Lecture 1

Introduction and Overview

What is Algebra?

History

Algebra is named in honor of Mohammed Ibn-e- Musa al-Khowârizmî. Around 825, he wrote a book entitled Hisb al-jabr u'l muqubalah, ("the science of reduction and cancellation"). His book, Al-jabr, presented rules for solving equations.

Algebra is a branch of Mathematics that uses mathematical statements to describe relationships between things that vary over time. These variables include things like the relationship between supply of an object and its price. When we use a mathematical statement to describe a relationship, we often use letters to represent the quantity that varies, since it is not a fixed amount. These letters and symbols are referred to as variables.

Algebra is a part of mathematics in which unknown quantities are found with the help of relations between the unknown and known.

In algebra, letters are sometimes used in place of numbers.

The mathematical statements that describe relationships are expressed using algebraic terms, expressions, or equations (mathematical statements containing letters or symbols to represent numbers). Before we use algebra to find information about these kinds of relationships, it is important to first introduce some basic terminology.

Algebraic Term

The basic unit of an algebraic expression is a term. In general, a term is either a product of a number and with one or more variables.

For example 4x is an algebraic term in which 4 is coefficient and x is said to be variable.

Study of Algebra

Today, algebra is the study of the properties of operations on numbers. Algebra generalizes arithmetic by using symbols, usually letters, to represent numbers or unknown quantities. Algebra is a problem-solving tool. It is like a tractor, which is a farmer's tool. Algebra is a mathematician's tool for solving problems. Algebra has applications to every human endeavor. From art to medicine to zoology, algebra can be a tool. People who say that they will never use algebra are people who do not know about algebra. Learning algebra is a bit like learning to read and write. If you truly learn algebra, you will use it. Knowledge of algebra can give you more power to solve problems and accomplish what you want in life. Algebra is a mathematicians' shorthand!

Algebraic Expressions

An expression is a collection of numbers, variables, and +ve sign or –ve sign, of operations that must make mathematical and logical behaviour.

For example $8x^2 + 9x - 1$ is an algebraic expression.

What is Linear Algebra?

One of the most important problems in mathematics is that of solving systems of linear equations. It turns out that such problems arise frequently in applications of mathematics in the physical sciences, social sciences, and engineering. Stated in its simplest terms, the world is not linear, but the only problems that we know how to solve are the linear ones. What this often means is that only recasting them as linear systems can solve non-linear problems. A comprehensive study of linear systems leads to a rich, formal structure to analytic geometry and solutions to 2x2 and 3x3 systems of linear equations learned in previous classes.

It is exactly what the name suggests. Simply put, it is the algebra of systems of linear equations. While you could solve a system of, say, five linear equations involving five unknowns, it might not take a finite amount of time. With linear algebra we develop techniques to solve m linear equations and n unknowns, or show when no solution exists. We can even describe situations where an infinite number of solutions exist, and describe them geometrically.

Linear algebra is the study of linear sets of equations and their transformation properties.

Linear algebra, sometimes disguised as matrix theory, considers sets and functions, which preserve linear structure. In practice this includes a very wide portion of mathematics!

Thus linear algebra includes axiomatic treatments, computational matters, algebraic structures, and even parts of geometry; moreover, it provides tools used for analyzing differential equations, statistical processes, and even physical phenomena.

Linear Algebra consists of studying matrix calculus. It formalizes and gives geometrical interpretation of the resolution of equation systems. It creates a formal link between matrix calculus and the use of linear and quadratic transformations. It develops the idea of trying to solve and analyze systems of linear equations.

Applications of Linear algebra

Linear algebra makes it possible to work with large arrays of data. It has many applications in many diverse fields, such as

- Computer Graphics,
- Electronics,
- Chemistry,
- Biology,
- Differential Equations,
- Economics,
- Business,
- Psychology,
- Engineering,
- Analytic Geometry,
- Chaos Theory,
- Cryptography,
- Fractal Geometry,
- Game Theory,
- Graph Theory,
- Linear Programming,
- Operations Research

It is very important that the theory of linear algebra is first understood, the concepts are cleared and then computation work is started. Some of you might want to just use the computer, and skip the theory and proofs, but if you don't understand the theory, then it can be very hard to appreciate and interpret computer results.

Why using Linear Algebra?

Linear Algebra allows for formalizing and solving many typical problems in different engineering topics. It is generally the case that (input or output) data from an experiment is given in a discrete form (discrete measurements). Linear Algebra is then useful for solving problems in such applications in topics such as Physics, Fluid Dynamics, Signal Processing and, more generally Numerical Analysis.

Linear algebra is not like algebra. It is mathematics of linear spaces and linear functions. So we have to know the term "linear" a lot. Since the concept of linearity is fundamental to any type of mathematical analysis, this subject lays the foundation for many branches of mathematics.

Objects of study in linear algebra

Linear algebra merits study at least because of its ubiquity in mathematics and its applications. The broadest range of applications is through the concept of vector spaces and their transformations. These are the central objects of study in linear algebra

1. The solutions of homogeneous systems of linear equations form paradigm examples of vector spaces. Of course they do not provide the only examples.
2. The vectors of physics, such as force, as the language suggests, also provide paradigmatic examples.
3. Binary code is another example of a vector space, a point of view that finds application in computer sciences.
4. Solutions to specific systems of differential equations also form vector spaces.
5. Statistics makes extensive use of linear algebra.
6. Signal processing makes use of linear algebra.
7. Vector spaces also appear in number theory in several places, including the study of field extensions.
8. Linear algebra is part of and motivates much abstract algebra. Vector spaces form the basis from which the important algebraic notion of module has been abstracted.

9. Vector spaces appear in the study of differential geometry through the tangent bundle of a manifold.
10. Many mathematical models, especially discrete ones, use matrices to represent critical relationships and processes. This is especially true in engineering as well as in economics and other social sciences.

There are two principal aspects of linear algebra: theoretical and computational. A major part of mastering the subject consists in learning how these two aspects are related and how to move from one to the other.

Many computations are similar to each other and therefore can be confusing without reasonable level of grasp of their theoretical context and significance. It will be very tempting to draw false conclusions.

On the other hand, while many statements are easier to express elegantly and to understand from a purely theoretical point of view, to apply them to concrete problems you will need to “get your hands dirty”. Once you have understood the theory sufficiently and appreciate the methods of computation, you will be well placed to use software effectively, where possible, to handle large or complex calculations.

Lecture 2
Background

Introduction to Matrices

<u>Matrix</u> A matrix is a collection of numbers or functions arranged into rows and columns.

Matrices are denoted by capital letters $A, B, \ldots, Y, Z$. The numbers or functions are called elements of the matrix. The elements of a matrix are denoted by small letters $a, b, \ldots, y, z$.

<u>Rows and Columns</u> The horizontal and vertical lines in a matrix are, respectively, called the rows and columns of the matrix.

<u>Order of a Matrix</u> The size (or dimension) of matrix is called as order of matrix. Order of matrix is based on the number of rows and number of columns. It can be written as $r \times c$; r means no. of row and c means no. of columns.

If a matrix has m rows and n columns then we say that the size or order of the matrix is $m \times n$. If A is a matrix having m rows and n columns then the matrix can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \ldots & \vdots & \ldots & \vdots \\ \ldots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The element, or entry, in the ith row and jth column of a $m \times n$ matrix A is written as a_{ij}

For example: The matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \end{pmatrix}$ has two rows and three columns. So order of A will be 2×3

<u>Square Matrix</u> A matrix with equal number of rows and columns is called square matrix.

For Example The matrix $A = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ has three rows and three columns. So it is a square matrix of order 3.

<u>Equality of matrices</u>

The two matrices will be equal if they must have

a) The same dimensions (i.e. same number of rows and columns)
b) Corresponding elements must be equal.

Example The matrices $A = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ equal matrices

(i.e A = B) because they both have same orders and same corresponding elements.

Column Matrix A column matrix X is any matrix having n rows and only one column. Thus the column matrix X can be written as

$$X = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{pmatrix} = [b_{i1}]_{n\times 1}$$

A column matrix is also called a column vector or simply a vector.

Multiple of matrix A multiple of a matrix A by a nonzero constant k is defined to be

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m\times n}$$

Notice that the product kA is same as the product Ak. Therefore, we can write $kA = Ak$.

It implies that if we multiply a matrix by a constant *k*, then each element of the matrix is to be multiplied by *k*.

Example 1

(a) $$5 \cdot \begin{bmatrix} 2 & -3 \\ 4 & -1 \\ 1/5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 20 & -5 \\ 1 & 30 \end{bmatrix}$$

(b) $$e^t \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} e^t \\ -2e^t \\ 4e^t \end{bmatrix}$$

Since we know that $kA = Ak$. Therefore, we can write

$$e^{-3t} \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2e^{-3t} \\ 5e^{-3t} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} e^{-3t}$$

Addition of Matrices Only matrices of the same order may be added by adding corresponding elements.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices then $A + B = [a_{ij} + b_{ij}]$

Obviously order of the matrix A + B is $m \times n$

Example 2 Consider the following two matrices of order 3×3

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$$

Since the given matrices have same orders, therefore, these matrices can be added and their sum is given by

$$A + B = \begin{pmatrix} 2+4 & -1+7 & 3+(-8) \\ 0+9 & 4+3 & 6+5 \\ -6+1 & 10+(-1) & -5+2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{pmatrix}$$

Example 3 Write the following single column matrix as the sum of three column vectors

$$\begin{pmatrix} 3t^2 - 2e^t \\ t^2 + 7t \\ 5t \end{pmatrix}$$

Solution

$$\begin{pmatrix} 3t^2 - 2e^t \\ t^2 + 7t \\ 5t \end{pmatrix} = \begin{pmatrix} 3t^2 \\ t^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 7t \\ 5t \end{pmatrix} + \begin{pmatrix} -2e^t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} t + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} e^t$$

Difference of Matrices The difference of two matrices A and B of same order $m \times n$ is defined to be the matrix $A - B = A + (-B)$

The matrix $-B$ is obtained by multiplying the matrix B with -1. So that $-B = (-1)\,B$

Multiplication of Matrices We can multiply two matrices if and only if, the number of columns in the first matrix equals the number of rows in the second matrix.
Otherwise, the product of two matrices is not possible.
OR
If the order of the matrix A is $m \times n$ then to make the product AB possible order of the matrix B must be $n \times p$. Then the order of the product matrix AB is $m \times p$. Thus

$$A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$$

If the matrices A and B are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & \cdots & a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2n}b_{np} \\ \vdots & \cdots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + a_{m2}b_{2p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

$$= \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{n \times p}$$

Example 4 If possible, find the products AB and BA, when

(a) $$A = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}$$

(b) $$A = \begin{pmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix}$$

Solution (a) The matrices A and B are square matrices of order 2. Therefore, both of the products AB and BA are possible.

$$AB = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 4 \cdot 9 + 7 \cdot 6 & 4 \cdot (-2) + 7 \cdot 8 \\ 3 \cdot 9 + 5 \cdot 6 & 3 \cdot (-2) + 5 \cdot 8 \end{pmatrix} = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}$$

Similarly $$BA = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 9 \cdot 4 + (-2) \cdot 3 & 9 \cdot 7 + (-2) \cdot 5 \\ 6 \cdot 4 + 8 \cdot 3 & 6 \cdot 7 + 8 \cdot 5 \end{pmatrix} = \begin{pmatrix} 30 & 53 \\ 48 & 82 \end{pmatrix}$$

Note From above example it is clear that generally a matrix multiplication is not commutative i.e. $AB \neq BA$.

(b) The product AB is possible as the number of columns in the matrix A and the number of rows in B is 2. However, the product BA is not possible because the number of column in the matrix B and the number of rows in A is not same.

$$AB = \begin{pmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \cdot (-4) + 8 \cdot 2 & 5 \cdot (-3) + 8 \cdot 0 \\ 1 \cdot (-4) + 0 \cdot 2 & 1 \cdot (-3) + 0 \cdot 0 \\ 2 \cdot (-4) + 7 \cdot 2 & 2 \cdot (-3) + 7 \cdot 0 \end{pmatrix} = \begin{pmatrix} -4 & -15 \\ -4 & -3 \\ 6 & -6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}, \quad BA = \begin{pmatrix} 30 & 53 \\ 48 & 82 \end{pmatrix}$$

Clearly $AB \neq BA.$

$$AB = \begin{pmatrix} -4 & -15 \\ -4 & -3 \\ 6 & -6 \end{pmatrix}$$

However**,** the product BA is not possible.

Example 5

(a)
$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ 1 & -7 & 9 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot(-3)+(-1)\cdot 6+3\cdot 4 \\ 0\cdot(-3)+4\cdot 6+5\cdot 6 \\ 1\cdot(-3)+(-7)\cdot 6+9\cdot 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 44 \\ -9 \end{pmatrix}$$

(b)
$$\begin{pmatrix} -4 & 2 \\ 3 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x+2y \\ 3x+8y \end{pmatrix}$$

Multiplicative Identity For a given any integer n, the $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the multiplicative identity matrix. If A is a matrix of order $n \times n$, then it can be verified that $I \cdot A = A \cdot I = A$

Example $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$**,** $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are identity matrices of orders 2 x 2 and 3 x 3 respectively and If $B = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}$ then we can easily prove that BI = IB = B

Zero Matrix or Null matrix A matrix whose all entries are zero is called zero matrix or null matrix and it is denoted by O.

For example $O=\begin{pmatrix}0\\0\end{pmatrix}$; $O=\begin{pmatrix}0&0\\0&0\end{pmatrix}$; $O=\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}$

and so on. If A and O are the matrices of same orders, then $A+O=O+A=A$

Associative Law The matrix multiplication is associative. This means that if A, B and C are $m\times p$, $p\times r$ and $r\times n$ matrices, then $A(BC)=(AB)C$

The result is a $m\times n$ matrix. This result can be verified by taking any three matrices which are confirmable for multiplication.

Distributive Law If B and C are matrices of order $r\times n$ and A is a matrix of order $m\times r$, then the distributive law states that

$$A(B+C)=AB+AC$$

Furthermore, if the product $(A+B)C$ is defined, then

$$(A+B)C=AC+BC$$

Remarks

It is important to note that some rules arithmetic for real numbers $\mathbb{R}$ do not carry over the matrix arithmetic.

For example, $\forall a,b,c$ and $d\in\mathbb{R}$

i) if $ab=cd$ and $a\neq 0$, then $b=c$ (Law of Cancellation)

ii) if $ab=0$, then least one of the factors a or b (or both) are zero.

However the following examples shows that the corresponding results are not true in case of matrices.

Example

Let $A=\begin{bmatrix}0&1\\0&2\end{bmatrix}$, $B=\begin{bmatrix}1&1\\3&4\end{bmatrix}$, $C=\begin{bmatrix}2&5\\3&4\end{bmatrix}$ and $D=\begin{bmatrix}1&7\\0&0\end{bmatrix}$, then one can easily check that

$AB=AC=\begin{bmatrix}3&4\\6&8\end{bmatrix}$. But $B\neq C$.

Similarly neither A nor B are zero matrices but $AD=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

But if D is diagonal say $D=\begin{bmatrix}1&0\\0&7\end{bmatrix}$, then $AD\neq DA$.

Determinant of a Matrix Associated with every square matrix A of constants, there is a number called the determinant of the matrix, which is denoted by $\det(A)$ or $|A|$. There is a special way to find the determinant of a given matrix.

Example 6 Find the determinant of the following matrix $A=\begin{pmatrix}3 & 6 & 2\\ 2 & 5 & 1\\ -1 & 2 & 4\end{pmatrix}$

Solution The determinant of the matrix A is given by

$$\det(A)=\begin{vmatrix}3 & 6 & 2\\ 2 & 5 & 1\\ -1 & 2 & 4\end{vmatrix}$$

We expand the $\det(A)$ by first row, we obtain

$$\det(A)=\begin{vmatrix}3 & 6 & 2\\ 2 & 5 & 1\\ -1 & 2 & 4\end{vmatrix}=3\begin{vmatrix}5 & 1\\ 2 & 4\end{vmatrix}-6\begin{vmatrix}2 & 1\\ -1 & 4\end{vmatrix}+2\begin{vmatrix}2 & 5\\ -1 & 2\end{vmatrix}$$

or $$\det(A)=3(20-2)-6(8+1)+2(4+5)=18$$

Transpose of a Matrix The transpose of $m\times n$ matrix A is denoted by A^{tr} and it is obtained by interchanging rows of A into its columns. In other words, rows of A become the columns of A^{tr}. Clearly A^{tr} is $n\times m$ matrix.

If $$A=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}, \text{ then } A^{tr}=\begin{pmatrix}a_{11} & a_{21} & \cdots & a_{m1}\\ a_{12} & a_{22} & \cdots & a_{m2}\\ \vdots & \vdots & \cdots & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{mn}\end{pmatrix}$$

Since order of the matrix A is $m\times n$, the order of the transpose matrix A^{tr} is $n\times m$.

Properties of the Transpose

The following properties are valid for the transpose;

- The transpose of the transpose of a matrix is the matrix itself:`
- The transpose of a matrix times a scalar (k) is equal to the constant times the transpose of the matrix: $(\underline{A}\underline{B}\underline{C})^T=\underline{C}^T\underline{B}^T\underline{A}^T$ $(k\underline{A})^T=k\underline{A}^T$
- The transpose of the sum of two matrices is equivalent to the sum of their transposes: $(\underline{A}+\underline{B})^T=\underline{A}^T+\underline{B}^T$
- The transpose of the product of two matrices is equivalent to the product of their transposes in reversed order: $(\underline{A}\underline{B})^T=\underline{B}^T\underline{A}^T$
- The same is true for the product of multiple matrices: $(\underline{A}\underline{B}\underline{C})^T=\underline{C}^T\underline{B}^T\underline{A}^T$

Example 7 (a) The transpose of matrix $A=\begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$ is $A^T = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

(b) If $X = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$, then $X^T = \begin{bmatrix} 5 & 0 & 3 \end{bmatrix}$

Multiplicative Inverse Suppose that A is a square matrix of order $n \times n$. If there exists an $n \times n$ matrix B such that $AB = BA = I$, then B is said to be the multiplicative inverse of the matrix A and is denoted by $B = A^{-1}$.

For example: If $A=\begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$ then the matrix $B=\begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix}$ is multiplicative inverse of A because AB = $\begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ =I

Similarly we can check that BA = I

Singular and Non-Singular Matrices A square matrix A is said to be a **non-singular** matrix if $\det(A) \neq 0$, otherwise the square matrix A is said to be **singular**. Thus for a singular matrix A we must have $\det(A) = 0$

Example: $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

$$|A| = 2(5-0) - 3(5-0) - 1(-3-2)$$
$$= 10 - 15 + 5 = 0$$

which means that A is singular.

Minor of an element of a matrix

Let A be a square matrix of order n x n. Then minor M_{ij} of the element $a_{ij} \in A$ is the determinant of $(n-1)\times(n-1)$ matrix obtained by deleting the *ith* row and *jth* column from A.

Example If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ is a square matrix. The Minor of $3 \in A$ is denoted by M_{12} and is defined to be $M_{12} = \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix}$ = 5-0 = 5

Cofactor of an element of a matrix

Let A be a non singular matrix of order $n \times n$ and let C_{ij} denote the cofactor (signed minor) of the corresponding entry $a_{ij} \in A$, then it is defined to be $C_{ij} = (-1)^{i+j} M_{ij}$

Example If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ is a square matrix. The cofactor of $3 \in A$ is denoted by C_{12} and is defined to be $C_{12} = = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} =$ - (5 - 0) = -5

Theorem If A is a square matrix of order $n \times n$ then the matrix has a multiplicative inverse A^{-1} if and only if the matrix A is non-singular.

Theorem Then inverse of the matrix A is given by $A^{-1} = \frac{1}{\det(A)}(C_{ij})^{tr}$

1. For further reference we take $n = 2$ so that A is a 2×2 non-singular matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Therefore $C_{11} = a_{22}$, $C_{12} = -a_{21}$, $C_{21} = -a_{12}$ and $C_{22} = a_{11}$. So that

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}^{tr} = \frac{1}{\det(A)}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

2. For a 3×3 non-singular matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ and so on.}$$

Therefore, inverse of the matrix A is given by $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$.

Example 8 Find, if possible, the multiplicative inverse for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$.

Solution The matrix A is non-singular because $\det(A) = \begin{vmatrix} 1 & 4 \\ 2 & 10 \end{vmatrix} = 10 - 8 = 2$

Therefore, A^{-1} exists and is given by $A^{-1} = \dfrac{1}{2}\begin{pmatrix} 10 & -4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix}$

Check

$$AA^{-1} = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix} = \begin{pmatrix} 5-4 & -2+2 \\ 10-10 & -4+5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$AA^{-1} = \begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 5-4 & 20-20 \\ -1+1 & -4+5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Example 9 Find, if possible, the multiplicative inverse of the following matrix

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Solution The matrix is singular because

$$\det(A) = \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = 2 \cdot 3 - 2 \cdot 3 = 0$$

Therefore, the multiplicative inverse A^{-1} of the matrix does not exist.

Example 10 Find the multiplicative inverse for the following matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}.$$

Solution Since $\det(A)=\begin{vmatrix}2 & 2 & 0\\ -2 & 1 & 1\\ 3 & 0 & 1\end{vmatrix}=2(1-0)-2(-2-3)+0(0-3)=12\neq 0$

Therefore, the given matrix is non singular. So, the multiplicative inverse A^{-1} of the matrix A exists. The cofactors corresponding to the entries in each row are

$$C_{11}=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}=1,\qquad C_{12}=-\begin{vmatrix}-2 & 1\\ 3 & 1\end{vmatrix}=5,\qquad C_{13}=\begin{vmatrix}-2 & 1\\ 3 & 0\end{vmatrix}=-3$$
$$C_{21}=-\begin{vmatrix}2 & 0\\ 0 & 1\end{vmatrix}=-2,\qquad C_{22}=\begin{vmatrix}2 & 0\\ 3 & 1\end{vmatrix}=2,\qquad C_{23}=-\begin{vmatrix}2 & 2\\ 3 & 0\end{vmatrix}=6$$
$$C_{31}=\begin{vmatrix}2 & 0\\ 1 & 1\end{vmatrix}=2,\qquad C_{32}=-\begin{vmatrix}2 & 0\\ -2 & 1\end{vmatrix}=-2,\qquad C_{33}=\begin{vmatrix}2 & 2\\ -2 & 1\end{vmatrix}=6$$

Hence $$\mathrm{A}^{-1}=\frac{1}{12}\begin{pmatrix}1 & -2 & 2\\ 5 & 2 & -2\\ -3 & 6 & 6\end{pmatrix}=\begin{pmatrix}1/12 & -1/6 & 1/6\\ 5/12 & 1/6 & -1/6\\ -1/4 & 1/2 & 1/2\end{pmatrix}$$

We can also verify that $A\cdot A^{-1}=A^{-1}\cdot A=I$

Derivative of a Matrix of functions

Suppose that

$$A(t)=\left[a_{ij}(t)\right]_{m\times n}$$

is a matrix whose entries are functions those are differentiable in a common interval, then derivative of the matrix $A(t)$ is a matrix whose entries are derivatives of the corresponding entries of the matrix $A(t)$. Thus

$$\frac{dA}{dt}=\left[\frac{da_{ij}}{dt}\right]_{m\times n}$$

The derivative of a matrix is also denoted by $A'(t)$.

Integral of a Matrix of Functions

Suppose that $A(t)=\left(a_{ij}(t)\right)_{m\times n}$ is a matrix whose entries are functions those are continuous on a common interval containing t, then integral of the matrix $A(t)$ is a matrix whose entries are integrals of the corresponding entries of the matrix $A(t)$. Thus

$$\int_{t_0}^{t} A(s)ds=\left(\int_{t_0}^{t} a_{ij}(s)ds\right)_{m\times n}$$

Example 11 Find the derivative and the integral of the following matrix $X(t)=\begin{pmatrix}\sin 2t\\ e^{3t}\\ 8t-1\end{pmatrix}$

Solution The derivative and integral of the given matrix are, respectively, given by

$$X'(t)=\begin{pmatrix}\frac{d}{dt}(\sin 2t)\\ \frac{d}{dt}(e^{3t})\\ \frac{d}{dt}(8t-1)\end{pmatrix}=\begin{pmatrix}2\cos 2t\\ 3e^{3t}\\ 8\end{pmatrix} \quad \text{and} \quad \int_0^t X(s)ds=\begin{pmatrix}\int_0^t \sin 2s ds\\ \int_0^t e^{3s}ds\\ \int_0^t 8s-1ds\end{pmatrix}=\begin{pmatrix}-1/2\cos 2t+1/2\\ 1/3e^{3t}-1/3\\ 4t^2-t\end{pmatrix}$$

Exercise

Write the given sum as a single column matrix

1. $3t\begin{pmatrix}2\\ t\\ -1\end{pmatrix}+(t-1)\begin{pmatrix}-1\\ -t\\ 3\end{pmatrix}-2\begin{pmatrix}3t\\ 4\\ -5t\end{pmatrix}$

2. $\begin{pmatrix}1 & -3 & 4\\ 2 & 5 & -1\\ 0 & -4 & -2\end{pmatrix}\begin{pmatrix}t\\ 2t-1\\ -t\end{pmatrix}+\begin{pmatrix}-t\\ 1\\ 4\end{pmatrix}-\begin{pmatrix}2\\ 8\\ -6\end{pmatrix}$

Determine whether the given matrix is singular or non-singular. If singular, find A^{-1}.

3. $A=\begin{pmatrix}3 & 2 & 1\\ 4 & 1 & 0\\ -2 & 5 & -1\end{pmatrix}$

4. $A=\begin{pmatrix}4 & 1 & -1\\ 6 & 2 & -3\\ -2 & -1 & 2\end{pmatrix}$

Find $\frac{dX}{dt}$

5. $X=\begin{pmatrix}\frac{1}{2}\sin 2t-4\cos 2t\\ -3\sin 2t+5\cos 2t\end{pmatrix}$

6. If $A(t)=\begin{pmatrix}e^{4t} & \cos \pi t\\ 2t & 3t^2-1\end{pmatrix}$ then find (a) $\int_0^2 A(t)dt$, (b) $\int_0^t A(s)ds$.

7. Find the integral $\int_1^2 B(t)dt$ if $B(t)=\begin{pmatrix}6t & 2\\ 1/t & 4t\end{pmatrix}$

Lecture 11

Matrix Operations

(i-j)th Element of a matrix

Let A be an $m \times n$ matrix, where m and n are number of rows and number of columns respectively, then a_{ij} represents the ***i***-th row and ***j***-th column entry of the matrix. For example a_{12} represents 1st row and 2nd column entry.

Similarly a_{32} represents 3rd row and 2nd column entry. The columns of ***A*** are vectors in $\boldsymbol{R^m}$ and are denoted by (boldface) $\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}$.

These columns are $A = [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \ldots \quad \boldsymbol{a_n}]$

The number a_{ij} is the ***i****-th* entry (from the top) of ***j****-th* column vector a_j.

$$\begin{array}{cc} & \begin{array}{c} Column \\ j \end{array} \\ Row \quad i & \begin{bmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} = A \\ & \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \boldsymbol{a_1} & \boldsymbol{a_j} & \boldsymbol{a_n} \end{array} \end{array}$$

Figure 1 Matrix notation.

Definitions

A **diagonal matrix** is a square matrix whose non-diagonal entries are zero.

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

The diagonal entries in $A = \left[a_{ij} \right]$ are $a_{11}, a_{22}, a_{33}, \cdots$ and they form the **main diagonal** of ***A.***

For example $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are all diagonal matrices.

Null Matrix or Zero Matrix

An $m \times n$ matrix whose entries are all zero is a Null or **zero matrix** and is always written as ***O***. A null matrix may be of any order.

For example $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3 x 3 3 x 2 4 x 5

are all Zero Matrices

Equal Matrices

Two matrices are said to be **equal** if they have the same size (i.e., the same number of rows and columns) and same corresponding entries.

Example 1 Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x+1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

The matrices ***A*** and ***B*** are equal if and only if $x+1 = 5$ or $x = 4$. There is no value of x for which ***A = C***, since ***A*** and ***C*** have different sizes.

If ***A*** and B are $m \times n$ matrices, then the **sum, *A + B***, is the $m \times n$ matrix whose columns are the sums of the corresponding columns in ***A*** and ***B***. Each entry in ***A + B*** is the sum of the corresponding entries in ***A*** and ***B***. The sum ***A + B*** is defined only when ***A*** and ***B*** are of the same size.

If *r* is a scalar and ***A*** is a matrix, then the **scalar multiple** ***rA*** is the matrix whose columns are *r* times the corresponding columns in ***A***.

Note: Negative of a matrix A is defined as $-A$ to mean $(-1)A$ and the difference of A and B is written as $A-B$, which means $A + (-1)B$.

Example 2 Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$

But ***A + C*** is not defined because ***A*** and ***C*** have different sizes.

$$2B = 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$

Theorem 1 Let ***A, B,*** and ***C*** are matrices of the same size, and let *r* and *s* are scalars.

a. ***A + B = B + A***

b. ***(A + B) + C = A + (B + C)***

c. ***A + 0 = A***

d. *r* ***(A + B)= r A + r B***

e. *(r + s)* ***A = r A + s A***

f. *r* ***(sA) = (rs) A***

Each equality in Theorem 1 can be verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal. Size is no problem because ***A, B,*** and ***C*** are equal in size. The equality of columns follows immediately from analogous properties of vectors.

For instance, if the *jth* columns of ***A, B,*** and ***C*** are a_j, b_j and c_j, respectively, then the *jth* columns of ***(A + B) + C*** and ***A + (B + C)*** are

$$(a_j + b_j) + c_j \qquad and \qquad a_j + (b_j + c_j)$$

respectively. Since these two vector sums are equal for each *j*, property (b) is verified.

Because of the associative property of addition, we can simply write ***A + B + C*** for the sum, which can be computed either as ***(A + B) + C*** or ***A + (B + C)***. The same applies to sums of four or more matrices.

Matrix Multiplication

Multiplying an m×n matrix with an n×p matrix results in an m×p matrix. If many matrices are multiplied together, and their dimensions are written in a list in order, e.g. m×n, n×p, p×q, q×r, the size of the result is given by the first and the last numbers (m×r).

It is important to keep in mind that this definition requires the number of columns of the first factor A to be the same as the number of rows of the second factor B. When this condition is satisfied, the sizes of A and B are said to conform for the product AB. If the sizes of A and B do not conform for the product AB, then this product is undefined.

Definition If ***A*** is an $m \times n$ matrix, and if ***B*** is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product ***AB*** is the $m \times p$ matrix whose columns are $Ab_1, \cdots, Ab_p$.

That is $$AB = A\begin{bmatrix} b_1 & b_2 & \ldots & b_p \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \ldots & Ab_p \end{bmatrix}$$

This definition makes equation (1) true for all $\boldsymbol{x}$ in $\boldsymbol{R}^p$. Equation (1) proves that the composite mapping ***(AB)*** is a linear transformation and that its standard matrix is ***AB***. Multiplication of matrices corresponds to composition of linear transformations.

A convenient way to determine whether ***A*** and ***B*** conform for the product ***AB*** and, if so, to find the size of the product is to write the sizes of the factors side by side as in Figure below (the size of the first factor on the left and the size of the second factor on the right).

$$\begin{array}{ccccc} \boldsymbol{A} & & \boldsymbol{B} & = & \boldsymbol{AB} \\ m \times s & & s \times n & & m \times n \end{array}$$

Inside

Outside

If the inside numbers are the same, then the product ***AB*** is defined and the outside numbers then give the size of the product.

<u>Example 3</u> Compute ***AB***, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

Solution: Here $\boldsymbol{B} = [\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \boldsymbol{b_3}]$, therefore

$$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

Then

$$AB = A[b_1 \quad b_2 \quad b_3] = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

$$\boldsymbol{Ab_1} \quad \boldsymbol{Ab_2} \quad \boldsymbol{Ab_3}$$

Note from the definition of ***AB*** that its first column, $\boldsymbol{Ab_1}$, is a linear combination of the columns of ***A***, using the entries in $\boldsymbol{b_1}$ as weights. The same holds true for each column of ***AB***. Each column of ***AB*** is a linear combination of the columns of ***A*** using weights from the corresponding column of ***B***.

Example 4 Find the product ***AB*** for

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Solution It follows from definition that the product ***AB*** is formed in a column-by-column manner by multiplying the successive columns of ***B*** by ***A***. The computations are

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$

$$\left[\begin{array}{c|c|c} 1 & 2 & 4 \\ 2 & 6 & 0 \end{array}\right] \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = 4c_1 + 0c_2 + 2c_3 = (4)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (0)\begin{bmatrix} 2 \\ 6 \end{bmatrix} + (2)\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

Similarly, $$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 6 \end{bmatrix} + (7)\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = (4)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (3)\begin{bmatrix} 2 \\ 6 \end{bmatrix} + (5)\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 26 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = (3)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 6 \end{bmatrix} + (2)\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 12 \end{bmatrix}$$

Thus, $$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Example 5 (An Undefined Product) Find the product ***BA*** for the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Solution The number of columns of B is not equal to number of rows of A so BA multiplication is not possible.

The matrix ***B*** has size 3×4 and the matrix ***A*** has size 2×3. The "inside" numbers are not the same, so the product ***BA*** is undefined.

Obviously, the number of columns of ***A*** must match the number of rows in ***B*** in order for a linear combination such as $\boldsymbol{Ab_1}$ to be defined. Also, the definition of ***AB*** shows that ***AB*** has the same number of rows as ***A*** and the same number of columns as ***B***.

<u>Example 6</u> If ***A*** is a 3×5 matrix and ***B*** is a 5×2 matrix, what are the sizes of ***AB*** and ***BA***, if they are defined?

Solution The product of matrices A and B of orders 3×5 and 5×2 will result in 3×2 matrix AB.

But for BA we have 5×2 and 3×5, here number of columns in1st matrix are 2 which is not equal to number of rows in 2nd matrix. So BA is not possible.

Since ***A*** has 5 columns and ***B*** has 5 rows, the product ***AB*** is defined and is a 3×2 matrix:

$$\begin{array}{ccccc} A & & B & & AB \\ \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} & & \begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix} & = & \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \end{array}$$

$$\underbrace{3\times \overbrace{5 \qquad\qquad 5}^{\text{Match}} \times 2}_{\text{Size of AB}} \qquad\qquad 3\times 2$$

The product ***BA*** is not defined because the 2 columns of ***B*** do not match the 3 rows of ***A***.

The definition of ***AB*** is important for theoretical work and applications, but the following rule provides a more efficient method for calculating the individual entries in ***AB*** when working small problems by hand.

<u>Row-Column Rule for Computing *AB*</u>

Explanation

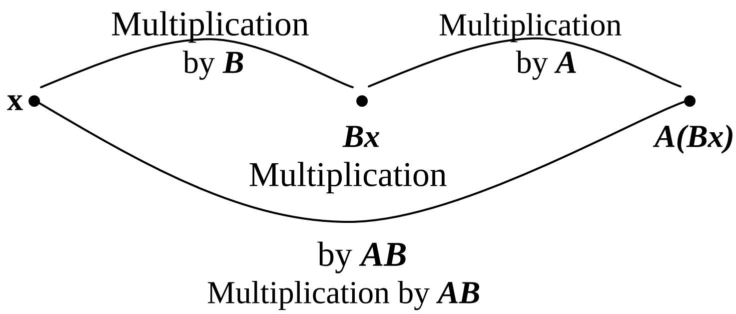
If a matrix ***B*** is multiplied with a vector ***x***, it transforms ***x*** into a vector ***Bx***. If this vector is then multiplied in turn by a matrix ***A***, the resulting vector is ***A (Bx)***.

$$\mathbf{x} \bullet \xrightarrow{\text{Multiplication by } \boldsymbol{B}} \bullet\, \boldsymbol{Bx} \xrightarrow{\text{Multiplication by } \boldsymbol{A}} \bullet\, \boldsymbol{A(Bx)}$$

Multiplication by ***B*** and then ***A***

Thus ***A(Bx)*** is produced from ***x*** by a composition of mappings. Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by ***AB***, so that

$$\boldsymbol{A\ (Bx) = (AB)\ x}\text{---}(1)$$



If ***A*** is $m \times n$, ***B*** is $n \times p$, and ***x*** is in $\boldsymbol{R^p}$, denote the columns of ***B*** by $b_1, \cdots, b_p$ and the entries in ***x*** by $x_1, \cdots, x_p$, then $Bx = x_1 b_1 + x_2 b_2 + \cdots + x_p b_p$

By the linearity of multiplication by ***A***,

$$\begin{aligned} A(Bx) &= A(x_1 b_1) + A(x_2 b_2) + \cdots + A(x_p b_p) \\ &= x_1 A b_1 + x_2 A b_2 + \cdots + x_p A b_p \end{aligned}$$

The vector ***A (Bx)*** is a linear combination of the vectors $Ab_1, \cdots, Ab_p$, using the entries in ***x*** as weights. If we rewrite these vectors as the columns of a matrix, we have

$$A(Bx) = \begin{bmatrix} Ab_1 & Ab_2 & \ldots & Ab_p \end{bmatrix} x$$

Thus multiplication by $\begin{bmatrix} Ab_1 & Ab_2 & \ldots & Ab_p \end{bmatrix}$ transforms ***x*** into ***A(Bx)***.
We have found the matrix we sought!

<u>Row-Column Rule for Computing *AB*</u>

If the product ***AB*** is defined, then the entry in row ***i*** and column ***j*** of ***AB*** is the sum of the products of corresponding entries from row ***i*** of ***A*** and column ***j*** of ***B***. If $\boldsymbol{(AB)_{ij}}$ denotes the *(i, j)* – entry in ***AB***, and if ***A*** is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

To verify this rule, let $B = \begin{bmatrix} b_1 & \ldots & b_p \end{bmatrix}$. Column ***j*** of ***AB*** is $\boldsymbol{Ab_j}$, and we can compute $\boldsymbol{Ab_j}$. The *ith* entry in $\boldsymbol{Ab_j}$ is the sum of the products of corresponding entries from row ***i*** of ***A*** and the vector $\boldsymbol{b_j}$, which is precisely the computation described in the rule for computing the *(i, j)* – entry of ***AB***.

Finding Specific Entries in a Matrix Product Sometimes we will be interested in finding a specific entry in a matrix product without going through the work of computing the entire column that contains the entry.

Example 7 Use the row-column rule to compute two of the entries in ***AB*** for the matrices in Example 3.

Solution: To find the entry in row 1 and column 3 of ***AB***, consider row 1 of ***A*** and column 3 of ***B***. Multiply corresponding entries and add the results, as shown below:

$$AB = \begin{array}{c} \rightarrow \\ \end{array}\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & \overset{\downarrow}{6} \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 2(6)+3(3) \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & \square & \square \end{bmatrix}$$

For the entry in row 2 and column 2 of ***AB***, use row 2 of ***A*** and column 2 of ***B***:

$$\begin{array}{c} \\ \rightarrow \end{array}\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & \overset{\downarrow}{3} & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 1(3)+-5(-2) & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 13 & \square \end{bmatrix}$$

Example 8 Use the dot product rule to compute the individual entries in the product of ***AB*** where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

Solution Since ***A*** has size 2×3 and ***B*** has size 3×4, the product ***AB*** is a 2×4 matrix of the form

$$AB = \begin{bmatrix} r_1(A) \times c_1(B) & r_1(A) \times c_2(B) & r_1(A) \times c_3(B) & r_1(A) \times c_4(B) \\ r_2(A) \times c_1(B) & r_2(A) \times c_2(B) & r_2(A) \times c_3(B) & r_2(A) \times c_4(B) \end{bmatrix}$$

where $r_1(A)$ and $r_2(A)$ are the row vectors of ***A*** and $c_1(B), c_2(B), c_3(B)$ *and* $c_4(B)$ are the column vectors of ***B***. For example, the entry in row 2 and column 3 of ***AB*** can be computed as

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \boxed{26} & \square \end{bmatrix}$$

$$(2 \times 4) + (6 \times 3) + (0 \times 5) = 26$$

and the entry in row 1 and column 4 of ***AB*** can be computed as

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1\times 3)+(2\times 1)+(4\times 2)=13$$

Here is the complete set of computations:

$$(AB)_{11} = (1\times 4)+(2\times 0)+(4\times 2)=12$$
$$(AB)_{12} = (1\times 1)+(2\times \text{-}1)+(4\times 7)=27$$
$$(AB)_{13} = (1\times 4)+(2\times 3)+(4\times 5)=30$$
$$(AB)_{14} = (1\times 3)+(2\times 1)+(4\times 2)=13$$
$$(AB)_{21} = (2\times 4)+(6\times 0)+(0\times 2)=8$$
$$(AB)_{22} = (2\times 1)+(6\times \text{-}1)+(0\times 7)=\text{-}4$$
$$(AB)_{23} = (2\times 4)+(6\times 3)+(0\times 5)=26$$
$$(AB)_{24} = (2\times 3)+(6\times 1)+(0\times 2)=12$$

Finding Specific Rows and Columns of a Matrix Product

The specific column of ***AB*** is given by the formula

$$AB = A\left[\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_n\right] = \left[A\boldsymbol{b}_1 \quad A\boldsymbol{b}_2 \quad \cdots \quad A\boldsymbol{b}_n\right]$$

Similarly, the specific row of ***AB*** is given by the formula $AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1B \\ a_2B \\ \vdots \\ a_mB \end{bmatrix}$

<u>**Example 9**</u> Find the entries in the second row of ***AB***, where

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

Solution: By the row-column rule, the entries of the second row of ***AB*** come from row 2 of ***A*** (and the columns of ***B***):

$$\begin{array}{r} \\ \\ \to \\ \\ \\ \end{array}\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}\begin{array}{c} \begin{array}{cc} \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix} \end{array} = \begin{bmatrix} \square & \square \\ -4+21-12 & 6+3-8 \\ \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ 5 & 1 \\ \square & \square \\ \square & \square \end{bmatrix}$$

Example 10 **(Finding a Specific Row and Column of AB)**

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$

Find the second column and the first row of ***AB***.

Solution $c_2(AB) = Ac_2(B) = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$

$$r_1(AB) = r_1(A)B = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

Properties of Matrix Multiplication

These are standard properties of matrix multiplication. Remember that I_m represents the $m \times m$ identity matrix and $I_m x = x$ for all ***x*** *belong to* $\boldsymbol{R^m}$.

Theorem 2 Let ***A*** be $m \times n$, and let ***B*** and ***C*** have sizes for which the indicated sums and products are defined.

- **a.** ***A (BC) = (AB) C*** (associative law of multiplication)
- **b.** ***A (B + C) = AB + AC*** (left distributive law)
- **c.** ***(B + C)A = BA + CA*** (right distributive law)
- ***d.*** *r* ***(AB) = (r A)B = A(r B)*** (for any scalar *r*)
- **e.** $I_m A = A = A I_m$ (identity for matrix multiplication)

Proof. Properties (b) to (e) are considered exercises for you. We start property (a) follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions), and it is known (or easy to check) that the composition of functions is associative.

Here is another proof of (a) that rests on the "column definition" of the product of two matrices. Let $C = \begin{bmatrix} c_1 & \dots & c_p \end{bmatrix}$

By definition of matrix multiplication $BC = \begin{bmatrix} Bc_1 & \dots & Bc_p \end{bmatrix}$

$$A(BC) = \begin{bmatrix} A(Bc_1) & \dots & A(Bc_p) \end{bmatrix}$$

From above, we know that ***A(Bx) = (AB)x*** for all ***x***, so

$$A(BC) = \begin{bmatrix} (AB)c_1 & \dots & (AB)c_p \end{bmatrix} = (AB)C$$

The associative and distributive laws say essentially that pairs of parentheses in matrix expressions can be inserted and deleted in the same way as in the algebra of real numbers. In particular, we can write ***ABC*** for the product, which can be computed as ***A(BC)*** or as ***(AB)C***. Similarly, a product ***ABCD*** of four matrices can be computed as

A(BCD) or ***(ABC)D*** or ***A(BC)D***, and so on. It does not matter how we group the matrices when computing the product, so long as the left-to-right order of the matrices is preserved.

The left-to-right order in products is critical because, in general, ***AB*** and ***BA*** are not the same. This is not surprising, because the columns of ***AB*** are linear combinations of the columns of ***A***, whereas the columns of ***BA*** are constructed from the columns of ***B***.

If ***AB = BA***, we say that ***A*** and ***B*** **commute** with one another.

<u>Example 11</u> Let $A=\begin{bmatrix}5 & 1\\ 3 & -2\end{bmatrix} and \quad B=\begin{bmatrix}2 & 0\\ 4 & 3\end{bmatrix}$

Show that these matrices don not commute, i.e. $AB \neq BA$.

Solution:

$$AB=\begin{bmatrix}5 & 1\\ 3 & -2\end{bmatrix}\begin{bmatrix}2 & 0\\ 4 & 3\end{bmatrix}=\begin{bmatrix}10+4 & 0+3\\ 6-8 & 0-6\end{bmatrix}=\begin{bmatrix}14 & 3\\ -2 & -6\end{bmatrix}$$

$$BA=\begin{bmatrix}2 & 0\\ 4 & 3\end{bmatrix}\begin{bmatrix}5 & 1\\ 3 & -2\end{bmatrix}=\begin{bmatrix}10+0 & 2-0\\ 20+9 & 4-6\end{bmatrix}=\begin{bmatrix}10 & 2\\ 29 & -2\end{bmatrix}$$

For emphasis, we include the remark about commutativity with the following list of important differences between matrix algebra and ordinary algebra of real numbers.

WARNINGS

1. In general, $AB \neq BA$. Clear from the Example # 11.

2. The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$.

For example: Consider the following three matrices

$$A=\begin{bmatrix}-3 & 2\\ -6 & 4\end{bmatrix} \quad B=\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix} \quad C=\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}$$

$$AB=\begin{bmatrix}9 & -10\\ 18 & -20\end{bmatrix}=\mathrm{A}C \quad \text{But } B \neq C$$

3. If a product ***AB*** is the zero matrix, you cannot conclude in general that either ***A = 0*** *or* ***B = 0***.

For example:

$$If \;\; AB=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \; then\,it\,can\,be\,either$$

$$A = \begin{bmatrix} 1 & 4 \\ 6 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad or$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & 4 \\ 6 & 2 \end{bmatrix}$$

Example

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 15-1 & 0+3 \\ 6-8 & 0-6 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

Powers of a Matrix If ***A*** is an $n \times n$ matrix and if *k* is a positive integer, $\boldsymbol{A^k}$ denotes the product of *k* copies of ***A***, $A^k = \underbrace{A \ldots A}_{k}$ Also, we interpret $\boldsymbol{A^0}$ as ***I***.

Transpose of a Matrix Given an $m \times n$ matrix ***A***, the **transpose** of ***A*** is the $n \times m$ matrix, denoted by $\boldsymbol{A^t}$, whose columns are formed from the corresponding rows of ***A***.
OR, if ***A*** is an *m x n* matrix, then transpose of ***A*** is denoted by $\boldsymbol{A^t}$, is defined to be the *nxm* matrix that is obtained by making the rows of ***A*** into columns; that is, the first column of $\boldsymbol{A^t}$ is the first row of ***A***, the second column of $\boldsymbol{A^t}$ is the second row of ***A***, and so forth.

Example 12 (Transpose of a Matrix)
The following is an example of a matrix and its transpose.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

Example 13 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$

Then $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^t = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^t = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$

Theorem 3 Let ***A*** and ***B*** denote matrices whose sizes are appropriate for the following sums and products.

$a.\ (A^t)^t = A$

$b.\ (A+B)^t = A^t + B^t$

$c.\ For\ any\ scalar\ r, (rA)^t = rA^t$

$d.\ (AB)^t = B^t A^t$

The generalization of (d) to products of more than two factors can be stated in words as follows.

> "The transpose of a product of matrices equals the product of their transposes in the reverse order."