Lecture 17

Introduction to Determinant

In algebra, the **determinant** is a special number associated with any square matrix. As we have studied in earlier classes, that the determinant of 2 x 2 matrix is defined as the product of the entries on the main diagonal minus the product of the entries off the main diagonal. The determinant of a matrix A is denoted by det (***A***) or |A|

For example: $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then det (***A***) = *ad-bc*.

or |A| = ad – bc

Example Find the determinant of the matrix $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$|A|=\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}=1\times 4-2\times 3=4-6=-2$$

To extend the definition of the det(***A***) to matrices of higher order, we will use subscripted entries for ***A***.

$$A=\begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

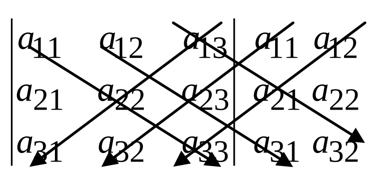
$$\det(\boldsymbol{A})=\begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix}=a_{11}b_{22}-a_{12}b_{21}$$

This is called a 2x2 determinant.

The determinant of a 3x3 matrix is also called a 3x3 determinant is defined by the following formula.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1)$$

For finding the determinant of the 3x3 matrix, we look at the following diagram:



We write 1st and 2nd columns again beside the determinant. The first arrow goes from a_{11} to $a_{33,}$ which gives us product: $a_{11}a_{22}a_{33}$. The second arrow goes from a_{12} to $a_{31,}$ which gives us product: $a_{12}a_{23}a_{31}$. The third arrow goes from a_{13} to $a_{32,}$ which gives us the product: $a_{13}a_{21}a_{32}$. These values are taken with positive signs.

The same method is used for the next three arrows that go from right to left downwards, but these products are taken as negative signs.

$$= a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example 2 Find the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$

$$detA = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= 1\times5\times9 + 2\times6\times7 + 3\times(-4)\times(-8) - 3\times5\times7 - 1\times6\times(-8) - 2\times(-4)\times9$$

$$= 45+84+96-105+48+72$$

$$= 240$$

We saw earlier that a 2×2 matrix is invertible if and only if its determinant is nonzero. In simple words, a matrix has its inverse if its determinant is nonzero. To extend this useful fact to larger matrices, we need a definition for the determinant of the $n\times n$ matrix. We can discover the definition for the 3×3 case by watching what happens when an invertible 3×3 matrix ***A*** is row reduced.

<u>Gauss' algorithm for evaluation of determinants</u>

1) Firstly, we apply it for 2×2 matrix say

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$$

$R_2' \to R_2 - 2R_1$ (Multiplying 1st row by 2 and then subtracting from 2nd row)

$$\sim \begin{bmatrix} 2 & 3 \\ 4-2(2) & 3-2(3) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix}$$

Now the determinant of this upper triangular matrix is the product of its entries on main diagonal that is

$Det(A) = 2(-3) - 0\times 3 \;\; = -6 - 0 \;\; = -6$

2) For 3×3 matrix say

$$B = \begin{bmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

By $R'_{12} \rightarrow R_{21}$ (Interchanging of 1st and 2nd rows)

$$\sim \begin{bmatrix} -1 & 1 & 3 \\ -2 & 2 & -3 \\ 2 & 0 & -1 \end{bmatrix}$$

$R'_2 \rightarrow R_2 - 2R_1$ (Multiplying 1st row by '-2' and then adding in the 2nd row)

$R'_3 \rightarrow R_3 + 2R_1$ (Multiplying 1st row by '2' and then adding in the 3rd row)

$$\sim \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & -9 \\ 0 & 2 & 5 \end{bmatrix}$$

By $R'_{23} \rightarrow R_{32}$ (Interchanging of 2nd and 3rd rows)

$$\sim \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -9 \end{bmatrix}$$

Now the determinant of this upper triangular matrix is the product of its entries on main diagonal and that is

$Det(B) = (-1)\cdot 2 \cdot (-9) = 18$

So in general,

<u>For a 1×1 matrix</u>

say, $A = [a_{ij}]$ - we define $\det A = a_{11}$.

<u>For 2×2 matrix</u>

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

By $R'_2 \rightarrow R_2 - \left(\dfrac{a_{21}}{a_{11}}\right) R_1$ provided that $a_{11} \neq 0$

$$\sim \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \dfrac{a_{21}}{a_{11}} a_{12} \end{bmatrix}$$

$\therefore \Delta = \det A =$ product of the diagonal entries

$$= a_{11}\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right) = a_{11}a_{22} - a_{12}a_{21}$$

For 3×3 matrix say

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

By $R_2' \to R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1,\ R_3' \to R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1$ provided that $a_{11} \neq 0$

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}} & \frac{a_{23}a_{11} - a_{13}a_{21}}{a_{11}} \\ 0 & \frac{a_{32}a_{11} - a_{12}a_{31}}{a_{11}} & \frac{a_{11}a_{33} - a_{13}a_{31}}{a_{11}} \end{bmatrix}$$

By $R_3' \to R_3 - \left(\frac{\frac{a_{32}a_{11} - a_{12}a_{31}}{a_{11}}}{\frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}}}\right)R_2$ provided that $\frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}} \neq 0$

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}} & \frac{a_{23}a_{11} - a_{13}a_{21}}{a_{11}} \\ 0 & 0 & \frac{a_{11}a_{33} - a_{13}a_{31}}{a_{11}} - \left(\frac{a_{23}a_{11} - a_{13}a_{21}}{a_{11}}\right)\left(\frac{\frac{a_{32}a_{11} - a_{12}a_{31}}{\not a_{11}}}{\frac{a_{22}a_{11} - a_{12}a_{21}}{\not a_{11}}}\right) \end{bmatrix} \because a_{11} \neq 0$$

Which is in echelon form.Now,

$\Delta = \det A =$ product of the diagonal entries

$$= \not{a}_{11}\left(\frac{a_{22}a_{11}-a_{12}a_{21}}{\not{a}_{11}}\right)\left(\frac{a_{11}a_{33}-a_{13}a_{31}}{a_{11}}-\left(\frac{a_{23}a_{11}-a_{13}a_{21}}{a_{11}}\right)\left(\frac{a_{32}a_{11}-a_{12}a_{31}}{a_{22}a_{11}-a_{12}a_{21}}\right)\right)$$

$$=\left(a_{22}a_{11}-a_{12}a_{21}\right)\left(\frac{a_{11}a_{33}-a_{13}a_{31}}{a_{11}}\right)-\left(a_{22}a_{11}-a_{12}a_{21}\right)\left(\frac{a_{23}a_{11}-a_{13}a_{21}}{a_{11}}\right)\left(\frac{a_{32}a_{11}-a_{12}a_{31}}{a_{22}a_{11}-a_{12}a_{21}}\right)$$

$$=\frac{1}{a_{11}}\left\{\left(a_{22}a_{11}-a_{12}a_{21}\right)\left(a_{11}a_{33}-a_{13}a_{31}\right)-\left(a_{23}a_{11}-a_{13}a_{21}\right)\left(a_{32}a_{11}-a_{12}a_{31}\right)\right\}$$

$$=\frac{1}{a_{11}}\left\{a_{11}^2a_{22}a_{33}-a_{11}a_{22}a_{13}a_{31}-a_{12}a_{21}a_{11}a_{33}+a_{12}a_{21}a_{13}a_{31}-a_{23}a_{11}^2a_{32}+a_{23}a_{11}a_{12}a_{31}+a_{13}a_{21}a_{32}a_{11}-a_{12}a_{21}a_{13}a_{31}\right\}$$

$$=\frac{1}{a_{11}}\left\{a_{11}^2a_{22}a_{33}-a_{11}a_{22}a_{13}a_{31}-a_{12}a_{21}a_{11}a_{33}-a_{23}a_{11}^2a_{32}+a_{23}a_{11}a_{12}a_{31}+a_{13}a_{21}a_{32}a_{11}\right\}$$

$$=\frac{a_{11}}{a_{11}}\left\{a_{11}a_{22}a_{33}-a_{22}a_{13}a_{31}-a_{12}a_{21}a_{33}-a_{23}a_{11}a_{32}+a_{23}a_{12}a_{31}+a_{13}a_{21}a_{32}\right\}$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}-a_{13}a_{22}a_{31}$$

□

Since ***A*** is invertible, Δ must be nonzero. *The converse is true as well*.

To generalize the definition of the determinant to larger matrices, we will use 2×2 determinants to rewrite the 3×3 determinant Δ described above. Since the terms in Δ can be grouped as:

$$\begin{aligned}\Delta &= (a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32})-(a_{12}a_{23}a_{31}-a_{12}a_{21}a_{33})+(a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}) \\ &= a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})\end{aligned},$$

$$\Delta = a_{11}\cdot\det\begin{bmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix}-a_{12}\cdot\det\begin{bmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{bmatrix}+a_{13}\cdot\det\begin{bmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix}$$

$$\Delta = a_{11}\cdot\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\cdot\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

For brevity, we write $\Delta = a_{11}\cdot\det A_{11}-a_{12}\cdot\det A_{12}+a_{13}\cdot\det A_{13}$ (3)

$$\det(A_{11})=\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix},\quad \det(A_{12})=\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}\quad\text{and}\quad \det(A_{13})=\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

where

$\boldsymbol{A_{11}}$ is obtained from ***A*** by deleting the first row and first column.
$\boldsymbol{A_{12}}$ is obtained from ***A*** by deleting the first row and second column.
$\boldsymbol{A_{13}}$ is obtained from ***A*** by deleting the first row and third column.
So in general, for any square matrix ***A***, let $\boldsymbol{A_{ij}}$ denote the sub-matrix formed by deleting the *ith* row and *jth* column of ***A***.

Let's understand it with the help of an example.

Example3

Find the determinant of the matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ 5 & 2 & 4 \\ 3 & 6 & 3 \end{bmatrix}$

Solution Given $A = \begin{bmatrix} 1 & 4 & 3 \\ 5 & 2 & 4 \\ 3 & 6 & 3 \end{bmatrix}$

$$
\begin{aligned}
|A| &= \begin{vmatrix} 1 & 4 & 3 \\ 5 & 2 & 4 \\ 3 & 6 & 3 \end{vmatrix} \\
&= 1\begin{vmatrix} 2 & 4 \\ 6 & 3 \end{vmatrix} - 4\begin{vmatrix} 5 & 4 \\ 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 5 & 2 \\ 3 & 6 \end{vmatrix} \\
&= 1(2\times3-4\times6)-4(5\times3-4\times3)+3(5\times6-2\times3) \\
&= 1(6-24)-4(15-12)+3(30-6) \\
&= 1(-18)-4(3)+3(24) \\
&= -18-12+72 \\
&= 42
\end{aligned}
$$

For instance, if $A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$

then A_{32} is obtained by crossing out row 3 and column 2,

$$
\begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \quad \text{so that} \quad A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}
$$

We can now give a recursive definition of a determinant.

When n = 3, det ***A*** is defined using determinants of the 2×2 submatrices A_{1j}.
When n = 4, det ***A*** uses determinants of the 3×3 submatrices A_{1j}
In general, an nxn determinant is defined by determinants of $(n-1)\times(n-1)$ sub matrices.

Definition

For $n \geq 2,$ the **determinant** of $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \times (\det A_{1j})$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \cdots, a_{1n}$ are from the first row of ***A***.

Here for a_{ij},

$i = 1, 2, 3, \ldots, n \quad (1 \leq i \leq n)$

$j = 1, 2, 3, \ldots, n \quad (1 \leq j \leq n)$

In symbols, $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + ... + (-1)^{1+n} a_{1n} \det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$

Example 4

Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

Solution

Here A is $n \times n = 3 \times 3$ matrix such that

$i = 1, 2, 3$

$j = 1, 2, 3$

$\because \det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$ and here $j = 1, 2, 3$

$$\therefore \det(A) = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j} = (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13}$$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$\det A = 1.\det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5.\det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0.\det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$\det A = 1.\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5.\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0.\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

= 1 [4(0) – (-1)(-2)] -5 [2(0) – 0(-1)] +0[2(-2) – 4(0)]

$$= 1(0 - 2) - 5(0 - 0) + 0(-4 - 0) = -2$$

Minor of an element

If ***A*** is a square matrix, then the **Minor** of entry a_{ij} (called the ijth minor of ***A***) is denoted by $\boldsymbol{M_{ij}}$ and is defined to be the determinant of the sub matrix that remains when the *ith* row and *jth* column of ***A*** are deleted.

In the above example, Minors are as follows:

$$M_{11} = \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix}, \quad M_{12} = \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}, \quad M_{13} = \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

Cofactor of an element

The number $C_{ij}=(-1)^{i+j}M_{ij}$ is called the cofactor of entry a_{ij}(or the *ijth* cofactor of A). When the + or – sign is attached to the Minor, then Minor becomes a cofactor.

In the above example, following are the Cofactors:

$$C_{11} = (-1)^{1+1} M_{11}, \quad C_{12} = (-1)^{1+2} M_{12}, \quad C_{13} = (-1)^{1+3} M_{13}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix}, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}, \quad C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

Example 5 Find the minor and the cofactor of the matrix $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

Solution Here $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5\times 8 - 6\times 4 = 40 - 24 = 16$$

and the corresponding cofactor is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

The minor of entry a_{32} is

$$\text{M}_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

and the corresponding cofactor is

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = -26$$

Alternate Definition

Given $A=[a_{ij}]$, the (i, j)-cofactor of ***A*** is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij} \qquad (4)$$

Then $\det A = a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n}$

This formula is called the *cofactor expansion across the first row* of ***A***.

Example 6 Expand a 3x3 determinant using cofactor concept $A = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix}$

Solution Using cofactor expansion along the first column;

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = (1)(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} + (-4)(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -8 & 9 \end{vmatrix} + (7)(-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

Now if we compare it with the formula (4),

$$= 1C_{11} + (-4)C_{21} + 7C_{31}$$

$$= (1)(-1)^2\begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} + (-4)(-1)^3\begin{vmatrix} 2 & 3 \\ -8 & 9 \end{vmatrix} + (7)(-1)^4\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= (1)(1)\begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} + (-4)(-1)\begin{vmatrix} 2 & 3 \\ -8 & 9 \end{vmatrix} + (7)(1)\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ -8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= 1(45-(-48)) + 4(18-(-24)) + 7(12-15)$$

$$= 1(45+48) + 4(18+24) + 7(12-15)$$

$$= (1)(93) + (4)(42) + (7)(-3) = 240$$

Using cofactor expansion along the second column,

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = (2)(-1)^{1+2} \begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + (5)(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-8)(-1)^{3+2} \begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix}$$

$$= (2)(-1)^{3} \begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + (5)(-1)^{4} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-8)(-1)^{5} \begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix}$$

$$= (2)(-1) \begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + (5)(1) \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-8)(-1) \begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix}$$

$$= -2 \begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8 \begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix}$$

$$= -2(-36-42) + 5(9-21) + 8(6-(-12))$$

$$= (-2)(-78) + (5)(-12) + (8)(18) = 240$$

Theorem 1 The determinant of an $n \times n$ matrix $\boldsymbol{A}$ can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix, regardless of the sign of a_{ij} itself. The factor $(-1)^{i+j}$ determines the following checkerboard pattern of signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Example 7 Use a cofactor expansion across the third row to compute det ***A***, where

$$A=\begin{bmatrix}1 & 5 & 0\\ 2 & 4 & -1\\ 0 & -2 & 0\end{bmatrix}$$

Solution Compute $\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$$= (-1)^{3+1}a_{31}\det A_{31} + (-1)^{3+2}a_{32}\det A_{32} + (-1)^{3+3}a_{33}\det A_{33}$$

$$= 0\begin{vmatrix}5 & 0\\ 4 & -1\end{vmatrix} - (-2)\begin{vmatrix}1 & 0\\ 2 & -1\end{vmatrix} + 0\begin{vmatrix}1 & 5\\ 2 & 4\end{vmatrix}$$

$$= 0 + 2(-1) + 0 = -2$$

Theorem 1 is helpful for computing the determinant of a matrix that contains many zeros. For example, if a row is mostly zeros, then the cofactor expansion across that row has many terms that are zero, and the cofactors in those terms need not be calculated.
The same approach works with a column that contains many zeros.

Example 8 Evaluate the determinant of $A=\begin{bmatrix}2 & 0 & 0 & 5\\ -1 & 2 & 4 & 1\\ 3 & 0 & 0 & 3\\ 8 & 6 & 0 & 0\end{bmatrix}$

Solution $\det(A) = \begin{vmatrix}2 & 0 & 0 & 5\\ -1 & 2 & 4 & 1\\ 3 & 0 & 0 & 3\\ 8 & 6 & 0 & 0\end{vmatrix}$

Expand from third column

$$\det(A) = 0\times C_{13} + 4\times C_{23} + 0\times C_{33} + 0\times C_{43}$$

$$= 0 \ + 4\times C_{23} \ + 0 + 0$$

$$= \ 4\times C_{23}$$

$$= 4\times(-1)^{2+3}\begin{vmatrix} 2 & 0 & 5 \\ 3 & 0 & 3 \\ 8 & 6 & 0 \end{vmatrix}$$

Expand from second column

$$= -4\left(0 \ + \ 0 \ + \ (-6)\begin{vmatrix} 2 & 5 \\ 3 & 3 \end{vmatrix}\right)$$

$$= (-4) \ (-6)\begin{vmatrix} 2 & 5 \\ 3 & 3 \end{vmatrix}$$

$$= -216$$

Example 9 Show that the value of the determinant is independent of θ

$$A = \begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \cos\theta - \sin\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

Solution Consider $A = \begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \cos\theta - \sin\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$

Expand the given determinant from 3rd column we have

$$= 0 - 0 + (-1)^{3+3}[\sin^2\theta + \cos^2\theta] = 1$$

Example 10 Compute det ***A***, where $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

Solution The cofactor expansion down the first column of ***A*** has all terms equal to zero except the first.

Thus $$\det A = 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} - 0.C_{21} + 0.C_{31} - 0.C_{41} + 0.C_{51}$$

Henceforth, we will omit the zero terms in the cofactor expansion.

Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there.

We have $$\det A = 3 \times 2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

This 3×3 determinant was computed above and found to equal –2.

Hence, det ***A*** = $3 \times 2 \times (-2) = -12$.

The matrix in this example was nearly triangular. The method in that example is easily adapted to prove the following theorem.

Triangular Matrix

A triangular matrix is a special kind of m x n matrix where the entries either below or above the main diagonal are zero.

$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ is upper triangular and 25×25 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 9 & 7 \end{bmatrix}$ is lower triangular matrices.

Determinants of Triangular Matrices

Determinants of the triangular matrices are also easy to evaluate regardless of size.

Theorem If ***A*** is triangular matrix, then det (***A***) is the product of the entries on the main diagonal.

Consider a 4×4 lower triangular matrix.

$$A=\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Keeping in mind that an elementary product must have exactly one factor from each row and one factor from each column, the only elementary product that does not have one of the six zeros as a factor is $(a_{11}a_{22}a_{33}a_{44})$. The column indices of this elementary product are in natural order, so the associated signed elementary product takes a +.

Thus, det (A)= $a_{11}\times a_{22}\times a_{33}\times a_{44}$

Example 11

$$\begin{vmatrix} -2 & 5 & 7 \\ 0 & 3 & 8 \\ 0 & 0 & 5 \end{vmatrix} = (-2)(3)(5) = -30$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ -7 & 6 & -1 & 0 \\ 3 & 8 & -5 & -2 \end{vmatrix} = (1)(9)(-1)(-2) = 18$$

$$\begin{vmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(1)(2)(3) = 6$$

The strategy in the above Example of looking for zeros works extremely well when an entire row or column consists of zeros. In such a case, the cofactor expansion along such a row or column is a sum of zeros. So, the determinant is zero. Unfortunately, most cofactor expansions are not so quickly evaluated.

Numerical Note By today's standards, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires over *n!* multiplications, and $25!\sim1.5x10^{25}$.

If a supercomputer could make one trillion multiplications per second, it would have to run for over 500,000 years to compute a 25×25 determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

Example 12 Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$

Solution Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a 3×3 matrix, which may be evaluated by an expansion down its first column,

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}2\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

$$= 2\cdot(-1)^{2+1}(-5)\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

The –1 in the next-to-last calculation came from the position of the –5 in the 3×3 determinant.

Exercises

Compute the determinants in exercises 1 to 6 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

1. $\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$

3. $\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

Use the method of Example 2 to compute the determinants in exercises 7 and 8. In exercises 9 to 11, compute the determinant of elementary matrix. In exercises 12 and 13, verify that det ***EA*** = (det ***E***) . (det ***A***), where ***E*** is the elementary matrix and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

7. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

8. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$

10. $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

14. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write 5***A***. Is det 5***A*** = 5 det ***A***?

15. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and *k* be a scalar. Find a formula that relates det (*k**A***)to *k* and det ***A***.

Lecture 18

Properties of Determinants

In this lecture, we will study the properties of the determinants. Some of them have already been discussed and you will be familiar with these. These properties become helpful, while computing the values of the determinants. The secret of determinants lies in how they change when row or column operations are performed.

Theorem 3 (Row Operations)**:** Let ***A*** be a square matrix.

a. If a multiple of one row of *A* is added to another row, the resulting determinant will remain same.
b. If two rows of ***A*** are interchanged to produce ***B***, then *det* ***B*** = –*det* ***A***.
c. If one row of ***A*** is multiplied by *k* to produce ***B***, then *det* ***B*** = *k* . *det* ***A***.

The following examples show how to use Theorem 3 to find determinants efficiently.

a. If a multiple of one row of *A* is added to another row, the resulting determinant will remain same.

Example

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$Multiplying\ 2nd\ row\ by\ non-zero\ scalar\ say\ 'k'\ as$

$ka_{21}\quad ka_{22}\quad ka_{23} ----adding\ this\ in\ 1st\ row\ then\ 'A'\ becomes$

$$= \begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad R_1' \to R_1 + kR_2$$

If each element of any row(column) can be expressed as sum of two elements then the resulting determinant can be expressed as sum of two determinants, so in this case

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{21} & ka_{22} & ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ By using property (c) of above theorem 3.

If any two rows or columns in a determinant are identical then value of this determinant is zero. So in this case $R_1 \equiv R_2$

$$\therefore\ \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k(0)$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = A$$

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

Example 1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 1 & 1 \\ 0 & 8 & 9 \end{bmatrix}$$

$$\text{Now, } \det A = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 1 & 1 \\ 0 & 8 & 9 \end{vmatrix} = 1(9-8) - 2(45-0) + 3(40-0) = 1 - 90 + 120 = 31$$

Now interchange column 1st with 2nd we get a new matrix, $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 1 \\ 8 & 0 & 9 \end{bmatrix}$

$$\det B = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 5 & 1 \\ 8 & 0 & 9 \end{vmatrix} = 2(45-0) - 1(9-8) + 3(0-40) = 90 - 1 - 120 = -31$$

c. If one row of A is multiplied by k to produce B, then $\det B = k\ .\ \det A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 1 \\ 0 & 8 & 9 \end{bmatrix}$$

$$|A| = 1(0-8) - 2(45-0) + 3(40-0)$$

$$= -8 - 90 + 120 = 22$$

Multipling R_1 *by* k*, we get say*

$$B = \begin{bmatrix} 1k & 2k & 3k \\ 5 & 0 & 1 \\ 0 & 8 & 9 \end{bmatrix}$$

$$|B| = k(40-0) - 2k(45-0) + 3k(40-0)$$
$$= 40k - 90k + 120k = 22k$$
$$= k|A|$$

Example 2

$$\text{Evaluate } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

Solution

$$\det A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix} \quad by\ R_2' \to R_2 + (-2)R_1, R_3' \to R_3 + (-3)R_1, R_4' \to R_4 + (-4)R_1$$

$$= \begin{vmatrix} -1 & -2 & -7 \\ -2 & -8 & -10 \\ -7 & -10 & -13 \end{vmatrix} \quad \text{expanding from Ist column}$$

$$= (-1)(-2)(-1)\begin{vmatrix} 1 & 2 & 7 \\ 1 & 4 & 5 \\ 7 & 10 & 13 \end{vmatrix} \quad taking\ (-1), (-2)\ and\ (-1)\ common\ from\ 1st, 2nd, 3rd\ rows$$

$$= (-2)\begin{vmatrix} 1 & 2 & 7 \\ 0 & 2 & -2 \\ 0 & -4 & -36 \end{vmatrix} \quad by\ R_2' \to R_2 + (-1)R_1, R_3' \to R_3 + (-7)R_1$$

$$= (-2)\begin{vmatrix} 2 & -2 \\ -4 & -36 \end{vmatrix} \quad \text{expanding by 1st column}$$

$$= (-2)(2)(-4)\begin{vmatrix} 1 & -1 \\ 1 & 9 \end{vmatrix} \quad \text{taking 2 and (-4) common from 1st and 2nd rows respectively.}$$

$$= 16\begin{vmatrix} 1 & -1 \\ 0 & 10 \end{vmatrix} \quad by\ R_2 + (-1)R_1$$

$$= 160$$

Example 3 Evaluate the determinant of the matrix $A = \begin{bmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{bmatrix}$

Solution

$$\det A = \begin{vmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 6 & 3 \\ 4 & 2 & 5 & 10 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix} \quad \text{interchanging } R_1 \text{ and } R_2 (R'_{12})$$

$$= -\begin{vmatrix} 1 & 1 & 6 & 3 \\ 0 & -2 & -19 & -2 \\ 0 & -4 & -42 & -16 \\ 0 & 2 & 5 & 8 \end{vmatrix} \quad By\ R_2' \to R_2 + (-4)R_1, R_3' \to R_3 + (-7)R_1$$

$$= -\begin{vmatrix} -2 & -19 & -2 \\ -4 & -42 & -16 \\ 2 & 5 & 8 \end{vmatrix} \quad \text{expanding from 1st column}$$

$$=(-1)^3 \begin{vmatrix} 2 & 19 & 2 \\ 4 & 42 & 16 \\ 2 & 5 & 8 \end{vmatrix} \quad \text{taking (-1) as a common factor from } R_1 \text{ and } R_2$$

$$= -\begin{vmatrix} 2 & 19 & 2 \\ 4 & 42 & 16 \\ 2 & 5 & 8 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 19 & 2 \\ 2 & 42 & 16 \\ 1 & 5 & 8 \end{vmatrix}$$

$$= (-2)\begin{vmatrix} 1 & 19 & 2 \\ 0 & 4 & 12 \\ 0 & -14 & 6 \end{vmatrix} \quad By\ R_2' \to R_2 + (-2)R_1, R_3' \to R_3 + (-1)R_1$$

$$= (-2)\begin{vmatrix} 1 & 19 & 2 \\ 0 & 4 & 12 \\ 0 & -14 & 6 \end{vmatrix} \quad R_2 + (-2)R_1, R_3 + (-1)R_1$$

$$= -2\begin{vmatrix} 4 & 12 \\ -14 & 6 \end{vmatrix} \text{ expand from Ist column}$$

$$= -2(24+168) = -384$$

Example 4 Without expansion, show that $\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = 0$

Solution

$$\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix}$$

$$= \begin{vmatrix} x & a+x-x & b+c \\ x & b+x-x & c+a \\ x & c+x-x & a+b \end{vmatrix} \quad By\ C_2' \to C_2 - C_1$$

$$= \begin{vmatrix} x & a & b+c \\ x & b & c+a \\ x & c & a+b \end{vmatrix}$$

Taking 'x' common from C_1

$$= x\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & a+b+c & b+c \\ 1 & b+c+a & c+a \\ 1 & c+a+b & a+b \end{vmatrix} \quad By\ C_2' \to C_2 + C_3$$

Now taking (a+b+c) common form C_2

$$= x(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$= 0$ as column Ist and 2nd are identical ($C_1 \equiv C_2$). So its value will be zero.

Example 5 Evaluate $A=\begin{vmatrix}2&3&1&0&1\\1&1&3&1&2\\2&1&2&3&4\\3&2&1&1&2\\4&1&1&0&0\end{vmatrix}$

Solution Interchanging R_1 and R_2, we get

$$A=-\begin{vmatrix}1&1&3&1&2\\2&3&1&0&1\\2&1&2&3&4\\3&2&1&1&2\\4&1&1&0&0\end{vmatrix}$$

$R_2' \to R_2-2R_1, R_3' \to R_3-2R_1, R_4' \to R_4-3R_1, R_5' \to R_5-4R_1$

$$=-\begin{vmatrix}1&1&3&1&2\\0&1&-5&-2&-3\\0&-1&-4&1&0\\0&-1&-8&-2&-4\\0&-3&-11&-4&-8\end{vmatrix}$$

expand from C_1

$$=-\begin{vmatrix}1&-5&-2&-3\\-1&-4&1&0\\-1&-8&-2&-4\\-3&-11&-4&-8\end{vmatrix}$$

$R_2' \to R_2+R_1, R_3' \to R_3+R_1, R_4' \to R_4+3R_1$

$$=-\begin{vmatrix}1&-5&-2&-3\\0&-9&-1&-3\\0&-13&-4&-7\\0&-26&-10&-17\end{vmatrix}$$

expand from C_1

$$=-\begin{vmatrix}-9&-1&-3\\-13&-4&-7\\-26&-10&-17\end{vmatrix}$$

taking (-1) common from Ist, 2nd and 3rd row

$$=\begin{vmatrix} 9 & 1 & 3 \\ 13 & 4 & 7 \\ 26 & 10 & 17 \end{vmatrix}$$

interchange Ist and 2nd Column(C'_{12})

$$=-\begin{vmatrix} 1 & 9 & 3 \\ 4 & 13 & 7 \\ 10 & 26 & 17 \end{vmatrix}$$

$C'_2 \to C_2 - 9C_1, C'_3 \to C_3 - 3C_1$

$$=-\begin{vmatrix} 1 & 0 & 0 \\ 4 & -23 & -5 \\ 10 & -64 & -13 \end{vmatrix}$$

expand from Ist row

$$=-\begin{vmatrix} -23 & -5 \\ -64 & -13 \end{vmatrix} = -(299-320) = 21$$

An Algorithm to evaluate the determinant

Algorithm means a sequence of a finite number of steps to get a desired result. The word Algorithm comes from the famous Muslim mathematician AL-Khwarizmi who invented the word algebra.

The step-by-step evaluation of det(***A***) of order ***n*** is obtained as follows:

Step 1: By an interchange of rows of ***A*** (and taking the resulting sign into account) bring a non zero entry to (1,1) the position (unless all the entries in the first column are zero in which case det ***A***=0).

Step 2: By adding suitable multiples of the first row to all the other rows, reduce the (n-1) entries, except (1,1) in the first column, to 0. Expand det(***A***) by its first column. Repeat this process or continue the following steps.

Step 3: Repeat step 1 and step 2 with the last remaining rows concentrating on the second column.

Step 4: Repeat step 1,step2 and step 3 with the remaining (n-2) rows, (n-3) rows and so on, until a triangular matrix is obtained.

Step5: Multiply all the diagonal entries of the resulting triangular matrix and then multiply it by its sign to get det(A)

Example 6 Compute det ***A***, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

Solution *The strategy is to reduce* ***A*** *to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries.* The first two row replacements in column 1 do not change the determinant:

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \quad By\ R_2' \to R_2 + 2R_1, R_3' \to R_1 + R_3$$

An interchange of rows 2 and 3 (R'_{23}), it reverses the sign of the determinant, so

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

Example 7 Compute det ***A***, where

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}.$$

Solution Taking'2' common from 1st row

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} \quad By\ R_2' \to R_2 - 3R_1, R_3' \to R_3 + 3R_1, R_4' \to R_4 - R_1$$

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (By\ R_4' \to R_4 - \tfrac{1}{2}R_3)$$

$$= 2.\{(1)(3)(-6)(1)\} = -36$$

Example 8 Show that $\begin{vmatrix} x & 2 & 2 & 2 \\ 2 & x & 2 & 2 \\ 2 & 2 & x & 2 \\ 2 & 2 & 2 & x \end{vmatrix} = (x+6)(x-2)^3$

Solution

$$\begin{vmatrix} x & 2 & 2 & 2 \\ 2 & x & 2 & 2 \\ 2 & 2 & x & 2 \\ 2 & 2 & 2 & x \end{vmatrix}$$

$$= \begin{vmatrix} x+6 & 2 & 2 & 2 \\ x+6 & x & 2 & 2 \\ x+6 & 2 & x & 2 \\ x+6 & 2 & 2 & x \end{vmatrix} \quad By\ \ C_1' \to C_1 + (C_2 + C_3 + C_4)$$

Taking (x+6) common from 1st column

$$= (x+6)\begin{vmatrix} 1 & 2 & 2 & 2 \\ 1 & x & 2 & 2 \\ 1 & 2 & x & 2 \\ 1 & 2 & 2 & x \end{vmatrix}$$

$$= (x+6)\begin{vmatrix} 1 & 2 & 2 & 2 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-2 & 0 \\ 0 & 0 & 0 & x-2 \end{vmatrix} \quad By\ R_2' \to R_2 - R_1, R_3' \to R_3 - R_1, R_4' \to R_4 - R_1$$

And this is the triangular matrix and its determinant is the prodcut of main diagonal's entries.

$$= (x+6)(x-2)^3$$

Example 9 Compute det ***A***, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Solution $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

$$\det A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} R_3' \to R_3 + 2R_1$$

$= \mathbf{0}$ as $R_2 \equiv R_3$

Example10 Compute det ***A***, where

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix} R_4' \to R_4 + R_2$$

$$= (-1)\begin{bmatrix} 2 & 1 & 2 & -1 \\ 0 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix} \quad By\ R_{12}'$$

Expanding from 1st row and 1st column

$$= -2\begin{vmatrix} 5 & -7 & 3 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$
$$= (-2)\{5(6+6) - (-7)(3-0) + 3(-9-0)\}$$
$$= 54$$

Remarks

Suppose that a square matrix ***A*** has been reduced to an echelon form ***U*** by row replacements and row interchanges.

If there are r interchanges, then $\det(A) = (-1)^r \det(U)$

Furthermore, all of the pivots are still visible in ***U*** (because they have not been scaled to ones). If ***A*** is invertible, then the pivots in ***U*** are on the diagonal (since ***A*** is row equivalent to the identity matrix). In this case, *det* ***U*** is the product of the pivots. If ***A*** is not invertible, then ***U*** has a row of zero and *det* ***U*** *= 0*.

$$U = \begin{bmatrix} \bullet & \circ & \circ & \circ \\ 0 & \bullet & \circ & \circ \\ 0 & 0 & \bullet & \circ \\ 0 & 0 & 0 & \bullet \end{bmatrix} \qquad U = \begin{bmatrix} \bullet & \circ & \circ & \circ \\ 0 & \bullet & \circ & \circ \\ 0 & 0 & \bullet & \circ \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\det U \neq 0 \qquad\qquad \det U = 0$$

Thus we have the following formula

$$\det A = \begin{cases} (-1)^r .(Product\, of\;\, pivots\, in\, U) & When\, A\, is\, invertible \\ 0 & When\, A\, is\, not\, invertible \end{cases} \qquad (1)$$

Example

Case-01 For 2×2 invertible matrix

Reducing given 2×2 invertible matrix into Echelon form as follows;

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$$

By interchanging 1st and 2nd rows(R'_{12})

$$\sim \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} \because \text{ one replacement of rows has occurred, } \therefore r = 1$$

$$\sim \begin{bmatrix} 3 & 2 \\ 0 & \frac{7}{3} \end{bmatrix} \text{ By } R_2' \to R_2 - \frac{4}{3}R_1 \text{, we have desired row-echelon form } U = \begin{bmatrix} 3 & 2 \\ 0 & \frac{7}{3} \end{bmatrix}.$$

Thus using the above formula as follows;

$\det A = (-1)^r .(Product\, of\;\; pivots\, in\, U) = (-1)^1 (3 \cdot \frac{7}{3}) = -7$

Case-02 For 2×2 non-invertible matrix
In this case say;

$$A = \begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix}$$

$\sim \begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix}$ $By\;\; R_2{}' \to R_2 - 2R_1$,desired row-echelon form is $U = \begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix}$

Here no interchange of rows has occurred. So, $r = 0$ and
$\therefore \det A = (-1)^r .(Product\, of\;\; pivots\, in\, U) = (-1)^0 (4 \cdot 0) = 0$

Theorem 5 If A is an $n \times n$ matrix, then *det* $\boldsymbol{A}^{\boldsymbol{T}}$ = *det* $\boldsymbol{A}$.

Example 11 If $A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$, find det($\boldsymbol{A}$) and det ($\boldsymbol{A}^{\boldsymbol{T}}$)

$$\det A = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{vmatrix} = 1(3-2) - 4(6-6) + 1(2-3) = 1 - 0 - 1 = 0$$

Now

$$A^t = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det A^t = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 1(3-2) - 2(12-1) + 3(8-1) = 1 - 22 + 21 = 0$$

Remark
Column operations are useful for both theoretical purposes and hand computations. However, for simplicity we'll perform only row operations in numerical calculations.

Theorem 6 (Multiplicative Property)

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices, then $\det(AB) = (\det A)(\det B)$.

Example 12 Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

Solution $AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$

and $\det AB = 25.13 - 20.14 = 325 - 280 = 45$

Since *det* ***A*** *= 9* and *det* ***B*** *= 5*, $(\det A)(\det B) = 9.5 = 45 = \det AB$

Remark

det (***A*** + ***B***) ≠ det ***A*** + det ***B***, in general.

For example**,**

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -3 \\ -1 & 5 \end{bmatrix}$. Then

$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A+B) = 0$

$\det A + \det B = \begin{vmatrix} 2 & 3 \\ 1 & -5 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 5 \end{vmatrix} = (-10-3) + (-10-3) = -26 \neq \det(A+B)$

Exercise

Find the determinants in exercises 1 to 6 by row reduction to echelon form.

1. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix}$

5. $\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$

Combine the methods of row reduction and cofactor expansion to compute the determinants in exercises 7 and 8.

7. $\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix}$

8. $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$

9. Use determinant to find out whether the matrix is invertible $\begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$

10. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 3×3 matrices, with det $\boldsymbol{A}$ = 4 and det $\boldsymbol{B}$ = -3. Use properties of determinants to compute

(a) det $\boldsymbol{AB}$ (b) det $7\boldsymbol{A}$ (c) det $\boldsymbol{B}^{\boldsymbol{T}}$ (d) det $\boldsymbol{A}^{\boldsymbol{T}}$
(e) det $\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}$

11 Show that

(a) $$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(b) $$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

12 Show that

(a) $$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(b) $$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

13. Show that $$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$