Example 1 Find the general solution of

$$y^{(4)}(t) + 6y'''(t) + 14y''(t) + 16y'(t) + 8y = 24$$

The particular integral of this fourth-order equation is simply

$$y_p = \frac{24}{8} = 3$$

Its characteristic equation is, by (16.51'),

$$r^4 + 6r^3 + 14r^2 + 16r + 8 = 0$$

which can be factored into the form

$$(r+2)(r+2)(r^2 + 2r + 2) = 0$$

From the first two parenthetical expressions, we can obtain the double roots $r_1 = r_2 = -2$, but the last (quadratic) expression yields the pair of complex roots $r_3, r_4 = -1 \pm i$, with $h = -1$ and $v = 1$. Consequently, the complementary function is

$$y_c = A_1 e^{-2t} + A_2 t e^{-2t} + e^{-t}(A_3 \cos t + A_4 \sin t)$$

and the general solution is

$$y(t) = A_1 e^{-2t} + A_2 t e^{-2t} + e^{-t}(A_3 \cos t + A_4 \sin t) + 3$$

The four constants A_1, A_2, A_3, and A_4 can be definitized, of course, if we are given four initial conditions.

Note that all the characteristic roots in this example either are real and negative or are complex and with a negative real part. The time path must therefore be convergent, and the intertemporal equilibrium is dynamically stable.

Convergence and the Routh Theorem

The solution of a high-degree characteristic equation is not always an easy task. For this reason, it should be of tremendous help if we can find a way of ascertaining the convergence or divergence of a time path without having to solve for the characteristic roots. Fortunately, there does exist such a method, which can provide a qualitative (though nongraphic) analysis of a differential equation.

This method is to be found in the *Routh theorem*,† which states that:

The real parts of all of the roots of the nth-degree polynomial equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

are negative if and only if the first n of the following sequence of determinants

$$|a_1|; \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}; \quad \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}; \quad \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}; \quad \cdots$$

all are positive.

In applying this theorem, it should be remembered that $|a_1| \equiv a_1$. Further, it is to be understood that we should take $a_m = 0$ for all $m > n$. For example, given a third-degree

† For a discussion of this theorem, and a sketch of its proof, see Paul A. Samuelson, *Foundations of Economic Analysis*, Harvard University Press, 1947, pp. 429–435, and the references there cited.

polynomial equation ($n = 3$), we need to examine the signs of the first *three* determinants listed in the Routh theorem; for that purpose, we should set $a_4 = a_5 = 0$.

The relevance of this theorem to the convergence problem should become self-evident when we recall that, in order for the time path $y(t)$ to converge regardless of what the initial conditions happen to be, all the characteristic roots of the differential equation must have negative real parts. Since the characteristic equation (16.51′) is an nth-degree polynomial equation, with $a_0 = 1$, the Routh theorem can be of direct help in the testing of convergence. In fact, we note that the coefficients of the characteristic equation (16.51′) are wholly identical with those of the given differential equation (16.51), so it is perfectly acceptable to substitute the coefficients of (16.51) directly into the sequence of determinants shown in the Routh theorem for testing, provided that we always take $a_0 = 1$. Inasmuch as the condition cited in the theorem is given on the "if and only if" basis, it obviously constitutes a necessary-and-sufficient condition.

mple 2

Test by the Routh theorem whether the differential equation of Example 1 has a convergent time path. This equation is of the fourth order, so $n = 4$. The coefficient are $a_0 = 1$, $a_1 = 6$, $a_2 = 14$, $a_3 = 16$, $a_4 = 8$, and $a_5 = a_6 = a_7 = 0$. Substituting these into the first four determinants, we find their values to be 6, 68, 800, and 6,400, respectively. Because they are all positive, we can conclude that the time path is convergent.

XERCISE 16.7

1. Find the particular integral of each of the following:
   (a) $y'''(t) + 2y''(t) + y'(t) + 2y = 8$
   (b) $y'''(t) + y''(t) + 3y'(t) = 1$
   (c) $3y'''(t) + 9y''(t) = 1$
   (d) $y^{(4)}(t) + y''(t) = 4$
2. Find the y_p and the y_c (and hence the general solution) of:
   (a) $y'''(t) - 2y''(t) - y'(t) + 2y = 4$
   [*Hint:* $r^3 - 2r^2 - r + 2 = (r-1)(r+1)(r-2)$]
   (b) $y'''(t) + 7y''(t) + 15y'(t) + 9y = 0$
   [*Hint:* $r^3 + 7r^2 + 15r + 9 = (r+1)(r^2 + 6r + 9)$]
   (c) $y'''(t) + 6y''(t) + 10y'(t) + 8y = 8$
   [*Hint:* $r^3 + 6r^2 + 10r + 8 = (r+4)(r^2 + 2r + 2)$]
3. On the basis of the signs of the characteristic roots obtained in Prob. 2, analyze the dynamic stability of equilibrium. Then check your answer by the Routh theorem.
4. Without finding their characteristic roots, determine whether the following differential equations will give rise to convergent time paths:
   (a) $y'''(t) - 10y''(t) + 27y'(t) - 18y = 3$
   (b) $y'''(t) + 11y''(t) + 34y'(t) + 24y = 5$
   (c) $y'''(t) + 4y''(t) + 5y'(t) - 2y = -2$

the Routh theorem that, for the second-order linear differential equation