\right\}. \end{cases} \tag{12}$$

The above result can be rewritten as:

$$\begin{cases} \alpha_i > 0, \quad \beta_i > 0,\\ \gamma_i = m_i + max\left\{0, \frac{d_{i+1}\beta_i-\Delta_i(\alpha_i+2\beta_i)}{\Delta_i}, \frac{d_i\alpha_i-\Delta_i(2\alpha_i+\beta_i)}{\Delta_i}\right\}, \quad m_i > 0. \end{cases} \tag{13}$$

The above discussion can be summarized as:

Theorem 1. *The rational cubic function(5)preserves theC^2monotone curve of monotone data in each subinterval$[x_i, x_{i+1}]$if and only if the shape parametersα_i, $\beta_i$and$\gamma_i$satisfy the Eq.(13).*

4. Numerical examples

Example 1. The world population is computed by the summing up all the living human on Earth. The data set taken in Table 1 represents the world population (in billions) as estimated by United States Census Bureau (USCB) from the year 1000 to the year 2011. The x-values represent the years and f-values indicate the population in that particular year. One can observe that the data set is monotone. The world population inherently monotone and we require the resulted rational cubic spline to preserve this property. Fig. 1, drawn with cubic Hermite interpolant [6], does not preserve the monotonicity of monotone data taken in Table 1. On the other hand, Fig. 2 when drawn by monotonicity preserving C^2 rational cubic interpolation developed in Section 3 preserves the shape of monotone data everywhere. The effect of shape parameters can be seen by noting the difference in smoothness of the monotone curves in Fig. 2 and Fig. 3. Table 2 demonstrates the numerical values which computed from the developed scheme of Fig. 3.

Example 2. A data set in Table 3 shows the observation of an experiment of watering the Great Northern beans. A solution with a combination of chemical flake (Potassium Hydroxide) and distilled water with a pH of 8.5 was used to water the beans. The beans were placed in vat, with a unique water and chemical solution. After 40 days, The beans plants were removed and weighted to see the effects of the solution, where the x-values are the days and f-values are the height of the beans. One can observed that the resulted data is monotone. Fig. 4, generated by cubic Hermite interpolant [6], does not preserve the monotonicity of monotone data sets taken in Table 3. On the other hand, Fig. 5, drawn with monotonicity preserving C^2 rational cubic interpolation developed in Section 3 preserves the shape of monotone data everywhere. A prominent difference in the smoothness with a visually pleasant view can be seen in these figures (Fig. 5 and Fig. 6) due to the liberty bestowed to the designer on the values of shape parameters. Table 4 demonstrates the numerical results which computed from the developed scheme of Fig. 6.

Table 1
A monotone data set.

i	x_i	f_i
1	1000	0.31
2	1250	0.40
3	1500	0.50
4	1920	1.86
5	1960	3.02
6	1980	4.44
7	1990	5.27
8	2000	6.06
9	2005	6.45
10	2011	7.02

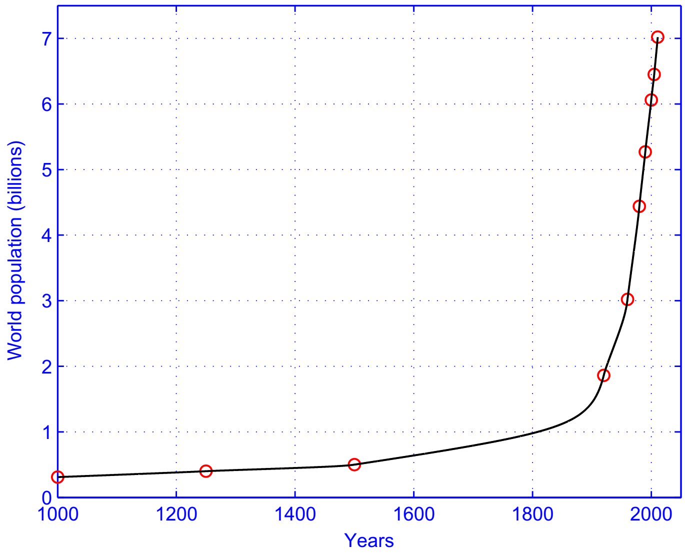

Fig. 2. Monotonicity-preserving rational cubic function with $\alpha_i = 0.25,\ \beta_i = 0.25$.

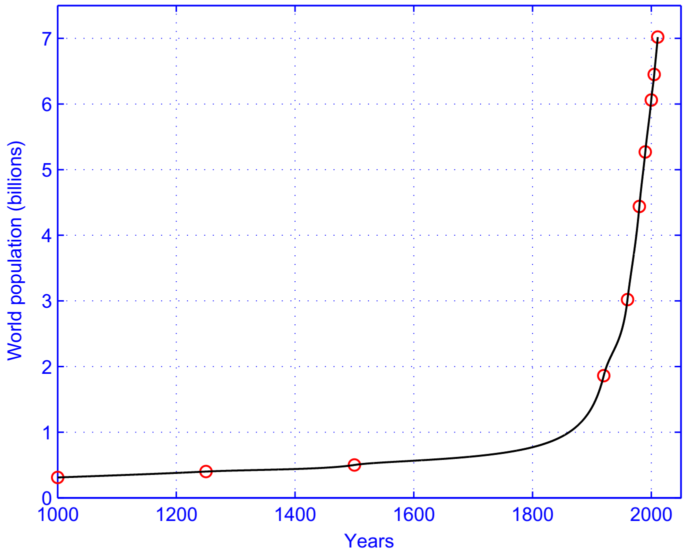

Fig. 3. Monotonicity-preserving rational cubic function with $\alpha_i = 2.5,\ \beta_i = 2.5$.

5. Interpolation error estimation

In this section, the error of interpolation is estimated when the function being interpolated is $f(x) \in C^3[x_0, x_n]$, using rational cubic function (5). It is to mention that the rational cubic function constructed in Section 2 is local, which allows

Table 2
Numerical results of Fig. 3.

i	d_i	Δ_i	α_i	β_i	γ_i
1	0.0003	0.0003	2.5	2.5	0.2667
2	0.0004	0.0004	2.5	2.5	6.2041
3	0.0014	0.0032	2.5	2.5	23.7940
4	0.0362	0.0290	2.5	2.5	4.3990
5	0.0752	0.0710	2.5	2.5	1.0290
6	0.1017	0.0830	2.5	2.5	0.6661
7	0.0931	0.0790	2.5	2.5	0.8364
8	0.0977	0.0780	2.5	2.5	1.1025
9	0.0990	0.0950	2.5	2.5	0.1468
10	0.1042	...	...	...	...

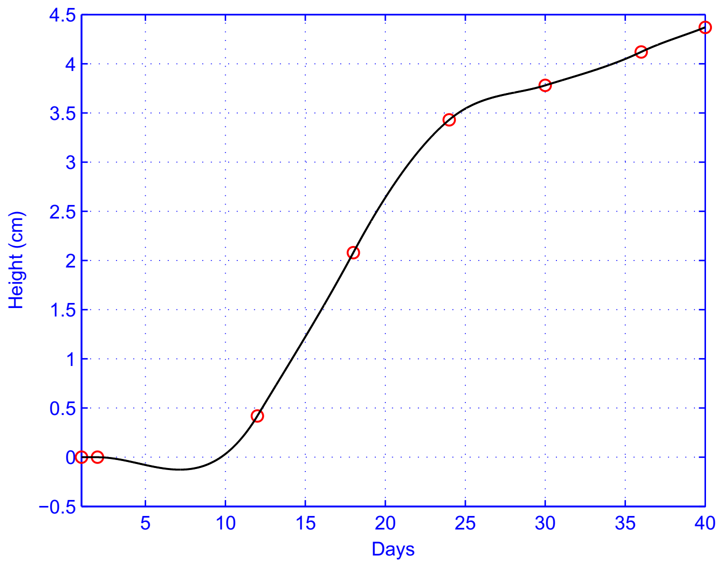

Fig. 4. Cubic Hermite curve.

Table 3
Height (in centimeters) of great northern beans.

i	1	2	3	4	5	6	7	8
x_i	1	2	12	18	24	30	36	40
f_i	0	0	0.42	2.08	3.43	3.78	4.12	4.37

investigating the error in an arbitrary subinterval $I_i = [x_i, x_{i+1}]$ without loss of generality. Using Peano Kernel Theorem [10] the error of interpolation in each subinterval $I_i = [x_i, x_{i+1}]$ is defined as:

$$R[f] = f(x) - S_i(x) = \frac{1}{2}\int_{x_i}^{x_{i+1}} f^{(3)}(\tau)R_x[(x-\tau)_+^2]d\tau. \tag{14}$$

It is assumed that the function being interpolated is $f(x) \in C^3[x_0, x_n]$. The absolute error in each subinterval $I_i = [x_i, x_{i+1}]$ is:

$$|f(x) - S_i(x)| \leqslant \frac{1}{2}\|f^{(3)}(\tau)\| \int_{x_i}^{x_{i+1}} \left|R_x[(x-\tau)_+^2]\right| d\tau, \tag{15}$$

where,

$$R_x[(x-\tau)_+^2] = \begin{cases} r(\tau, x), & x_i < \tau < x, \\ s(\tau, x), & x < \tau < x_{i+1} \end{cases} \tag{16}$$

with,

$$r(\tau, x) = (x-\tau)^2 - \frac{\left[\left\{(\alpha_i + 2\beta_i + \gamma_i)(x_{i+1}-\tau)^2 - 2h_i\beta_i(x_{i+1}-\tau)\right\}\theta^2(1-\theta) + \beta_i(x_{i+1}-\tau)^2\theta^3\right]}{q_i(\theta)},$$

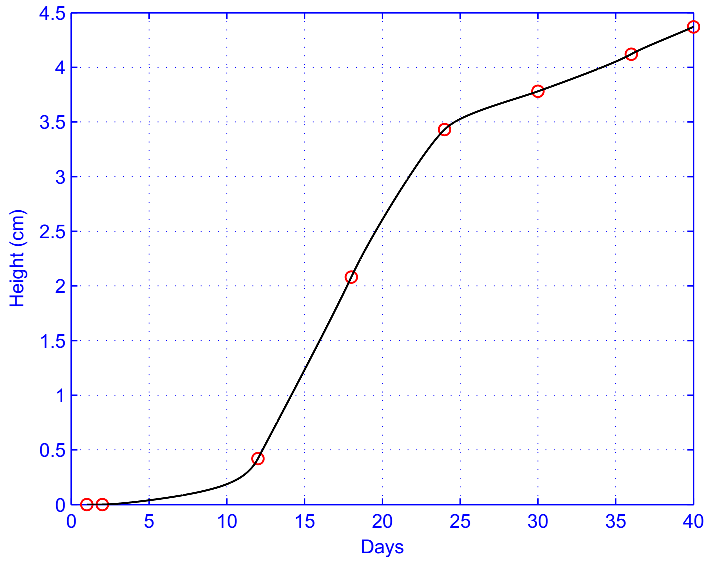

Fig. 5. Monotone C^2 rational cubic curve with $\alpha_i = 0.5,\ \beta_i = 0.5$.

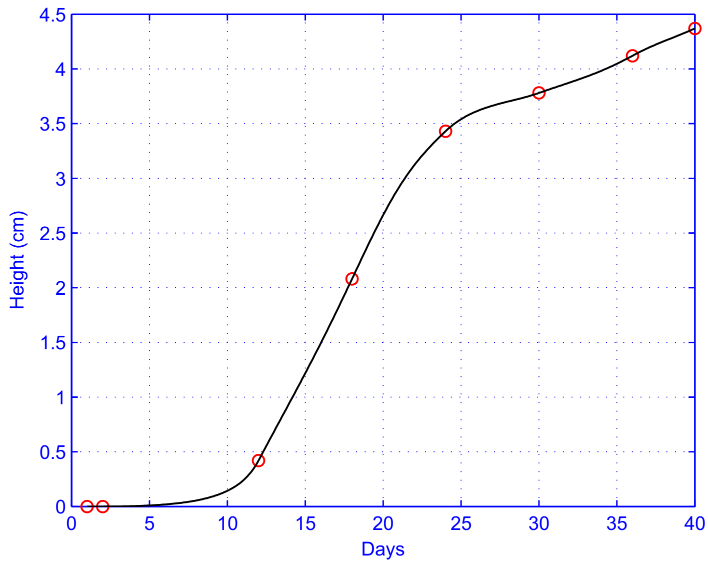

Fig. 6. Monotone C^2 rational cubic curve with $\alpha_i = 2.5,\ \beta_i = 2.5$.

Table 4
Numerical results of Fig. 6.

i	1	2	3	4	5	6	7	8
d_i	0	0	0.2686	0.3030	0.1357	0.0507	0.0773	0.0648
Δ_i	0	0.0420	0.2766	0.2250	0.0583	0.0566	0.0625	...
α_i	2.5	2.5	2.5	2.5	2.5	2.5	2.5	...
β_i	2.5	2.5	2.5	2.5	2.5	2.5	2.5	...
γ_i	0.05	16.99	0.7497	0.3138	4.9878	1.4720	1.5283	...

$$s(\tau,x) = -\frac{\left[\left\{(\alpha_i+2\beta_i+\gamma_i)(x_{i+1}-\tau)^2 - 2h_i\beta_i(x_{i+1}-\tau)\right\}\theta^2(1-\theta) + \beta_i(x_{i+1}-\tau)^2\theta^3\right]}{q_i(\theta)},$$

where $R_x[(x-\tau)_+^2]$ is called the Peano Kernel of integral. To derive the error analysis, first of all we need to examine the properties of the kernel functions $r(\tau,x)$ and $s(\tau,x)$, and then to find the values of following integrals

$$\int_{x_i}^{x_{i+1}} \left|R_x[(x-\tau)_+^2]\right| d\tau = \int_{x_i}^{x} |r(\tau,x)| d\tau + \int_{x}^{x_{i+1}} |s(\tau,x)| d\tau, \tag{17}$$

By simple computation, the roots of $r(x,x)$ in $[0,1]$ are: $\theta = 0,\ \theta = 1$ and $\theta^* = 1 - \beta_i/\rho_i,\ \rho_i = \alpha_i + \beta_i + \gamma_i$. The roots of $r(\tau,x) = 0$ are: $\tau_k = x - \frac{\theta h_i(\theta\rho_i + (-1)^{1+k}H)}{\alpha_i + \theta\rho_i},\ k = 1,2$

where,

$$H = \sqrt{\alpha_i^2 + \gamma_i(\alpha_i + \rho_i\theta)}.$$

The roots of $s(\tau,x) = 0$ are: $\tau_3 = x_{i+1}, \tau_4 = x_{i+1} - \frac{2(1-\theta)\beta_i h_i}{\beta_i + (1-\theta)\rho_i}$.

Following three cases arise:

Case 3. For $0 \leqslant \frac{\beta_i}{\rho_i} \leqslant 1,\ 0 < \theta < \theta^*$, the Eq. (15) takes the form

$$\begin{aligned} |f(x) - S_i(x)| &\leqslant \frac{1}{2}\|f^{(3)}(\tau)\| \int_{x_i}^{x_{i+1}} \left|R_x[(x-\tau)_+^2]\right| d\tau = \frac{1}{2}\|f^{(3)}(\tau)\| \left\{ \int_{x_i}^{x} (-r(\tau,x)) d\tau + \int_{x}^{\tau_4} (-s(\tau,x)) d\tau + \int_{\tau_4}^{x_{i+1}} s(\tau,x) d\tau \right\} \\ &= \|f^{(3)}(\tau)\| \xi_1(\alpha_i, \beta_i, \gamma_i, \theta), \end{aligned}$$

where,

$$\xi_1(\alpha_i,\beta_i,\gamma_i,\theta) = \begin{cases} \frac{h_i^3(1-\theta)^3\theta^2}{6q_i(\theta)}\{\rho_i(1-\theta) - 2\beta_i\} + \frac{h_i^3(1-\theta)\theta^2}{6q_i(\theta)}(\rho_i - 2\beta_i) \\ + \frac{4\beta_i^3 h_i^3\theta^2(1-\theta)^3}{3q_i(\theta)(\beta_i + (1-\theta)\rho_i)^2} + \frac{h_i^3(1-\theta)^3\theta^2}{6q_i(\theta)}\{2\beta_i - \rho_i(1-\theta)\} \\ + \frac{\beta_i h_i^3\theta^3}{6q_i(\theta)} - \frac{\theta^3 h_i^3}{6} \end{cases}. \tag{18}$$

Case 4. For $0 \leqslant \frac{\beta_i}{\rho_i} \leqslant 1,\ \theta^* < \theta < 1$, the Eq. (15) takes the form

$$\begin{aligned} |f(x) - S_i(x)| &\leqslant \frac{1}{2}\|f^{(3)}(\tau)\| \int_{x_i}^{x_{i+1}} \left|R_x[(x-\tau)_+^2]\right| d\tau = \frac{1}{2}\|f^{(3)}(\tau)\| \left\{ \int_{x_i}^{\tau_1} (-r(\tau,x)) d\tau + \int_{\tau_1}^{x} r(\tau,x) d\tau + \int_{x}^{x_{i+1}} s(\tau,x) d\tau \right\} \\ &= \|f^{(3)}(\tau)\| \xi_2(\alpha_i, \beta_i, \gamma_i, \theta), \end{aligned}$$

where,

$$\xi_2(\alpha_i,\beta_i,\gamma_i,\theta) = \begin{cases} \frac{(\rho_i\theta - H)^3\theta^3 h_i^3}{3(\alpha_i + \theta\rho_i)^3} - \frac{h_i^3\theta^2}{3q_i(\theta)}\left[(1-\theta) + \frac{\theta(\rho_i\theta - H)}{\alpha_i + \theta\rho_i}\right]^3((1-\theta)\rho_i + \beta_i) \\ + \frac{h_i^3\beta_i\theta^2(1-\theta)}{q_i(\theta)}\left[(1-\theta) + \frac{\theta(\rho_i\theta - H)}{\alpha_i + \theta\rho_i}\right]^2 + \frac{h_i^3\theta^2(1-\theta)}{6q_i(\theta)}(\rho_i - 2\beta_i) \\ - \frac{\theta^3 h_i^3}{6} + \frac{h_i^3\theta^2(1-\theta)^3}{6q_i(\theta)}(\rho_i(1-\theta) - 2\beta_i) + \frac{h_i^3\beta_i\theta^3}{6q_i(\theta)} \\ + \frac{h_i^3\theta^2(1-\theta)^3}{6q_i(\theta)}(2\beta_i - \rho_i(1-\theta)) \end{cases}. \tag{19}$$

Case 5. For $\frac{\beta_i}{\rho_i} > 1,\ 0 < \theta < 1$, the Eq. (15) takes the form

$$|f(x) - S_i(x)| \leqslant \frac{1}{2}\|f^{(3)}(\tau)\| \int_{x_i}^{x_{i+1}} \left|R_x[(x-\tau)_+^2]\right| d\tau = \frac{1}{2}\|f^{(3)}(\tau)\| \left\{ \int_{x_i}^{x} (r(\tau,x)) d\tau + \int_{x}^{x_{i+1}} s(\tau,x) d\tau \right\} = \|f^{(3)}(\tau)\| \xi_3(\alpha_i, \beta_i, \gamma_i, \theta),$$

where,

$$\xi_3(\alpha_i,\beta_i,\gamma_i,\theta) = \begin{cases} \frac{\theta^2 h_i^3}{6q_i}\{\rho_i(1-\theta) - \beta_i(2-3\theta)\} - \frac{\theta^3 h_i^3}{6} \\ + \frac{h_i^3\theta^2(1-\theta)^3}{6q_i(\theta)}(\rho_i(1-\theta) - 2\beta_i) + \frac{h_i^3\theta^2(1-\theta)^3}{6q_i(\theta)}(2\beta_i - \rho_i(1-\theta)) \end{cases}. \tag{20}$$

Theorem 2. *The error of interpolating rational cubic function (5), for $f(x) \in C^3[x_0, x_n]$, in each subinterval $I_i = [x_i, x_{i+1}]$ is*

$$\begin{aligned} &|f(x) - S_i(x)| \leqslant \frac{1}{2}\|f^{(3)}(\tau)\| \int_{x_i}^{x_{i+1}} \left|R_x[(x-\tau)_+^2]\right| d\tau = \|f^{(3)}(\tau)\| c_i, \\ &c_i = \max_{0 \leqslant \theta \leqslant 1} \xi(\alpha_i, \beta_i, \gamma_i, \theta), \end{aligned}$$

Table 5
Values of c_i for $\alpha_i = 0.5,\ \beta_i = 0.5$ with some values of γ_i.

i	α_i	β_i	γ_i	c_i
1	0.5	0.5	0.01	0.0105
2	0.5	0.5	0.1	0.0119
3	0.5	0.5	0.5	0.0180
4	0.5	0.5	1.0	0.0243
5	0.5	0.5	1.5	0.0292
6	0.5	0.5	5.0	0.0458
7	0.5	0.5	10.0	0.0533
8	0.5	0.5	100.0	0.0628
9	0.5	0.5	500.0	0.0638
10	0.5	0.5	700.0	0.0639
11	0.5	0.5	1000.0	0.0640
12	0.5	0.5	1500.0	0.0640
13	0.5	0.5	2000.0	0.0640

Table 6
Values of c_i for $\alpha_i = 2.5,\ \beta_i = 2.5$ with some values of γ_i.

i	α_i	β_i	γ_i	c_i
1	2.5	2.5	0.01	0.0104
2	2.5	2.5	0.1	0.0107
3	2.5	2.5	0.5	0.0119
4	2.5	2.5	1.0	0.0151
5	2.5	2.5	1.5	0.0243
6	2.5	2.5	5.0	0.0331
7	2.5	2.5	10.0	0.0582
8	2.5	2.5	100.0	0.0628
9	2.5	2.5	500.0	0.0632
10	2.5	2.5	700.0	0.0634
11	2.5	2.5	1000.0	0.0637
12	2.5	2.5	1500.0	0.0638
13	2.5	2.5	2000.0	0.0638

Table 7
Values of c_i for $\alpha_i = 100.0,\ \beta_i = 100.0$ with some values of γ_i.

i	α_i	β_i	γ_i	c_i
1	100.0	100.0	0.01	0.0104
2	100.0	100.0	0.1	0.0104
3	100.0	100.0	0.5	0.0104
4	100.0	100.0	1.0	0.0104
5	100.0	100.0	1.5	0.0105
6	100.0	100.0	5.0	0.0108
7	100.0	100.0	10.0	0.0112
8	100.0	100.0	100.0	0.0180
9	100.0	100.0	500.0	0.0363
10	100.0	100.0	700.0	0.0410
11	100.0	100.0	1000.0	0.0458
12	100.0	100.0	1500.0	0.0505
13	100.0	100.0	2000.0	0.0533

where

$$\xi(\alpha_i,\beta_i,\gamma_i,\theta) = \begin{cases} \max \xi_1(\alpha_i,\beta_i,\gamma_i,\theta) & 0 \leqslant \theta \leqslant \theta^*, \\ \max \xi_2(\alpha_i,\beta_i,\gamma_i,\theta) & \theta^* \leqslant \theta \leqslant 1, \\ \max \xi_3(\alpha_i,\beta_i,\gamma_i,\theta) & 0 \leqslant \theta \leqslant 1, \end{cases}$$

where, $\xi_1(\alpha_i,\beta_i,\gamma_i)$, $\xi_2(\alpha_i,\beta_i,\gamma_i)$ *and* $\xi_3(\alpha_i,\beta_i,\gamma_i)$ *are defined in Section* 5 *by* Eqs. (18)–(20) *respectively.*

Remark 4. The rational cubic interpolation (5) reduces to standard cubic Hermite interpolant for the values of shape parameters $\alpha_i = 1,\ \beta_i = 1$ and $\gamma_i = 0$. In this special case the functions $\xi_1(\alpha_i,\beta_i,\gamma_i,\theta)$, $\xi_2(\alpha_i,\beta_i,\gamma_i,\theta)$ and $\xi_3(\alpha_i,\beta_i,\gamma_i,\theta)$ become

$$\xi_1(\alpha_i,\beta_i,\gamma_i,\theta) = 4\theta^2(1-\theta)^3/3(3-2\theta)^2, \quad 0 \leqslant \theta \leqslant 1/2$$
$$\xi_2(\alpha_i,\beta_i,\gamma_i,\theta) = 4\theta^3(1-\theta)^2/3(1+2\theta)^2, \quad 1/2 \leqslant \theta \leqslant 0$$
$$\xi_3(\alpha_i,\beta_i,\gamma_i,\theta) = 0 \quad 0 \leqslant \theta \leqslant 1$$

It follows that the error coefficient $c_i = 1/96$ which is standard value for cubic Hermite spline interpolation.

Theorem 3. *For any given values of shape parameters*α_i, $\beta_{i>0}$ *and* $\gamma_{i\geq 0}$, *the optimal error*c_i *in* Theorem 2 *satisfies* $0 < c_i \leqslant 0.0640$.

6. Concluding remarks

In this paper, a C^2 rational cubic function has been developed for the smooth and pleasing visualization of shaped data. Three families of shape parameters are used in its representations to maintain the shapes of monotone data. Data dependent sufficient conditions are imposed on single shape parameter to insure the monotonicity. Remaining two of these shape parameters provide liberty to designer to easily control the shape of the curve by simply their values. The proposed interpolating method is appropriate to such shape preserving problems in which only data points are given. No additional knots are inserted between any two knots in the interval where the interpolant loses the desired shape of the data. The values of derivative parameters are calculated by solving the single system of linear equations, which is computationally economically as compared to scheme developed by Fiorot and Tabka [5] in which three tri-diagonal systems of linear equations for finding the values of derivative parameters.

The proposed scheme is C^2, smoother, local in comparison with global scheme [7], computationally economical and visually pleasing as compared to schemes developed in [1,8,9,11]. It works for both equally and unequally spaced data. Experimental and interpolation error analysis suggest that the proposed C^2 rational cubic interpolation appear to produce smoother graphical results. The optimal error is calculated with some fixed values of free parameters and different values of constrained parameters which are shown in Table 5, Table 6 and Table 7. The comparison of error between the proposed C^2 scheme and the existing schemes is notable.

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