Chapter 7

THE FOURIER TRANSFORM AND ITS APPLICATIONS

In this chapter we discuss another well-known integral transform which goes by the name of *Fourier transform.* After discussing its theory we will turn to its applications.

7.1 Definition and Basic Properties

Given an integrable function $f(x)$ for $-\infty < x < \infty$. We can associate with it another function $F(k)$ of variable k, $(-\infty < k < +\infty$, by the relation

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{ikx} f(x)\, dx \tag{7.1.1}$$

The function $F(k)$ is called the *Fourier transform* of $f(x)$, and $f(x)$ is called the inverse Fourier transform of $F(k)$. It can be shown that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} F(k)\, dk \tag{7.1.2}$$

Notation and Convention

$\frac{1}{2\sqrt{\pi}}\left(e^{ik\infty} - e^{-ik\infty}\right.$

If we write

$$F(k) = c_1 \int_{-\infty}^{+\infty} e^{\iota k x} f(x)\, dx$$

and

$$f(x) = c_2 \int_{-\infty}^{+\infty} e^{-\iota k x} F(k)\, dk$$

then the following forms of coefficients are mutually consistent.

$$\text{(i)} \quad c_1 = \frac{1}{\sqrt{2\pi}}, \quad c_2 = \frac{1}{\sqrt{2\pi}}$$

$$\text{(ii)} \quad c_1 = 1, \quad c_2 = \frac{1}{2\pi}$$

$$\text{(iii)} \quad c_1 = \frac{1}{2\pi}, \quad c_2 = 1$$

Also

$$F(k) = \mathcal{F}\{f(x)\}$$

where the operator $\mathcal{F}$ is called the *Fourier transform operator.*

It is also possible to define the Fourier transform and its inverse in such a way that the coefficients c_1, c_2 in each are unity. In this definition they are given by the relations

$$F(k) = \int_{-\infty}^{+\infty} e^{2\pi \iota k x} f(x)\, dx$$

and

$$f(x) = \int_{-\infty}^{+\infty} e^{-2\pi \iota k x} F(k)\, dk$$

These relations can be obtained from (7.1.1) and (7.1.2) by making the transformations $x' = \sqrt{2\pi}\, x$ and $k' = \sqrt{2\pi}\, k$ and then reverting to the unprimed symbols.

Various choices of pairs of variables such as $(x,\, k)$, $(x,\, p)$, $(x,\, \xi)$, $(t,\, \omega)$ are used by different authors. In order to indicate the associated variable the following notation is also used for the Fourier transform:

$$\mathcal{F}\{f(x),\ x \to k\},\ \mathcal{F}\{f(x),\ x \to \xi\}, \mathcal{F}\{f(t),\ t \to \omega\}$$

for the Fourier transforms $F(k)$, $F(\xi)$, $F(\omega)$ of $f(x)$, $f(x)$ and $f(t)$ respectively.

7.1.1 The Fourier transform and its inverse

If the function $f(x)$ or $F(k)$ is continuous or piecewise continuous over $(-\infty, +\infty)$ and bounded then Fourier transform and inverse Fourier transform exist.

If the function $f(x)$ is absolutely integrable *i.e.* the integral $\int_{-\infty}^{+\infty} |f(x)|\, dx$ exists, then the Fourier transform exists. This is a sufficient condition. Similarly for the inverse Fourier transform.

Linearity of $\mathcal{F}$ and $\mathcal{F}^{-1}$ Operators

The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear i.e.

$$\mathcal{F}\{c_1 f_1(x) + c_2 f_2(x)\} = c_1\mathcal{F}\{f_1(x)\} + c_2\mathcal{F}\{f_2(x)\}$$

and

$$\mathcal{F}^{-1}\{c_1 F_1(k) + c_2 F_2(k)\} = c_1\mathcal{F}^{-1}\{F_1(k)\} + c_2\mathcal{F}^{-1}\{F_2(k)\}$$

7.1.2 Fourier series and Fourier transform

The Fourier series representation of a periodic piecewise smooth function over the interval $(-l, l)$ leads to the integral representation of the same function as $l \to \infty$ and the index n in the Fourier series $\to \infty$. The condition of periodicity is replaced by the condition of absolute integrability for the function $f(x)$ over $(-\infty, \infty)$. This can be seen as follows.

We start with the complex from of the Fourier series representation for the function $f(x)$, as explained in chapter 1.

$n\pi i x t/\ell$

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n\, e^{in\pi x/\ell}, \qquad -\ell < x < \ell \tag{7.1.3}$$

where the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(x)\, e^{in\pi x/\ell}\, dx \tag{7.1.4}$$

Now we consider the situation in which $\ell \to \infty$. Let $n\pi/\ell = k$ then $n = \ell k/\pi$ and the increment Δn in n will be given by $\ell\,\Delta k/\pi$ i.e.

$$\Delta n = \ell\,\Delta k/\pi \text{ or } \Delta k = \pi/\ell$$

where $\Delta n = 1$. In the limit $\ell \to \infty$, $\Delta k \to 0$. In view of this we can rewrite (7.1.3) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \Delta n e^{\iota n\pi x/\ell} = \sum_k c_l(k)\frac{\ell}{\pi}\Delta k e^{\iota kx} \tag{7.1.5}$$

where we have put $c_n = c(\ell k/\pi) = c_\ell(k)$ to show the dependence of the coefficients c_n on ℓ and k.

Similarly (7.1.4) can be written as

$$c(\ell k/\pi) \equiv c_\ell(k) = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(x)\, e^{\iota kx}\, dx$$

or

$$\frac{\ell}{\pi}c_\ell(k) = \frac{1}{2\pi}\int_{-\ell}^{+\ell} f(x)\, e^{-\iota kx}\, dx \tag{7.1.6}$$

Equations (7.1.5) and (7.1.6) correspond to each other in the same way as equations (7.1.3) and (7.1.4) do. Now we let $\ell \to \infty$ so that k now incomes a continuous variable, and assuming that the sum goes over into the Riemann integral, we have from (7.1.5)

$$f(x) = \int_{-\infty}^{+\infty} c(k)\, e^{\iota kx}\, dk \tag{7.1.7}$$

where $c(k) = \lim_{\ell\to\infty}(l/\pi)\, c(\ell k/\pi)$. Also from (7.1.6)

$$c(k) = \frac{1}{2\pi}\int_{-\ell}^{+\ell} f(x)\, e^{-\iota kx}\, dx \tag{7.1.8}$$

To conform to the notation followed in this book we further set

$$c(k) = \frac{1}{\sqrt{2\pi}}F(-k)$$

then (7.1.7) and (7.1.8) become

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, e^{\iota kx}\, dx$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(k)\, e^{-\iota kx}\, dk$$

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7.2 Fourier Transforms of Some Simple Functions

In this section we make calculations to evaluate the Fourier transforms of some simple functions.

7.2.1 Illustrative examples

Example 1

(Fourier transform of the Gaussian Function)

Find the Fourier transform of the Gaussian function

$$g(x) = Ne^{-\alpha x^2}$$

where N and α are constants, and $\alpha > 0$.

Solution

From definition

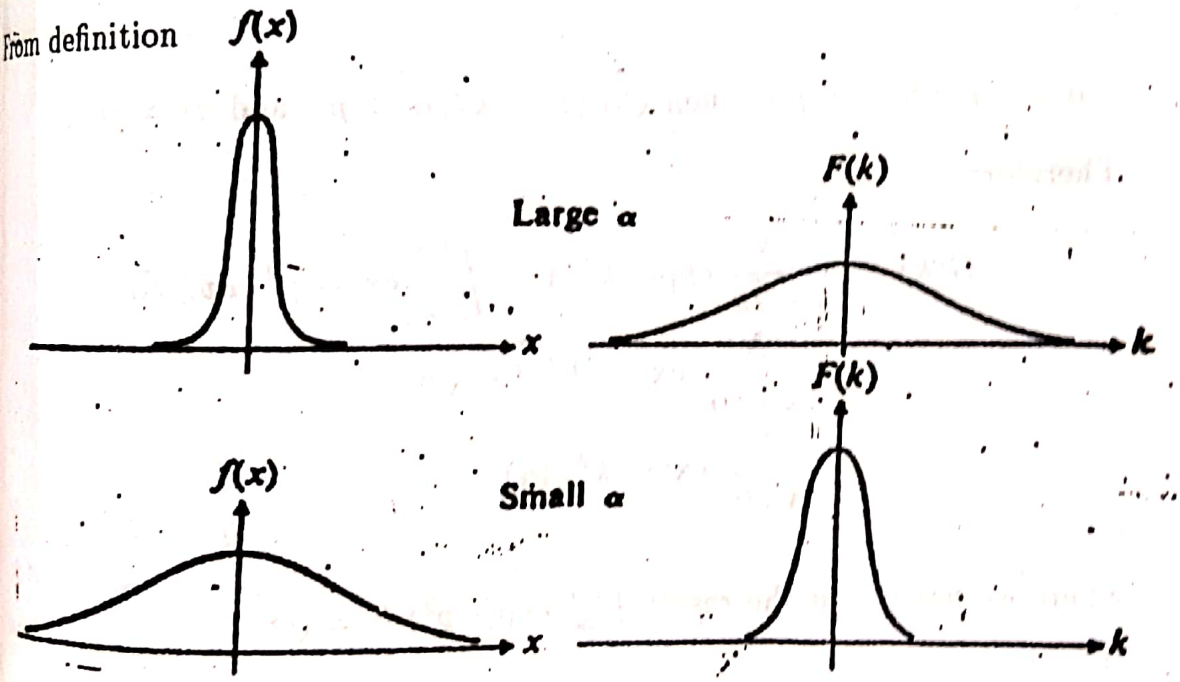

Figure 7.1:

$$\mathcal{F}\{g(x)\} = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx}\, g(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx}\, Ne^{-\alpha x^2}\, dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx - \alpha x^2}\, dx$$

Now

$$\begin{aligned}
\iota\, k\, x - \alpha x^2 &= -\alpha(x^2 - \frac{\iota k x}{\alpha}) \\
&= -\alpha\left[x^2 - \iota k x/\alpha + (\iota k/2\alpha)^2 - (\iota k/2\alpha)^2\right] \\
&= -\alpha\left[(x - \iota k/2\alpha)^2 + k^2/4\alpha^2\right]
\end{aligned}$$

Therefore

$$G(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\alpha(x - \iota k/2\alpha)^2\right] \exp\left(-k^2/4\alpha\right) dx$$

or

$$G(k) = \frac{N}{\sqrt{2\pi}} \exp(-k^2/4\alpha) \int_{-\infty}^{+\infty} \exp\left[-\alpha(x - \iota k/2\alpha)^2\right] dx$$

Let

$$\alpha(x - \iota k/2\alpha)^2 = p^2, \quad \text{then } \sqrt{\alpha}\,(x - \iota\, k/2\alpha) = p, \quad \text{and } dx = dp/\sqrt{\alpha}$$

Therefore

$$\begin{aligned}
G(k) &= \frac{N}{\sqrt{2\pi}} \exp(-k^2/4\alpha) \int_{-\infty}^{+\infty} \exp(-p^2)\, dp/\sqrt{\alpha} \\
&= \frac{N}{\sqrt{2\pi\alpha}} \exp(-k^2/4\alpha)\sqrt{\pi} \\
&= \frac{N}{\sqrt{2\alpha}} \exp(-k^2/4\alpha)
\end{aligned}$$

where we have used the result $\int_{-\infty}^{+\infty} \exp(-p^2)\, dp = \sqrt{\pi}$.

Hence

$$\mathcal{F}\left\{N \exp(-\alpha\, x^2)\right\} = \frac{1}{\sqrt{2\alpha}} N \exp -k^2/4\alpha$$

We note that the function $g(x) = N \exp(-\alpha\, x^2)$, $(\alpha > 0)$, will be sharply-peaked for large values of α.

+ Example 2

Find the Fourier transform of $g(x) = a/(x^2 + a^2)$, $a > 0$

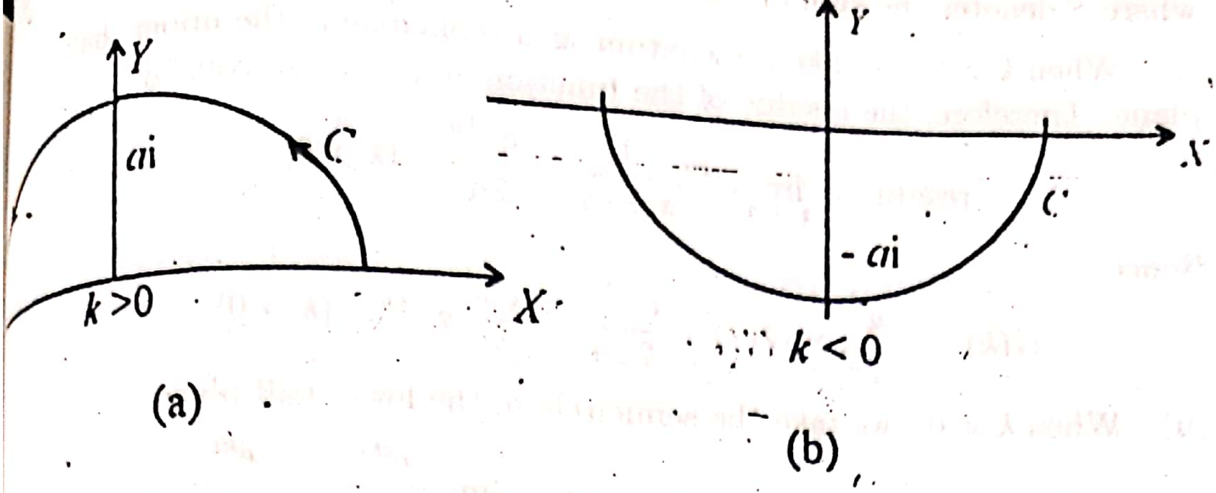

Figure 7.2:

Solution

The function $g(x)$ as well as its derivative are continuous over the interval $(-\infty,\ +\infty)$, and the integral $\int_{-\infty}^{+\infty} g(x)\,dx$ is absolutely integrable. Therefore the Fourier transform of the given function must exist.

$$\begin{aligned}\mathcal{F}\{g(x)\} &= G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx} \frac{a}{x^2+a^2}\,dx \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{\iota kz}}{z^2+a^2}\,dz \\ &= \frac{a}{\sqrt{2\pi}} \oint_C \frac{e^{\iota kz}}{z^2+a^2}\,dz\end{aligned}$$

where C is a closed contour consisting of the X-axis and a semicircle (of infinite radius) in the upper or lower half-plane. Now

$$\iota kz = \cancel{\iota k(\mathrm{Re}z + \iota\,\mathrm{Im}z)} = \iota\, k\,(x + \iota\, y) = \iota k\, x - ky$$

Therefore $e^{\iota\, kz} = e^{\iota kx - ky} \to 0$ when $y \to \infty$ for $k > 0$.

The same will also $\to 0$ for $k < 0$ if $y \to -\infty$. We want to choose the contour C in such a way that the integral $\int_C g(z)\,dz$ is zero. In one case the contour will lie in the upper-half plane whereas in the other case it will lie in the lower-half plane, (see figs. 7.2 a, b). Therefore

$$G(k) = \frac{a}{\sqrt{2\pi}} \oint_C \frac{e^{\iota kz}}{z^2+a^2}\,dz = \frac{a}{\sqrt{2\pi}} \times\ 2\pi\ \iota \times\ S$$

where S denotes the sum of residues of the poles in the contour.

(i). When $k > 0$, we take the contour as a semicircle in the upper- half plane. Therefore, the residue of the function at $z = \iota a$ is given by

$$\text{residue} = \lim_{z \to \iota a} e^{\iota k z} \frac{1}{z + \iota a} = \frac{e^{-ka}}{2\iota a}, \quad (k > 0)$$

Hence

$$G(k) = \frac{a}{\sqrt{2\pi}} \times (2\pi\iota) \times \frac{e^{-ka}}{2\iota a} = \frac{\sqrt{\pi}}{2} e^{-ka}, \quad (k > 0)$$

(ii) When $k < 0$, we take the semicircle in the lower-half plane.

$$\text{Residue of the function at } -\iota a = \lim_{x \to a\iota} \frac{e^{\iota k z}}{z - \iota a} = \frac{e^{ka}}{-2\iota a}$$

Therefore

$$G(k) = \frac{a}{\sqrt{2\pi}} (-2\pi\iota) \frac{e^{ka}}{-2\iota a} = \sqrt{\frac{\pi}{a}}$$

Combining the two results, we have

$$G(k) = \sqrt{\frac{\pi}{2}} e^{-|k|a}, \text{ for all } k$$

Example 3

Find the Fourier transform of the box function

$$f(x) = \begin{cases} 1, & |x| \le a, \; a > 0 \\ 0, & |x| > a \end{cases}$$

$\rightarrow -a < x < a$

$x > a$

$x < -a$

Solution

$$\begin{aligned}
\mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota k x} f(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} 0 \times e^{\iota k x}\, dx \\
&+ \int_{-a}^{+a} 1 \times e^{\iota k x}\, dx + \int_{+a}^{+\infty} 0 \times e^{\iota k x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(0 + \frac{e^{\iota k x}}{\iota k}\right)\Bigg]_{-a}^{+a} = \frac{1}{\sqrt{2\pi}} \frac{e^{\iota a k} - e^{\iota k a}}{\iota k} \\
&= \frac{1}{\sqrt{2\pi}} \frac{e^{\iota a k} - e^{-\iota a k}}{\iota k} = \frac{2}{\sqrt{2\pi}} \frac{e^{\iota a k} - e^{-\iota a k}}{2\iota k} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin ka}{k}
\end{aligned}$$

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$$= \frac{1}{\sqrt{2\pi}} \frac{e^{\iota ak} - e^{-\iota ak}}{\iota k} = \frac{2}{\sqrt{2\pi}} \frac{e^{\iota ak} - e^{-\iota ak}}{2\iota k}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin ka}{k}$$

7.3 Properties of Fourier Transformation

1. Linearity property

It is a linear transformation; both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear.

2. Conjugation property

If $f(x)$ is real, then $F(-k) = \overline{F(k)}$, (where the bar symbol denotes the complex conjugate).

Proof

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx} f(x)\, dx$$

and therefore

$$\overline{F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\iota kx} f(x)\, dx$$

Also

$$F(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\iota kx} f(x)\, dx$$

which proves that $F(-k) = \overline{F(k)}$.

3. Real and Complex Values of the F.T.

(a) If $f(x)$ is real and even, $F(k)$ is real.

(b) If $f(x)$ is real and odd, $F(k)$ is pure imaginary.

(c) If $f(x)$ is complex, then $\mathcal{F}\{\overline{f(-x)}\} = \overline{F(k)}$.

Proof of (3) a

We have to prove that if $f(x)$ is even, then $F(k)$ is real.

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx} f(x)\, dx$$

When f(x) is even, i.e. $f(x) = f(-x)$, then

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx} f(-x)\, dx$$

Let $-x = x'$ or $dx = -dx'$, therefore

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-\iota kx'} f(x')\,(-dx')$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\iota kx'} f(x')\,(+dx') = F(-k)$$

Hence $F(k) = \overline{F(k)}$, which shows that $F(k)$ is real.

Proof of (3 b)

$$F(k) = \frac{1}{\sqrt{(2\pi)}} \int_{\infty}^{\infty} e^{-\iota kx} f(x)\, dx$$

When $f(x)$ is odd i.e. $f(x) = -f(-x)$, we have

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\iota kx} [-f(-x)]\, dx$$

Let $x' = -x$, then $dx' = -dx$, and

$$F(k) = \frac{-1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} e^{-\iota x'k} f(x')\,(-dx')$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\iota kx'} f(x')\, dx'$$

or $F(k) = -F(-k) = -\overline{F(k)}$, which shows that $F(k)$ is pure imaginary.

Proof of 3(c)

$$\mathcal{F}\{\bar{f}(-x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx} \bar{f}(-x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\iota kx'} \bar{f}(x')\, dx', \quad (x' = -x)$$

$$= \text{complex conjugate of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx'} f(x')\, dx'$$

$= \text{complex conjugate of } F(k) = \overline{F(k)}$

4. **Attenuation property**

$$\mathcal{F}\{e^{ax} f(x)\} = F(k - a\iota)$$

It can be proved directly from the definition.

$$\begin{aligned}
\mathcal{F}\{e^{ax} f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx} e^{ax} f(x)\, dx \\
&= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} e^{(\iota k + \iota)x} f(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota(k - \iota a)x} f(x)\, dx \\
&= F(k - a\iota)
\end{aligned}$$

5. **Shifting properties**

(i) $\mathcal{F}\{f(x-a)\} = e^{\iota ka} F(k)$

(ii) $\mathcal{F}\{e^{\iota ax} f(x)\} = F(k+a)$

Proof of (i)

$$\mathcal{F}\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx} f(x-a)\, dx$$

Put $x - a = x'$, which implies $dx = dx'$. Then

$$\begin{aligned}
\mathcal{F}\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota k(x'+a)} f(x')\, dx' \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota ka}\, e^{\iota kx'} f(x')\, dx' \\
&= e^{\iota ka} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx'} f(x')\, dx' \\
&= e^{\iota ka} F(k)
\end{aligned}$$

Proof of (ii)

$$\begin{aligned}
\mathcal{F}\{e^{\iota ax} f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota ax}\, e^{\iota kx} f(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota(k+a)x} f(x)\, dx \\
&= F(k+a)
\end{aligned}$$

6. Scaling property

If c is a non-zero constant, then

$$\mathcal{F}\{f(cx)\} = \frac{1}{|c|}F(k/c)$$

Proof

Let $c > 0$, then

$$\begin{aligned}
\mathcal{F}\{f(cx)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota kx} f(cx)\,dx, \quad \text{let } cx = x' \quad x = \tfrac{1}{c}x' \\
&= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} e^{\iota kx'/c} f(x')\,dx'/c \quad dx = \tfrac{1}{c}dx' \\
&= \frac{1}{c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota(k/c)x} f(x)\,dx \\
&= \frac{1}{c}F(k/c), \quad c > 0
\end{aligned}$$

If $c < 0$, then we can show that by letting $c = -p$, $p > 0$

$$\mathcal{F}\{f(cx)\} = -\frac{1}{c}F(k/c)$$

Combining the two results, we have

$$\mathcal{F}\{f(cx)\} = \frac{1}{|c|}F(k/c)$$

7. Modulation property of Fourier transform

$$\begin{aligned}
\mathcal{F}\{\cos\alpha x f(x)\} &= \mathcal{F}\{(\frac{e^{\iota\alpha x} + e^{-\iota\alpha x}}{2})\, f(x)\} \\
&= \frac{1}{2}\mathcal{F}\{e^{\iota\alpha x} f(x)\} + \frac{1}{2}\mathcal{F}\{e^{\iota\alpha x} f(x)\} \\
&= \frac{1}{2}[F(k + \alpha) + F(k - \alpha)]
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{F}\{\sin\alpha x f(x)\} &= \mathcal{F}\{(\frac{e^{\iota\alpha x} - e^{-\iota\alpha x}}{2\iota})\, f(x)\} \\
&= \frac{1}{2\iota}\mathcal{F}\{e^{\iota\alpha x} f(x)\} - \frac{1}{2\iota}\mathcal{F}\{e^{-\iota\alpha x} f(x)\} \\
&= \frac{1}{2\iota}[F(k + \alpha) - F(k - \alpha)]
\end{aligned}$$

8. Boundedness and Continuity of the F.T.

If $f(x)$ is piece-wise smooth and absolutely integrable on the interval $(-\infty, +\infty)$, then its Fourier transform $F(k)$ is bounded and continuous.

Proof

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx} f(x)\, dx$$

Therefore

$$|F(k)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |f(x)|\, dx$$

Since by assumption the integral on RHS exists, we denote it by J, and obtain

$$|F(k)| \le (2\pi)^{-1/2} J$$

which proves that $F(k)$ is bounded. To prove continuity of $F(k)$, we have

$$\begin{aligned} F(k+h) - F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[e^{i(k+h)x} - e^{ikx}\right] f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx}\left(e^{ihx} - 1\right) f(x)\, dx \\ &= I(k, h) \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{h\to 0} [F(k+h) - F(k)] &= \frac{1}{\sqrt{2\pi}} \lim_{h\to 0} \int_{-\infty}^{\infty} e^{ikx}\left(e^{ihx} - 1\right) f(x)\, dx \\ &= \lim_{h\to 0} I(k, h) \end{aligned}$$

The interchange between the operations of limit and integration will be justified if the integral is uniformly convergent. Now

$$\begin{aligned} |I(k, h)| &\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, |e^{ihx} - 1|\, dx \\ &\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \left[(\cos hx - 1)^2 + \sin^2 hx\right]^{1/2} dx \\ &\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \sqrt{2}\, dx \\ &\le \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx \end{aligned}$$

which implies that $I(k, h)$ is uniformly convergent. Hence

$$\lim_{h\to 0}[F(k+h)-F(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, e^{\iota kx} \lim_{h\to 0}\left(e^{\iota hx}-1\right) dx = 0$$

Therefore $F(k)$ is continuous.

+ 9. Riemann- Lebesgue Theorem/Lemma

If $f(x)$ is piece-wise smooth and absolutely integrable function, then $\lim_{|k|\to\infty} F(k) = 0$.

Proof

By definition

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota kx} f(x)\, dx$$

Integrating on RHS by parts, we have

$$F(k) = \frac{1}{\sqrt{2\pi}}\left[\left\{\frac{e^{\iota kx} f(x)}{\iota k}\right\}_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{e^{\iota kx}}{\iota k} f'(x)\, dx\right]$$

or

$$|F(k)| \le \frac{1}{\sqrt{2\pi}}\left[\lim_{x\to+\infty}\frac{|f(x)|}{|k|} - \lim_{x\to-\infty}\frac{|f(x)|}{|k|}\right.$$
$$\left. - \int_{-\infty}^{+\infty}|f'(x)|\frac{1}{|k|}\, dx\right] \qquad (7.3.1)$$

Since $f(x)$ is absolutely integrable,

$$\lim_{x\to\pm\infty}|f(x)| = 0$$

Therefore from (7.3.1) we have

$$|F(k)| \le \frac{1}{|k|\sqrt{2\pi}}\int_{-\infty}^{+\infty}|f'(x)|\, dx \qquad (7.3.2)$$

Now since $f(x)$ is piecewise smooth, $f'(x)$ is piecewise continuous, and therefore the RHS of (7.3.2) is finite. Hence

$$\lim_{|k|\to\infty}|F(k)| \le \lim_{|k|\to\infty}\frac{1}{|k|}\frac{1}{\sqrt{2\pi}}, \text{ a finite positive number}$$

or

$$\lim_{|k|\to\infty}|F(k)| \le 0$$

Hence the theorem.

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7.4 Fourier Transforms of Derivatives and other Functions

7.4.1 Fourier transforms of derivatives

The Fourier transforms of derivatives of a function $f(x)$ whose Fourier transform exists are given by

$$\mathcal{F}\{f'(x)\} = (-\iota k)F(k) \tag{7.4.1}$$

where $f(x)$ is supposed to tend to zero as $x \to \pm\infty$.

$$\mathcal{F}\{f''(x)\} = (-\iota k)^2 F(k) \tag{7.4.2}$$

where $f(x)$, $f'(x)$ are supposed to tend to 0 as $x \to \pm\infty$. and

$$\mathcal{F}\{f^n(x)\} = (-\iota k)^n F(k) \tag{7.4.3}$$

where $f(x)$, $f'(x)$, $\cdots$ $f^{n-1}(x) \to 0$ as $x \to \pm\infty$.

Proof

For (7.4.1) we have

$$\begin{aligned}
\mathcal{F}\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\iota kx} f'(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}}\left[e^{\iota kx} f(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} f(x)(\iota k)e^{\iota kx}\, dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[0 + (-\iota k)\int_{-\infty}^{\infty} f(x)\, e^{\iota kx}\, dx\right] \\
&= (-\iota k)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{\iota kx} dx = (-\iota k)\cdot F(k)
\end{aligned}$$

For (7.4.2)

$$\begin{aligned}
\mathcal{F}\{f''(x)\} &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} e^{\iota kx} f''(x)\, dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[e^{\iota kx} f'(x)\Big|_{-\infty}^{+\infty} - \iota k\int_{-\infty}^{+\infty} e^{\iota kx} f'(x)\, dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[0 + (-\iota k)\int_{-\infty}^{+\infty} e^{\iota kx}\, f'(x)\, dx\right]
\end{aligned}$$

$$= (-\iota k)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota kx} f'(x)\,dx$$
$$= (-\iota k)\mathcal{F}\{f'(x)\} = (-\iota k)(-\iota k)F(k)$$

or finally

$$\boxed{\mathcal{F}\{f''(x)\} = (-\iota k)^2 F(k) = -k^2 F(k)}$$

For (7.4.3) we have

$$\begin{aligned}
\mathcal{F}\{f^n(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota kx} f^n(x)\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{\iota kx} f^{n-1}(x)\right]_{-\infty}^{+\infty} - (\iota k)\int_{-\infty}^{+\infty} e^{\iota kx} f^{n-1}(x)\,dx\right\} \\
&= \frac{(-\iota k)}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} e^{\iota kx} f^{n-1}(x)\,dx\right] \\
&= \frac{(-\iota k)}{\sqrt{2\pi}}\left\{\left[e^{\iota kx} f^{n-2}(x)\right]_{-\infty}^{+\infty} - (\iota k)\int_{-\infty}^{+\infty} e^{\iota kx} f^{n-2}(x)\,dx\right\} \\
&= \frac{(-\iota k)^2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota kx} f^{n-2}(x) dx
\end{aligned}$$

Continuing this process we get

$$\mathcal{F}\{f^n(x)\} = (-\iota k)^n \int_{-\infty}^{+\infty} f(x)\, e^{\iota kx}\, dx = (-\iota k)^n F(k)$$

7.4.2 Fourier transform of functions of the form $x^n f(x)$

Let n be a poritive integer and $f(x)$ a piecewise continuous function on the interval $[-\ell, +\ell]$ for every positive ℓ. Suppose that $\int_{-\infty}^{+\infty} |x^n f(x)| dx$ converges. Then

$$\mathcal{F}\{x^n f(x)\} == (-\imath)^n \left(\frac{d}{dk}\right)^n F(k) = \imath^n F^{(n)}(k)$$

Proof

Using the definition

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota kx} f(x) dx$$

and differentiating successively w.r.t. k we obtain

$$F'(k) = \imath\,\mathcal{F}\{x\,f(x)\}, \quad F''(k) = \imath^2\mathcal{F}\{x^2\,f(x)\}$$
$$F^{(n)}(k) = \imath^n\mathcal{F}\{x^n\,f(x)\}$$

From the last relation we obtain

$$\mathcal{F}\left\{x^n\,f(x)\right\} = \frac{1}{\imath^n}\,F^{(n)}(k) = \imath^{-n}\,F^{(n)}(k)$$

7.4.3 Fourier transform of an integral

We suppose that $f(x)$ is piecewise continuous on $(-\infty, +\infty)$ and that $\int_{-\infty}^{+\infty}|f(x)|dx < \infty$. Also let $F(0) = 0$, with $F(k) = \mathcal{F}\left\{f(x)\right\}$. Then

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,dx'\right\} = \frac{1}{\imath k}\,F(k)$$

Proof

Let

$$g(x) = \int_{-\infty}^{x} f(x')\,dx' \quad (1)$$

Then by the Leibniz rule (see appendix B), $g'(x) = f(x)$, whenever $f(x)$ is continuous. Form the defining relation for $g(x)$, $g(-\infty) = 0$.

Now using the definitions

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\imath kx}\,f(x)dx, \ \cancel{\text{and}} \Rightarrow \ F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,dx$$

~~we have~~

$$\text{Now } (1) \Rightarrow \lim_{x\to\infty} g(x) = \int_{-\infty}^{+\infty} f(x)\,dx = \sqrt{2\pi}\,F(0) = 0$$

Now using the result

$$\mathcal{F}\{g'(x)\} = \imath k\,\mathcal{F}\left\{g(x)\right\}$$

we have

$$\begin{aligned} F(k) &= \mathcal{F}\{f(x)\} = \mathcal{F}\{g'(x)\} = \imath k\mathcal{F}\{g(x)\} \\ &= \imath k\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,dx'\right\} \end{aligned}$$

which gives

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,dx'\right\} = \frac{1}{\imath k}\,F(k)$$

7.5 Convolution and Convolution Theorem

Definition

If $f(x)$ and $g(x)$ are functions of x defined over the interval $(-\infty, +\infty)$ then their convolution, denoted by $f \star g$ is defined as follows

$$f \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta) g(x - \eta)\, d\eta$$

We can show that $g \star f = f \star g$.

Now consider

$$\begin{aligned} f \star g &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)\, g(x - \eta)\, d\eta \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x - x')\, g(x')\, dx', \quad \text{where } x - \eta = x' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x')\, f(x - x')\, dx' = g \star f \end{aligned}$$

$\Gamma(\alpha)$

$(\alpha-1)!$

In a similar manner we can prove the following properties of the convolution.

$\alpha(\alpha-1)!$

$$f \star (g \star h) = (f \star g) \star h$$

and

$$f \star (g + h) = f \star g + f \star h$$

7.5.1 Convolution theorem

If $F(k)$ and $G(k)$ are Fourier transforms of $f(x)$ and $g(x)$, then

$$\mathcal{F}\{f \star g\} = F(k)\, G(k)$$

or

$$\mathcal{F}^{-1}\{F(k)\, G(k)\} = f(x) \star g(x)$$

Proof

$$\begin{aligned} \mathcal{F}^{-1}\{F(k)G(k)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, F(k)G(k)\, dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k) \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} g(x')\, dx' \right\} d \end{aligned}$$

$L(t^n) = \frac{n!}{s^{n+1}}$

where we have used the definition of the Fourier transform of $g(x)$. Changing the order of integration we have

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} e^{-ik(x-x')}F(k)dk\right\}g(x')\,dx'$$

Now by definition

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ik(x-x')}\,F(k)\,dk = f(x-x')$$

We find that

$$\mathcal{F}^{-1}\{F(k)\,G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-x')\,g(x')\,dx' = f\star g$$

Hence

$$\mathcal{F}^{-1}\{F(k)\,G(k)\} = f(x)\star g(x)$$

which is equivalent to

$$F(k)G(k) = \mathcal{F}\{f\star g\}$$

7.6 Parseval's Theorems

These theorems are named after the French mathematician Marc Antoine des Chenes Parseval (1755-1836). Note that there are similar theorems in the theory of Fourier series. These theorems are also referred to as Parseval's identities.

The Parseval first and the second theorem may be stated as follows:

$$\int_{-\infty}^{+\infty} F(k)\,\overline{F(k)}\,dk = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx \quad\text{or}\quad \int_{-\infty}^{\infty}|F(k)|^2\,dk = \int_{-\infty}^{+\infty}|f(x)|^2\,dx$$

and

$$\int_{-\infty}^{+\infty} F(k)\,G(k)\,dk = \int_{-\infty}^{+\infty} f(u)g(-u)\,du = \int_{-\infty}^{+\infty} f(x)g(-x)\,dx$$

Proof

We prove the second theorem. The first follows from it as a special case.
By convolution theorem

$$\mathcal{F}^{-1}\{F(k)\,G(k)\} = f(x) \star g(x)$$

or

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} F(k) G(k)\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, g(x-u)\, du$$

Putting $x = 0$ on both sides, we have

$$\int_{-\infty}^{+\infty} F(k)\, G(k)\, dk = \int_{-\infty}^{+\infty} f(u) g(-u)\, du = \int_{-\infty}^{+\infty} f(x)\, g(-x)\, dx$$

which is Parseval's second theorem. To derive the first theorem from it, we take $g(-x) = \overline{f(x)} i.e. g(x) = \overline{f(-x)}$.

Therefore

$$\mathcal{F}\{g(x)\} = \mathcal{F}\{\bar{f}(-x)\} \text{ or } G(k) = \overline{F(k)}$$

Hence

$$\int_{-\infty}^{+\infty} F(k)\, \overline{F(k)}\, dk = \int_{-\infty}^{\infty} f(x)\, \overline{f(x)}\, dx \text{ or } \int_{-\infty}^{\infty} |F(k)|^2\, dk = \int_{-\infty}^{\infty} |f(x)|^2$$

which can equivalently be written as

$$||F|| = ||f||$$

q.4.A

7.7 The Fourier Integral Theorem

If $f(x)$ is a real function defined over $(-\infty, +\infty)$, and the integral $\int_{-\infty}^{+\infty} f(x)\, dx$ is absolutely convergent, then

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cancel{dk} \int_{-\infty}^{+\infty} \cos k(\xi - x) f(\xi)\, d\xi \;\; dk$$

Proof

Since the integral $\int_{-\infty}^{\infty} f(x)\, dx$ is absolutely convergent, its Fourier transform and the inverse both exist. Therefore

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k)\, dk$$

Splitting the infinite integral in parts and using the fact that $f(x)$ is real, we have

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-\iota kx} F(k)\, dk + \int_{-\infty}^{0} e^{-\iota kx} F(k)\, dk\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\iota kx} F(k)\, dk + \int_{\infty}^{0} e^{\iota k' x} F(-k')\,(-dk'), \quad k' = -k \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-\iota kx} F(k)\, dk + \int_0^{\infty} e^{\iota k' x} F(-k')\, dk'\right]
\end{aligned}
$$

Since $f(x)$ is real, by conjugation property, we must have $F(-k) = \overline{F(k)}$. Therefore we can write

$$
f(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[e^{-\iota kx} F(k) + e^{\iota kx} \overline{F(k)}\right] dk \qquad 1
$$

Now

$$
F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota kx'} f(x')\, dx'
$$

Therefore

$$
e^{-\iota kx} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\iota k(x-x')} f(x')\, dx' \qquad 2
$$

and taking complex conjugate of both sides

$$
e^{\iota kx} \overline{F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\iota k(x-x')} f(x')\, dx' \qquad 3
$$

2 & 3

Adding ~~the last two~~ equations, we have

$$
\begin{aligned}
e^{-\iota kx} F(k) + e^{\iota kx} \bar{F}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x') \left[e^{\iota k(x-x')} + e^{-\iota k(x-x')}\right] dx' \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, 2\cos k(x-x')\, dx'
\end{aligned}
$$

Substituting in the equation for $f(x)$, we obtain

$$
f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{+\infty} f(x') \cos k(x-x')\, dx'\, dk
$$

7.8 Fourier Sine and Cosine Transforms

If a function $f(x)$ is defined over the interval $[0, \infty)$, then we can define its Fourier sine transform $F_s(k)$ or $\mathcal{F}_s\{f(x)\}$ as

$$\boxed{F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin kx \, dx} \tag{7.8.1}$$

Similarly the Fourier cosine transform is defined as

$$F_c(k) \equiv \mathcal{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos kx \, dx \tag{7.8.2}$$

The inverse relations to (7.8.1) and (7.8.2) are given by

$$f(x) \equiv \mathcal{F}_s^{-1}\{F_s(k)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(k) \sin kx \, dk \tag{7.8.3}$$

and

$$f(x) \equiv \mathcal{F}_c^{-1}\{F_c(k)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(k) \cos kx \, dk \tag{7.8.4}$$

7.8.1 Justification for the Definitions

The above definitions follow directly from the definition of complex or exponential Fourier transform. Let a real function $f(x)$ be defined over $[0, \infty)$. Its even extension $f_e(x)$ over the whole real line can be defined as

$$f_e(x) = \begin{cases} f(x), & \text{for } 0 \le x < \infty \\ f(-x), & \text{for } -\infty < x < 0 \end{cases}$$

Now the Fourier transform of $f_e(x)$ is given by

$$\begin{aligned}
\mathcal{F}\{f_e(x)\} \equiv F_e(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx} f_e(x) \, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^{+\infty} e^{ikx} f(x) + \int_{-\infty}^{0} e^{ikx} f(-x)\right] dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left[e^{ikx} + e^{-ikx}\right] f(x) \, dx
\end{aligned}$$

or

$$F_e(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos kx \, f(x) \, dx$$

Similarly if we define the odd extension of $f(x)$ over the whole real line as

$$f_o(x) = \begin{cases} f(\ x), & \text{for } 0 \le x < \infty \\ -f(-x), & \text{for } \infty < x < 0 \end{cases}$$

and perform similar calculations, we at the definition of the Fourier sine transform given above.

7.9 Fourier Sine and Cosine Transforms of Derivatives

To calculate Fourier sine and cosine transforms of first order derivative, we assume that (i) $f(x)$ is real and (ii) $|f(x)| \to 0$ as $x \to \infty$. Then

$$\begin{aligned} \mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos kx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[\cos kx \, f(x)]_0^\infty + k \int_0^\infty f(x) \sin kx \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \times 0 - \sqrt{\frac{2}{\pi}} f(0) + k\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx \end{aligned}$$

Therefore

$$\mathcal{F}_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}} f(0) + kF_s(k) \qquad (7.9.1)$$

Similarly

$$\begin{aligned} \mathcal{F}_s\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin kx \, dx \\ &= \sqrt{\frac{2}{\pi}} f(x) \sin kx]_0^\infty - k \int_0^\infty f(x) \cos kx \, dx \\ &= -\sqrt{\frac{2}{\pi}} \times 0 - \sqrt{\frac{2}{\pi}} \times 0 - k\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx \end{aligned}$$

$$\mathcal{F}_s\{f'(x) = -k\, F_c(k) \qquad (7.9.2)$$

To calculate Fourier sine and cosine transforms of second order derivatives, we assume further that (iii) $|f'(x)| \to 0$, as $x \to \infty$, then

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \sin kx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\sin kx\ f'(x)\right]_0^\infty - k\int_0^\infty f'(x)\cos kx\, dx\Big]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - k\int_0^\infty f'(x)\cos kx\, dx\right]$$

$$= -k\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos kx\ dx$$

$$= -k\,\mathcal{F}_c\{f'(x)\}$$

$$= -k\left[-\sqrt{\frac{2}{\pi}}f(0) + kF_s(k)\right]$$

Therefore

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}k\,f(0) - k^2\,F_s(k)$$

Similarly

$$F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[f'(x)\cos kx\right]_0^\infty + k\int_0^\infty f'(x)\ \sin kx dx\Big]$$

$$= 0 - \sqrt{\frac{2}{\pi}}f'(0) + k\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin kx\ dx$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + k\mathcal{F}_s\{f'(x)\}$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + k[-kF_c(k)]$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) - k^2F_c(k)$$

7.10 Illustrative Examples

Example 1

Calculate Fourier sine transform of the function $f(x) = e^{-x}\cos x$.

Solution

Also F.C

By definition

$$\mathcal{F}_s\{e^{-x}\cos x\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x \sin kx dx$$

Using the result $\sin A \cos B = (1/2)[\sin(A+B)+\sin(A-B)]$, we have

$$\sin kx . \cos x = \frac{1}{2}[\sin(k+1)x + \sin(k-1)x]$$

Therefore

$$\mathcal{F}_s\{e^{-x}\cos x\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\{\sin(k+1)x+\sin(k-1)x\}\,dx$$

Now we use the formula

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\}$$

and obtain

$$\begin{aligned}\int_0^\infty e^{-x}\sin(k+1)x\,dx &= \frac{e^{-x}}{1+(k+1)^2}\times \\ &\times \{[-\sin(k+1)x-(k+1)\cos(k+1)\,x\}]_0^\infty \\ &= 0-\frac{1}{k^2+2k+2}(0-(k+1)) \\ &= \frac{k+1}{k^2+2k+2}\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^\infty e^{-x}\sin(k-1)x\,dx &= \frac{e^{-x}}{1+(k-1)^2}\times \\ &\times \{-\sin(k-1)x-(k-1)\cos(k-1)\}]_0^\infty \\ &= 0-\frac{1}{k^2-2k+2}(0-(k-1)) \\ &= \frac{k-1}{k^2-2k+2}\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{F}_s\{e^{-x}\cos x\} &= \frac{1}{\sqrt{2\pi}}\left[\frac{k+1}{k^2+2k+2}+\frac{k-1}{k^2-2k+2}\right] \\ &= \frac{1}{\sqrt{2\pi}}\frac{2k^3}{k^4+4} = \frac{2}{\sqrt{\pi}}\frac{k^3}{k^4+4}\end{aligned}$$

Example 2

Calculate Fourier sine transform of the function Ah F.C.

$$f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi, \\ 0, & x > \pi \end{cases}$$

Solution

$$\begin{aligned}
\mathcal{F}_s\{f(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin kx \, dx + \sqrt{\frac{2}{\pi}} \int_\pi^\infty f(x) \sin kx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin kx \, dx + \sqrt{\frac{2}{\pi}} \int_\pi^\infty 0 \sin kx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin kx \, dx \\
&= -\frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\pi [\cos(k+1)x - \cos(k-1)x] \, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\pi [-\cos(k+1)x + \cos(k-1)x] \, dx \\
&= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{\sin(k+1)x}{k+1} + \frac{\sin(k-1)x}{k-1} \right\} \Bigg]_0^\pi \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{\sin(k+1)\pi}{k+1} + \frac{\sin(k-1)\pi}{k-1} \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2-1} [-(k-1)\sin(k+1)\pi + (k+1)\sin(k-1)\pi] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2-1} [(k-1)\sin k\pi - (k+1)\sin k\pi] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2-1} (-2\sin k\pi) \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin k\pi}{1-k^2}
\end{aligned}$$

Example 3

Show that

$$\text{(a)} \quad \mathcal{F}_s\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{2a^2 k}{(a^2+k^2)^2}$$