Lecture 4

Row Reduction and Echelon Forms

To analyze system of linear equations, we shall discuss how to refine the row reduction algorithm. While applying the algorithm to any matrix, we begin by introducing a non zero row or column (i.e. contains at least one nonzero entry) in a matrix,

Echelon form of a matrix

A rectangular matrix is in *echelon form* (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Reduced Echelon Form of a matrix

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* (or reduced row echelon form):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Examples of Echelon Matrix form

The following matrices are in echelon form. The leading entries ($\circ$) may have any nonzero value; the started entries (*) may have any values (including zero).

1. $\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$

2. $\begin{bmatrix} \circ & * & * & * \\ 0 & \circ & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 0 & \circ & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \circ & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \circ & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \circ & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \circ & * \end{bmatrix}$

4. $\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Examples of Reduced Echelon Form

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

1. $\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
6. $\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Note A matrix may be row reduced into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form obtained from a matrix, is unique.

Theorem 1 (Uniqueness of the Reduced Echelon Form) Each matrix is row equivalent to one and only one reduced echelon matrix.

Pivot Positions

A pivot position in a matrix ***A*** is a location in ***A*** that corresponds to a leading entry in an echelon form of ***A***.

Note When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries.

Pivot column

A **pivot column** is a column of ***A*** that contains a pivot position.

Example 2 Reduce the matrix ***A*** below to echelon form, and locate the pivot columns

$$A=\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution Leading entry in first column of above matrix is zero which is the pivot position. A nonzero entry, or pivot, must be placed in this position. So interchange first and last row.

$$\begin{bmatrix} 1^{\;Pivot} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot Column

Since all entries in a column below a leading entry should be zero. For this add row 1 in row 2, and multiply row 1 by 2 and add in row 3.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \qquad \boxed{\begin{array}{l} R_1 + R_2 \\ 2R_1 + R_3 \end{array}}$$

Next pivot column

Add –5/2 times row 2 to row 3, and add 3/2 times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \qquad \begin{array}{l} -\dfrac{5}{2}R_2 + R_3 \\ \dfrac{3}{2}R_2 + R_4 \end{array}$$

Interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \textit{General} \quad \textit{form} \qquad \begin{bmatrix} \circ & * & * & * & * \\ 0 & \circ & * & * & * \\ 0 & 0 & 0 & \circ & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot column

This is in echelon form and thus columns 1, 2, and 4 of ***A*** are pivot columns.

Pivot positions

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot columns

<u>Pivot element</u>

A pivot is a nonzero number in a pivot position that is used as needed to create zeros via row operations

The Row Reduction Algorithm consists of four steps, and it produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form.

The algorithm is explained by an example.

<u>Example 3</u> Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution

STEP 1 Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{array}{l} \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \\ \quad \text{Pivot column} \end{array}$$

STEP 2 Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{l} \quad \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

STEP 3 Use row replacement operations to create zeros in all positions below the pivot

Subtract Row 1 from Row 2. i.e. $R_2 - R_1$

$$\begin{array}{l} \quad \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

STEP 4 Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1 –3 to the sub-matrix, which remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, we'll select as a pivot the "top" entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Next pivot column

According to step 3 "*All entries in a column below a leading entry are zero*". For this subtract 3/2 time R_2 from R_3

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad R_3 - \frac{3}{2}R_2$$

When we cover the row containing the second pivot position for step 4, we are left with a new sub matrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

This is the Echelon form of the matrix.
To change it in reduced echelon form we need to do one more step:

STEP 5 Make the leading entry in each nonzero row 1. Make all other entries of that column to 0.

Divide first Row by 3 and 2^{nd} Row by 2

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \frac{1}{2}R_2 \;,\; \frac{1}{3}R_1$$

Multiply second row by 3 and then add in first row.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad 3R_2 + R_1$$

Subtract row 3 from row 2, and multiply row 3 by 5 and then subtract it from first row

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{matrix} R_2 - R_3 \\ R_1 - 5R_3 \end{matrix}$$

This is the matrix is in reduced echelon form.

Solutions of Linear Systems

When this algorithm is applied to the augmented matrix of the system it gives solution set of linear system.
Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 \quad -5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \quad \text{which means } x_3 \text{ is free} \end{aligned} \qquad (1)$$

The variables x_1 and x_2 corresponding to pivot columns in the above matrix are called basic variables. The other variable, x_3 is called a free variable.

Whenever a system is consistent, the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

In (4), we can solve the first equation for x_1 and the second for x_2. (The third equation is ignored; it offers no restriction on the variables.)

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 &\quad is \quad free \end{aligned} \qquad (2)$$

By saying that x_3 is "free", we mean that we are free to choose any value for x_3. When $x_3 = 0$, the solution is (1, 4, 0); when $x_3 = 1$, the solution is (6, 3, 1 etc).

Note The solution in (2) is called a **general solution** of the system because it gives an explicit description of all solutions.

Example 4 Find the general solution of the linear system whose augmented matrix has been reduced to $\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

Solution The matrix is in echelon form, but we want the reduced echelon form before solving for the basic variables. The symbol "~" before a matrix indicates that the matrix is row equivalent to the preceding matrix.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

By $R_1 + 2R_3$ and $R_2 + R_3$ We get

$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

By $\frac{1}{2}R_2$ we get

$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

By $R_1 - 2R_2$ we get

$$\sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

The matrix is now in reduced echelon form.
The associated system of linear equations now is

$$\begin{aligned} x_1 + 6x_2 \quad + 3x_4 &= 0 \\ x_3 - 4x_4 &= 5 \\ x_5 &= 7 \end{aligned} \tag{6}$$

The pivot columns of the matrix are 1, 3 and 5, so the basic variables are x_1, x_3, and x_5. The remaining variables, x_2 and x_4, must be free.

Solving for the basic variables, we obtain the general solution:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases} \qquad (7)$$

Note that the value of x_5 is already fixed by the third equation in system (6).

Exercise

1. Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

2. Find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

Find the general solutions of the systems whose augmented matrices are given in Exercises 3-12

3. $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & 0 & 3 & 6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 & 6 & 9 \\ -1 & 1 & -2 & -1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 3 & 9 & -4 & 1 \end{bmatrix}$

7. $\begin{pmatrix} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 5 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{pmatrix}$

9. $\begin{pmatrix} 2 & -4 & 3 \\ -6 & 12 & -9 \\ 4 & -8 & 6 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

11. $\begin{pmatrix} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

13. $\begin{bmatrix} 1 & 4 & 2 \\ -3 & h & -1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & h & 3 \\ 2 & 8 & 1 \end{bmatrix}$

Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answer for each part.

15. $x_1 + hx_2 = 1$
$2x_1 + 3x_2 = k$

16. $x_1 - 3x_2 = 1$
$2x_1 + hx_2 = k$