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Shape preserving positive surface data visualization by spline functions

Muhammad Abbas[1],Ahmad Abd Majid[2]

Mohd Nain Hj Awang[3],Jamaludin Md Ali[4]

[124]School of Mathematical Sciences,
[3]School of Distance Education,
University Sains Malaysia,11800 USM,Penang, Malaysia
[2]majid@cs.usm.my; [3]mnain@usm.my; [4]jamaluma@cs.usm.my

Abstract

In this paper, a visualization of positive data is made in such a fashion where it presents a smooth, pleasant and eye catching view of the positive surface to viewer. An attempt has been made in order to extend a rational cubic function into a rational bi-cubic function for the preservation of positive data arranged over rectangular grid in the vision of positive surface. Moreover, rational bi-cubic function has six free parameters which are arranged in such a manner, two of them are constrained parameters to preserve the positivity of data and remaining four are left for user choice to refine the positive surface as desired. The scheme under discussion is C^1, simple, local, computationally economical and time saving as compared to existing schemes. Numerical examples are provided to demonstrate that the anticipated scheme is interactive and smooth.

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1 Introduction

Spline interpolation is very powerful tool in Computer Graphics, Computer Aided Geometric Design and Engineering as well. Therefore, in these fields,

[1]Corresponding Author: m.abbas@uos.edu.pk (Muhammad Abbas)

it is often needed to generate a positivity preserving interpolating curves and surfaces according to the given positive data. The aspiration of this paper is to preserve the hereditary attribute that is the positivity of data. The most important attribute of interpolation schemes that build good judgment to study is its positivity. In particular, we have discussed the problems of data visualizing where most important constrained must be established.

Ordinary spline scheme is also a very authoritative tool in designing. The rational spline, specifically the rational cubic spline provides significant results in shape control, design and preserve the data. Graphical representation of visualization of scientific data provides an understanding and insight into data. The research in fields of computer graphics, data plots, meteorological monitoring, maps, image processing, drawing, high computing performance and many other fields are based on scientific visualization [6]. Positivity is a very important aspect of shape. The entities only taken are positive values in many physical situations. For instance, probability distribution and population graph is always printed in positive values. It is also noticed in the inspection of gas leakage during process of experiments and amount of rainfall, area, density and concentration of sugar in blood [8]. Therefore, a responsibility is empowered to positive interpolation to preserve the shape of positive data.

Many authors have solved the problems of visualization of the positive data and the references therein. But most of them have solved the problems of scattered data, only some authors have consider the arrangement of positive data over a rectangle grid, see for details ([1],[4],[7],[8]). The above developed schemes are discussed in following manner. Brodlie, et al [1] developed sufficient conditions in the term of the first and mixed partial derivatives at the rectangular grid points to preserve the shape of positive data by using a piecewise bi-cubic function arranging the data over rectangular mesh. Butt and Brodlie [9] developed a piecewise rational cubic interpolant to preserve the shape of positive data by interval subdivision technique. In [9], the authors inserted extra knots in the interval where the function lost the positivity. Asim and Brodlie [3] developed a piecewise rational cubic function to preserve the positivity of positive data. In [3], the function does not preserve the positivity in the interval then the authors inserted extra knot to improve this matter.

Goodman, et al [10] constructed non planar shape preserving interpolating curve scheme. They obtained a curve through an optimization process involving some fairness criteria, in order to achieve curve by G^2 piecewise rational cubic function. Goodman surveyed the shape preserving interpolating algorithms for 2D data [11]. Hussain and Maria [6] constructed a local positivity preserving scheme for positive data by making constraints on free parameters in the account of rational bi-cubic partially blended patches. In [6], the authors also developed the constraints on parameters to preserve the shape of data that is lying above the plane. The user did not enjoy the liberty to refine the curves and surface as desired in these schemes.

Srafraz, et al [4] developed a rational cubic function with two free parameters to preserve the shape of data. In [4], the authors extended the rational cubic function into partially blended rational bi-cubic function to preserve the shape of positive surface data. They derived data dependent constraints on these parameters to preserve the shape of the positive data. The schemes did not require the modification of data and no freedom for user to refine the curves and surfaces. May be, these schemes are unsuitable for interactive design. Sarfraz and Hussain [5] developed a piecewise rational cubic function with two free parameters to preserve the positivity of positive data. They derived data dependent sufficient conditions on these shape parameters which provide very pleasing curves but have no freedom to user for the refinement of curves as desire. Hussain and Maria [7] developed a rational bi-cubic spline to preserve the shape of positive surface data and the surface data that was lying above a plane. In [7], Simple data depended constraints were derived on free parameters to preserve the shape of positive surface data and constraint surface data. This scheme did not give the flexibility to the user for the refinement of the surfaces. Hussain and Sarfraz [8] developed a piecewise rational cubic function with four families of parameters to preserve the shape of positive data. Further, the authors extended a rational cubic function into rational bi-cubic function with eight free parameters arranging a date over the rectangular grid. In [8], simple sufficient conditions were derived on these free parameters to preserve the shape of positive data. The constrained parameters are dependent to each other in rational bi-cubic positive surface data scheme. The scheme seems to be computationally expansive.

This paper is related to shape preserving positive surface data by using the rational bi-cubic functions. In this scheme we extend the rational cubic function with three free parameters [2] into rational bi-cubic function with six free parameters to preserve the shape of positive surface data arranged over the rectangular grid. Out of these six parameters, two are constrained parameters to preserve the shape of positive surface data and the remaining four are left for user choice for the refinement of surfaces as desired. Simple data dependent sufficient constraints are derived on these parameters which guarantee to preserve the shape of data and the result is pleasant and smooth. Our scheme has a number of attributes over the existing schemes. It is C^1 surface scheme. There is no need of extra knots to insert in the interval where positivity is not preserved. The constrained parameters are not dependent on each other unlike [8] because the scheme has only one constrained parameter. This scheme is local, simple, computationally economical and time saving as compared to existing schemes ([4], [5], [7]). Our method looks like to a proposed method in [8] but have short summary of constraints on single shape parameter and allow to the user to refine surfaces as desired.

The rest of this paper is completed in such a away. A rational cubic function is rewritten in section (2) . In section (3), the rational cubic function is extended into a rational bi-cubic function with six free parameters. Shape preserving 3D positive data rational bi-cubic interpolation is constructed in section (4) which generates the pleasant and attractive positive surface. Derivative Approximation method is discussed in section (5). Some Numerical Demonstration of 3D positive data for surface are given in section (6). Finally, the conclusion of this work is discussed in section (7).

2 Review of rational cubic function

Let $\{(x_i, f_i), i = 0, 1, 2, ..., n\}$ be the give set of data points defined over the arbitrary interval $[a, b]$, where $a = x_0 < x_1 < x_2 < ... < x_n = b$. The piecewise rational cubic function [2] with three shape parameters in each subinterval $I_i = [x_i, x_{i+1}]\,, i = 0, 1, 2, ..., n-1$ is defined as:

$$S(x) = S_i(x) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i A_i}{(1-\theta)^3 u_i + \theta(1-\theta)\,w_i + \theta^3 v_i}, \tag{1}$$

where $\theta = \frac{x-x_i}{h_i}, \theta \in [0,1]\,, h_i = x_{i+1} - x_i$ and u_i, v_i, w_i are the positive free parameters. The following interpolatory conditions are bestow for the C^1 continuity of the piecewise rational cubic function (1),

$$S_i(x_i) = f_i\,, \qquad S_i(x_{i+1}) = f_{i+1}\,, \qquad S_i'(x_i) = d_i\,, \qquad S_i'(x_{i+1}) = d_{i+1} \tag{2}$$

where $S_i'(x)$ denotes the derivative with respect to x and d_i denotes the derivative values are estimated at knots. The C^1 continuity conditions defined in (2) claims the following values of unknowns $A_i,\ i = 0, 1, 2, 3$

$$A_0 = u_i f_i, A_1 = w_i f_i + u_i h_i d_i, A_2 = w_i f_{i+1} - v_i h_i d_{i+1}, A_3 = v_i f_{i+1} \tag{3}$$

The C^1 piecewise rational cubic function (1) is reformulated after using(3) as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{4}$$

with,

$$p_i(\theta) = u_i f_i(1-\theta)^3 + (w_i f_i + u_i h_i d_i)\theta(1-\theta)^2 + (w_i f_{i+1} - v_i h_i d_{i+1})\theta^2(1-\theta) + v_i f_{i+1}\theta^3,$$

$$q_i(\theta) = u_i(1-\theta)^3 + w_i\theta(1-\theta) + v_i\theta^3.$$

When we set the values of free parameters as $u_i = 1,\ v_i = 1 \,\&\, w_i = 3$, then it is very interesting to note that the C^1 piecewise rational cubic function reduces to standard cubic Hermite spline.

Abbas, et al [2] developed the following result as:

Theorem 2.1. *The C^1 piecewise rational cubic function (4) to preserve the positivity of positive data, if in each subinterval $I_i = [x_i, x_{i+1}], i = 0, 1, 2, ..., n.$, the constrained parameter w_i satisfy the following sufficient conditions.*

$$w_i > max \left\{ 0, \frac{3u_i f_i - u_i h_i d_i}{f_i}, \frac{3v_i f_{i+1} + v_i h_i d_{i+1}}{f_{i+1}} \right\}, u_i > 0, v_i > 0.$$

The above constrained can be rearranged as:

$$w_i = r_i + max \left\{ 0, \frac{3u_i f_i - u_i h_i d_i}{f_i}, \frac{3v_i f_{i+1} + v_i h_i d_{i+1}}{f_{i+1}} \right\}, r_i > 0, u_i > 0, v_i > 0.$$

Example 2.2. *A positive data set taken in Table (1) is browed from [4]. Figure (1) is generated by Cubic Hermite Spline scheme, it is to be noted that does not preserve the shape of positive data. To improve this flaw, Abbas, et al [2] developed a rational cubic function with three free parameters to preserve the positivity of positive data which looks pleasant and smooth as shown in Figure (2) with free parameters $u_i =$ 0.1 & $v_i =$ 0.1 and Figure (3) with free parameters $u_i =$ 0.5 & $v_i =$ 0.5.*

Table 1: Positive Data set

i	1	2	3	4	5	6	7
x_i	0	2	4	10	28	30	32
f_i	20.8	8.8	4.2	0.5	3.9	6.2	9.6

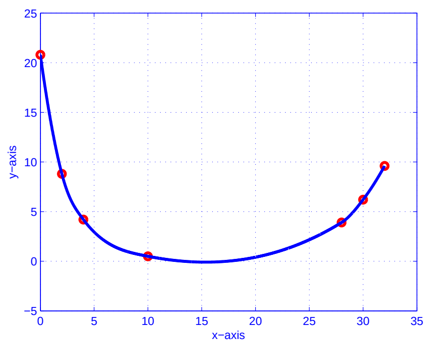

Figure 1: Cubic Hermite Spline

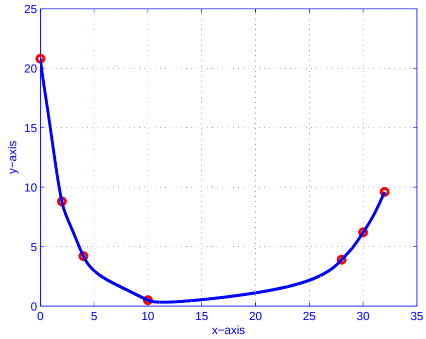

Figure 2: Rational cubic function with free parameters $u_i = 0.1$ & $v_i = 0.1$.

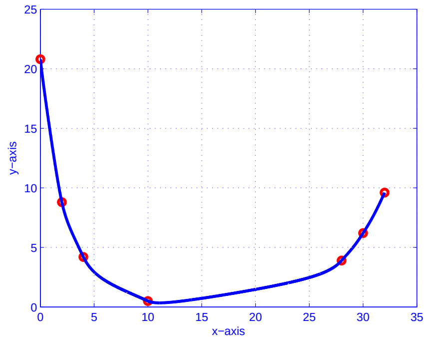

Figure 3: Rational cubic function with free parameters $u_i = 0.5$ & $v_i = 0.5$.

3 Rational bi-cubic spline function

A C^1 piecewise rational cubic function (4) extended into rational bi-cubic function $S(x,y)$ over the rectangular Domain $\Omega = [x_0, x_m] \times [y_0, y_n]$. Let $\pi : a = x_0 < x_1 < x_2 < ... < x_m = b$ be the partition of arbitrary interval $[a,b]$ & $\hat{\pi} : c = y_0 < y_1 < y_2 < ... < y_n = d$ be the partition of arbitrary interval $[c,d]$. The rational bi-cubic function in each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0,1,2,...,m-1$; $j = 0,1,2,...,n-1$ is defined as:

$$S(x,y) = S_{i,j}(x,y) = B_i(\theta) F(i,j) \hat{B}_j^T(\phi), \tag{5}$$

where

$$F(i,j) = \begin{pmatrix} F_{i,j} & F_{i,j+1} & F^y_{i,j} & F^y_{i,j+1} \\ F_{i+1,j} & F_{i+1,j+1} & F^y_{i+1,j} & F^y_{i+1,j+1} \\ F^x_{i,j} & F^x_{i,j+1} & F^{xy}_{i,j} & F^{xy}_{i,j+1} \\ F^x_{i+1,j} & F^x_{i+1,j+1} & F^{xy}_{i+1,j} & F^{xy}_{i+1,j+1} \end{pmatrix}, \tag{6}$$

$$B_i(\theta) = \begin{bmatrix} b_0(\theta) & b_1(\theta) & b_2(\theta) & b_3(\theta) \end{bmatrix}, \tag{7}$$

$$\hat{B}_j(\varphi) = \begin{bmatrix} \hat{b}_0(\varphi) & \hat{b}_1(\varphi) & \hat{b}_2(\varphi) & \hat{b}_3(\varphi) \end{bmatrix}, \tag{8}$$

with,

$$b_0(\theta) = \frac{(1-\theta)^3 u_{i,j} + \theta(1-\theta)^2 w_{i,j}}{q_i(\theta)},$$
$$b_1(\theta) = \frac{\theta^3 v_{i,j} + \theta^2(1-\theta) w_{i,j}}{q_i(\theta)},$$
$$b_2(\theta) = \frac{\theta(1-\theta)^2 h_i u_{i,j}}{q_i(\theta)},$$
$$b_3(\theta) = \frac{-\theta^2(1-\theta) h_i v_{i,j}}{q_i(\theta)},$$
$$\hat{b}_0(\varphi) = \frac{(1-\varphi)^3 \hat{u}_{i,j} + \varphi(1-\varphi)^2 \hat{w}_{i,j}}{q_j(\varphi)},$$
$$\hat{b}_1(\varphi) = \frac{\varphi^3 \hat{v}_{i,j} + \varphi^2(1-\varphi) \hat{w}_{i,j}}{q_j(\varphi)},$$
$$\hat{b}_2(\varphi) = \frac{\varphi(1-\varphi)^2 \hat{h}_j \hat{u}_{i,j}}{q_j(\varphi)},$$
$$\hat{b}_3(\varphi) = \frac{-\varphi^2(1-\varphi) \hat{h}_j \hat{v}_{i,j}}{q_j(\varphi)},$$

$$\begin{cases} q_i(\theta) = u_{i,j}(1-\theta)^3 + w_{i,j}\theta(1-\theta) + v_{i,j}\theta^3, \\ q_j(\varphi) = \hat{u}_{i,j}(1-\varphi)^3 + \hat{w}_{i,j}\varphi(1-\varphi) + \hat{v}_{i,j}\varphi^3, \\ \theta = \dfrac{x-x_i}{h_i}, \varphi = \dfrac{y-y_j}{\hat{h}_j}, h_i = x_{i+1} - x_i, \hat{h}_j = y_{j+1} - y_j. \end{cases}$$

The $F^x_{i,j}, F^y_{i,j}$ are the partial derivative with respect to x and y respectively and $F^{xy}_{i,j}$ be the mixed partial derivative with respect to xy.

Substituting the Equations (6)-(8) into (5), then the rational bi-cubic function (5) can be represented as:

$$S(x,y) = \frac{P_{ij}(\theta,\phi)}{Q_{ij}(\theta)}, \tag{9}$$

with,
$P_{ij}(\theta,\phi) = (1-\theta)^3\delta_{i,j} + \theta(1-\theta)^2\eta_{i,j} + \theta^2(1-\theta)\kappa_{i,j} + \theta^3\lambda_{i,j}$,
$Q_{ij}(\theta) = (1-\theta)^3 u_{i,j} + \theta(1-\theta) w_{i,j} + \theta^3 v_{i,j}$
where,

$$\delta_{i,j} = \frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k A_k}{q_j(\phi)}, \tag{10}$$

with,
$A_0 = \hat{u}_{i,j} u_{i,j} F_{i,j}$,
$A_1 = u_{i,j}(F_{i,j}\hat{w}_{i,j} + F^y_{i,j}\hat{h}_j\hat{u}_{i,j})$,
$A_2 = u_{i,j}(F_{i,j+1}\hat{w}_{i,j} - F^y_{i,j+1}\hat{h}_j\hat{v}_{i,j})$,
$A_3 = u_{i,j}\hat{v}_{i,j}F_{i,j+1}$,
$q_j(\phi) = \hat{u}_{i,j}(1-\phi)^3 + \hat{w}_{i,j}\phi(1-\phi) + \hat{v}_{i,j}\phi^3$.

$$\eta_{i,j} = \frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k B_k}{q_j(\phi)}, \tag{11}$$

with,
$B_0 = \hat{u}_{i,j}(w_{i,j}F_{i,j} + h_i u_{i,j} F^x_{i,j})$,
$B_1 = \hat{w}_{i,j}(w_{i,j}F_{i,j} + h_i u_{i,j}F^x_{i,j}) + \hat{h}_j\hat{u}_{i,j}(w_{i,j}F^y_{i,j} + h_i u_{i,j}F^{xy}_{i,j})$,
$B_2 = \hat{w}_{i,j}(w_{i,j}F_{i,j+1} + h_i u_{i,j}F^x_{i,j+1}) + \hat{h}_j\hat{v}_{i,j}(w_{i,j}F^y_{i,j+1} + h_i u_{i,j}F^{xy}_{i,j+1})$,
$B_3 = \hat{v}_{i,j}(w_{i,j}F_{i,j+1} + h_i u_{i,j}F^x_{i,j+1})$,
$q_j(\phi) = \hat{u}_{i,j}(1-\phi)^3 + \hat{w}_{i,j}\phi(1-\phi) + \hat{v}_{i,j}\phi^3$.

$$\kappa_{i,j} = \frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k C_k}{q_j(\phi)}, \tag{12}$$

with,
$C_0 = \hat{u}_{i,j}(F_{i+1,j}w_{i,j} - h_i v_i F^x_{i+1,j})$,
$C_1 = \hat{w}_{i,j}(F_{i+1,j}w_{i,j} - h_i v_i F^x_{i+1,j}) + \hat{h}_j\hat{u}_{i,j}(F^y_{i+1,j}w_{i,j} - h_i v_i F^{xy}_{i+1,j})$,
$C_2 = \hat{w}_{i,j}(F_{i+1,j+1}w_{i,j} - h_i v_i F^x_{i+1,j+1}) + \hat{h}_j\hat{v}_{i,j}(F^y_{i+1,j+1}w_{i,j} - h_i v_i F^{xy}_{i+1,j+1})$,
$C_3 = \hat{v}_{i,j}(F_{i+1,j+1}w_{i,j} - h_i v_i F^x_{i+1,j+1})$,
$q_j(\phi) = \hat{u}_{i,j}(1-\phi)^3 + \hat{w}_{i,j}\phi(1-\phi) + \hat{v}_{i,j}\phi^3$.

$$\lambda_{i,j} = \frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k D_k}{q_j(\phi)}, \tag{13}$$

with,
$D_0 = \hat{u}_{i,j}v_{i,j}F_{i+1,j}$,
$D_1 = v_{i,j}(F_{i+1,j}\hat{w}_{i,j} + F^y_{i+1,j}\hat{h}_j\hat{u}_{i,j})$,
$D_2 = v_{i,j}(F_{i+1,j+1}\hat{w}_{i,j} - F^y_{i+1,j+1}\hat{h}_j\hat{v}_{i,j})$,
$D_3 = v_{i,j}\hat{v}_{i,j}F_{i+1,j+1}$,
$q_j(\phi) = \hat{u}_{i,j}(1-\phi)^3 + \hat{w}_{i,j}\phi(1-\phi) + \hat{v}_{i,j}\phi^3$.

4 Shape preserving positive surface data rational bi-cubic spline interpolation

In section (3), the rational bi-cubic function does not permit to preserve the positivity of positive data over the rectangular grid $D = [x_0, x_m] \times [y_0, y_n]$

without any mathematical treatment. For this purpose, we are required some mathematically actions on the free parameters $w_{i,j}$ and $\hat{w}_{i,j}$ which guarantee to preserve the positivity. They are explained in the following way.

Theorem 4.1. *The rational bi-cubic functions (5) declare the surfaces which preserve the positivity, if in each rectangular patch* $[x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0, 1, 2, ..., m-1$; $j = 0, 1, 2, ..., n-1$, *the free parameters* $u_{i,j} > 0, v_{i,j} > 0, \hat{u_{i,j}} > 0, \hat{v_{i,j}}$ *and* $\hat{w}_{i,j}$, $w_{i,j}$ *satisfy the following sufficient conditions.*

$$\hat{w}_i > Max \left\{ \begin{array}{l} 0, \dfrac{-F^y_{i,j}\hat{h}_j\hat{u}_{i,j}}{F_{i,j}}, \dfrac{F^y_{i,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i,j+1}}, \\ \dfrac{-F^y_{i+1,j}\hat{h}_j\hat{u}_{i,j}}{F_{i+1,j}}, \dfrac{F^y_{i+1,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i+1,j+1}} \end{array} \right\}.$$

$$w_i > Max \left\{ \begin{array}{l} 0, \dfrac{h_i v_i F^x_{i+1,j}}{F_{i+1,j}}, \dfrac{h_i v_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \dfrac{h_i v_i F^x_{i+1,j+1}}{F_{i+1,j+1}}, \\ \dfrac{h_i v_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}, \dfrac{-h_i u_{i,j} F^x_{i,j}}{F_{i,j}}, \dfrac{-h_i u_{i,j} F^{xy}_{i,j}}{F^y_{i,j}}, \\ \dfrac{-h_i u_{i,j} F^x_{i,j+1}}{F_{i,j+1}}, \dfrac{-h_i u_{i,j} F^{xy}_{i,j+1}}{F^y_{i,j+1}} \end{array} \right\}.$$

The above results can be rearranged as:

$$\hat{w}_i = k_{i,j} + max \left\{ \begin{array}{l} 0, \dfrac{-F^y_{i,j}\hat{h}_j\hat{u}_{i,j}}{F_{i,j}}, \dfrac{F^y_{i,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i,j+1}}, \\ \dfrac{-F^y_{i+1,j}\hat{h}_j\hat{u}_{i,j}}{F_{i+1,j}}, \dfrac{F^y_{i+1,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i+1,j+1}} \end{array} \right\}, k_{i,j} > 0.$$

$$w_i = m_{i,j} + max \left\{ \begin{array}{l} 0, \dfrac{h_i v_i F^x_{i+1,j}}{F_{i+1,j}}, \dfrac{h_i v_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \dfrac{h_i v_i F^x_{i+1,j+1}}{F_{i+1,j+1}}, \\ \dfrac{h_i v_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}, \dfrac{-h_i u_{i,j} F^x_{i,j}}{F_{i,j}}, \dfrac{-h_i u_{i,j} F^{xy}_{i,j}}{F^y_{i,j}}, \\ \dfrac{-h_i u_{i,j} F^x_{i,j+1}}{F_{i,j+1}}, \dfrac{-h_i u_{i,j} F^{xy}_{i,j+1}}{F^y_{i,j+1}} \end{array} \right\}, m_{i,j} > 0.$$

Proof. Let $\{(x_i, y_j, F_{i,j}), i = 0, 1, 2, 3, ...m; j = 0, 1, 2, 3, ..., n\}$ is positive data defined over the rectangular grid $I = [x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0, 1, 2, ..., m-1$; $j = 0, 1, 2, ..., n-1$ such that $F_{i,j} > 0\, \forall i, j$. The intention is to construct a piecewise rational bi-cubic function $S(x, y)$ on $D = [x_0, x_m] \times [y_0, y_n]$, such that

$$S(x_i, y_j) = F_{i,j} > 0, i = 0, 1, 2, ..., m;\ j = 0, 1, 2, ..., n,$$

The rational bi-cubic function (5) preserve the positivity if $S(x,y) > 0$,
So $S(x,y) > 0$,if both $P_{ij}(\theta,\phi) > 0$ and $Q_{ij}(\theta) > 0$. such that,

$$P_{ij}(\theta,\phi) = (1-\theta)^3\delta_{i,j} + \theta(1-\theta)^2\eta_{i,j} + \theta^2(1-\theta)\kappa_{i,j} + \theta^3\lambda_{i,j} > 0. \tag{14}$$

$$Q_{ij}(\theta) = (1-\theta)^3 u_{i,j} + \theta(1-\theta)w_{i,j} + \theta^3 v_{i,j} > 0. \tag{15}$$

$$u_{i,j} > 0, v_{i,j} > 0 \ and \ w_{i,j} > 0. \tag{16}$$

$$\hat{u}_{i,j} > 0, \hat{v}_{i,j} > 0 \ and \ \hat{w}_{i,j} > 0. \tag{17}$$

Equations (16) and (17) are sufficient for $Q_{ij}(\theta) > 0$. Now, the cubic polynomial in (14) will be positive if

$$\delta_{i,j} > 0, \eta_{i,j} > 0, \kappa_{i,j} > 0 \, and \ \lambda_{i,j} > 0. \tag{18}$$

$\delta_{i,j} > 0$, if $A_i > 0, i = 0,1,2,3$ and $q_j(\phi) > 0$. Equation (17) is sufficient $q_j(\phi) > 0$, it is obvious that $A_0 > 0$, $A_3 > 0$ and $A_1 > 0, A_2 > 0$, if,

$$\hat{w}_{i,j} > Max\left\{\frac{-F^y_{i,j}\hat{h}_j\hat{u}_{i,j}}{F_{i,j}}, \frac{F^y_{i,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i,j+1}}\right\} \tag{19}$$

Now $\lambda_{i,j} > 0$, if $D_i > 0, i = 0,1,2,3$ and $q_j(\phi) > 0$. It is obvious that $D_0 > 0$, $D_3 > 0$ and $D_1 > 0$, $D_2 > 0$, if,

$$\hat{w}_{i,j} > Max\left\{\frac{-F^y_{i+1,j}\hat{h}_j\hat{u}_{i,j}}{F_{i+1,j}}, \frac{F^y_{i+1,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i+1,j+1}}\right\} \tag{20}$$

$\eta_{i,j} > 0$, if $B_i > 0, i = 0,1,2,3$ and $q_j(\phi) > 0$. Equation (17) is sufficient $q_j(\phi) > 0$, it is obvious that $B_0 > 0$, $B_3 > 0$ and $B_1 > 0$, $B_2 > 0$ if,

$$w_{i,j} > Max\left\{\frac{-h_i u_{i,j}F^x_{i,j}}{F_{i,j}}, \frac{-h_i u_{i,j}F^{xy}_{i,j}}{F^y_{i,j}}, \frac{-h_i u_{i,j}F^x_{i,j+1}}{F_{i,j+1}}, \frac{-h_i u_{i,j}F^{xy}_{i,j+1}}{F^y_{i,j+1}}\right\} \tag{21}$$

$\kappa_{i,j} > 0$, if $C_i > 0, i = 0,1,2,3$ and $q_j(\phi) > 0$. Equation (17) is sufficient for $q_j(\phi) > 0$, clearly $C_0 > 0, C_3 > 0$ and $C_1 > 0, C_2 > 0$, if,

$$w_{i,j} > Max\left\{\frac{h_i v_i F^x_{i+1,j}}{F_{i+1,j}}, \frac{h_i v_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \frac{h_i v_i F^x_{i+1,j+1}}{F_{i+1,j+1}}, \frac{h_i v_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}\right\} \tag{22}$$

Equations (17), (19) and (20) can be summarized as:

$$\hat{w}_{i,j} > Max\left\{0, \frac{-F^y_{i,j}\hat{h}_j\hat{u}_{i,j}}{F_{i,j}}, \frac{F^y_{i,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i,j+1}}, \frac{-F^y_{i+1,j}\hat{h}_j\hat{u}_{i,j}}{F_{i+1,j}}, \frac{F^y_{i+1,j+1}\hat{h}_j\hat{v}_{i,j}}{F_{i+1,j+1}}\right\} \tag{23}$$

Equations (16), (21) and (22) can be expressed as:

$$w_i > Max \left\{ \begin{array}{l} 0, \dfrac{h_i v_i F^x_{i+1,j}}{F_{i+1,j}}, \dfrac{h_i v_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \dfrac{h_i v_i F^x_{i+1,j+1}}{F_{i+1,j+1}}, \\ \dfrac{h_i v_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}, \dfrac{-h_i u_{i,j} F^x_{i,j}}{F_{i,j}}, \dfrac{-h_i u_{i,j} F^{xy}_{i,j}}{F^y_{i,j}}, \\ \dfrac{-h_i u_{i,j} F^x_{i,j+1}}{F_{i,j+1}}, \dfrac{-h_i u_{i,j} F^{xy}_{i,j+1}}{F^y_{i,j+1}} \end{array} \right\}. \tag{24}$$

□

5 Derivative approximation

Mostly the derivatives values at the knots are not given.These values are estimated by some approximation methods. In this paper, these values are calculated by following method in such a way that the smoothness of the interpolant (5) is maintained.

5.1 Arithmetic mean method for 3D data

$$\begin{aligned}
F^x_{1,j} &= \Delta_{1,j} + \frac{(\Delta_{1,j} - \Delta_{2,j})h_1}{(h_1 + h_2)}, \\
F^x_{m,j} &= \Delta_{m-1,j} + \frac{(\Delta_{m-1,j} - \Delta_{m-2,j})h_{m-1}}{(h_{m-1} + h_{m-2})}, \\
F^x_{i,j} &= \frac{(\Delta_{i,j} + \Delta_{i-1,j})}{2}, i = 2, 3, \cdots, m-1, j = 1, 2, \cdots, n, \\
F^y_{i,1} &= \hat{\Delta}_{i,1} + \frac{(\hat{\Delta}_{i,1} - \hat{\Delta}_{i,2})\hat{h}_1}{(\hat{h}_1 + \hat{h}_2)}, \\
F^y_{i,n} &= \hat{\Delta}_{i,n-1} + \frac{(\hat{\Delta}_{i,n-1} - \hat{\Delta}_{i,n-2})\hat{h}_{n-1}}{(\hat{h}_{n-1} + \hat{h}_{n-2})}, \\
F^y_{i,j} &= \frac{(\hat{\Delta}_{i,j} + \hat{\Delta}_{i,j-1})}{2}, i = 1, 2, \cdots, m, j = 2, 3, \cdots, n-1, \\
F^{xy}_{i,j} &= \frac{1}{2}\{\frac{F^y_{i+1,j} - F^y_{i-1,j}}{h_{i-1} + h_i} + \frac{F^x_{i,j-1} - F^x_{i,j+1}}{h_{j-1} + h_j}\}, i = 1, 2, \cdots, m-1; j = 1, 2, \cdots, n-1
\end{aligned}$$

where $\Delta_{i,j} = \dfrac{F_{i+1,j} - F_{i,j}}{h_i}$ and $\hat{\Delta}_{i,j} = \dfrac{F_{i,j+1} - F_{i,j}}{\hat{h}_j}$. The above method is computationally easy on the pocket as well as provide suitable choice for visualization of positive data.

6 Demonstrations

In this section, a numerical demonstration of positivity preserving scheme given in section (4) is presented.

Example 6.1. *The data set up to 4 decimal places taken in Table (2) is generated by the following function [8].*

$$F_1(x,y) = 4/((x^2+y^2)^2-1), \quad -3 \le x, y \le 3, \ x, y \ne 0$$

Figure (4) is generated by Bi-cubic Hermite Spline. Figure (5) is presented the xz view of Figure (4). It is to be noted that the Figures (4) and (5) do not preserve the shape of positive data. We have developed a rational bi-cubic function in section (4) for the improvement of above blemish. Figure (6) is produced by the scheme developed in section (4) with the free parameters $u_{i,j} = 0.05, v_{i,j} = 0.05, \hat{u}_{i,j} = 0.05,$ *and* $\hat{v}_{i,j} = 0.05$ *to preserve the shape of same positive data. In this scheme, we have four free parameters for the user to refine the surface as desired. Figure (7) present the xz-view of Figure (6) also preserved the positivity of positive data.*

Table 2: Positive surface Data set

y/x	-3	-2	-1	1	2	3
-3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124
-2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
-1	0.0404	0.1667	1.3333	1.3333	0.1667	0.0404
1	0.0404	0.1667	1.333	1.3333	0.1667	0.0404
2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124

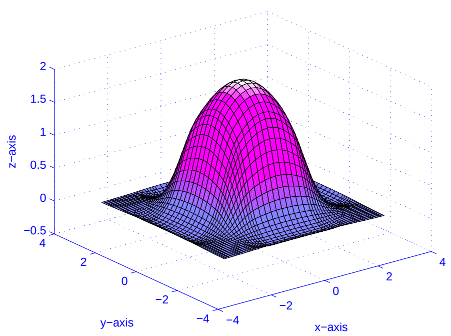

Figure 4: Bi-cubic Hermite Spline

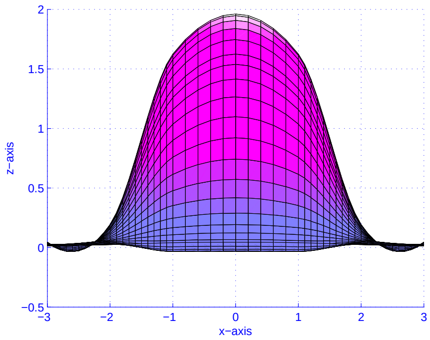

Figure 5: Bi-cubic Hermite Spline in xz-view of Figure (4)

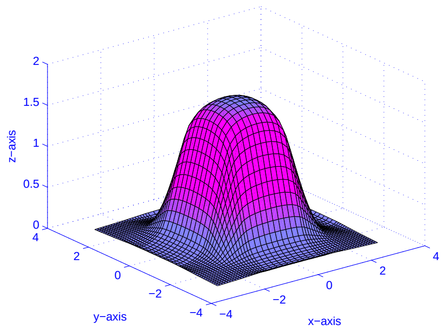

Figure 6: Rational bi-cubic function with $u_{i,j} = v_{i,j} = \hat{u}_{i,j} = \hat{v}_{i,j} = 0.05$

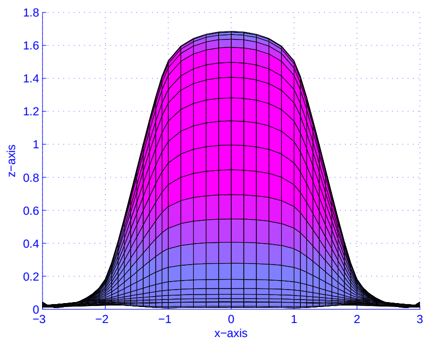

Figure 7: Rational bi-cubic function in xz-view of Figure (6)

Example 6.2. *The data set taken in Table (3) is produced by the following function [4].*

$$F_2(x, y) = (x^2 - y^2)^2 + 1, -3 \leq x, y \leq 3$$

The Bi-cubic Hermite Spline scheme losses the positivity of positive data as shown in Figure (8). Figure (9) present the xz-view of Figure (8) which shows *very clearly that the scheme does not preserve the positivity of positive data. We have developed a rational bi-cubic function in section (4) to improve the above flaw. Figure (10) is produced by the scheme developed in section (4) with the free parameters* $u_{i,j} = 0.05, v_{i,j} = 0.05, \hat{u}_{i,j} = 0.05,$ *and* $\hat{v}_{i,j} = 0.05$ *to preserve the shape of same positive data. Figure (11) present the xz-view of Figure (10) also preserved the positivity of positive data.*

Table 3: Positive surface Data set

y/x	-3	-2	-1	0	1	2	3
-3	1	26	65	82	65	26	1
-2	26	1	10	17	10	1	26
-1	65	10	1	2	1	10	65
0	82	17	2	1	2	17	82
1	65	10	1	2	1	10	65
2	26	1	10	17	10	1	26
3	1	26	65	82	65	26	1

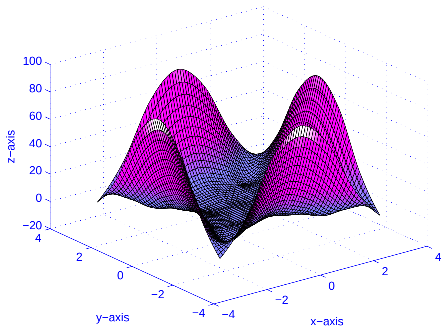

Figure 8: Bi-cubic Hermite Spline

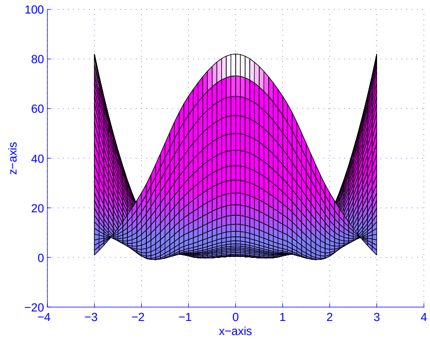

Figure 9: Bi-cubic Hermite Spline in xz-view of Figure (8)

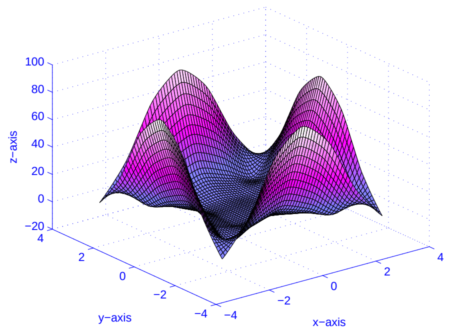

Figure 10: Rational bi-cubic function with $u_{i,j} = v_{i,j} = \hat{u}_{i,j} = \hat{v}_{i,j} = 0.05$

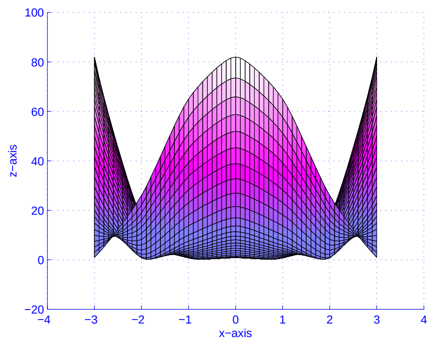

Figure 11: Rational bi-cubic function in xz-view of Figure (10)

7 Conclusion

In this paper, we have extended the rational cubic function [2] into rational bi-cubic function with six free parameters in each rectangular patch to preserve the shape of positive 3D data over the rectangular grid in the view of positive surface. In this developed scheme, the free parameters are arranged in such a way that two of them are constrained parameters (not depend on the other parameter like [8]) to preserve the shape of 3D positive data while the remaining are left free for user 's choice to refine the surface as desired. The developed surface scheme has been demonstrated through different numerical examples and observed that the scheme is not only local and computationally economical but also visually pleasant. Also the scheme in section (4) is more flexible and suitable for interactive CAD system. Our method of manipulation is so vigorous.

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