Separable Variables

The differential equation in (15.22)

$$f(y, t)\,dy + g(y, t)\,dt = 0$$

may happen to possess the convenient property that the function f is in the variable y alone, while the function g involves only the variable t, so that the equation reduces to the special form

$$f(y)\,dy + g(t)\,dt = 0 \tag{15.23}$$

In such an event, the variables are said to be *separable*, because the terms involving y—consolidated into $f(y)$—can be mathematically separated from the terms involving t, which are collected under $g(t)$. To solve this special type of equation, only simple integration techniques are required.

Example 1

Solve the equation $3y^2\,dy - t\,dt = 0$. First let us rewrite the equation as

$$3y^2\,dy = t\,dt$$

Integrating the two sides (each of which is a differential) and equating the results, we get

$$\int 3y^2\,dy = \int t\,dt \qquad \text{or} \qquad y^3 + c_1 = \frac{1}{2}t^2 + c_2$$

Thus the general solution can be written as

$$y^3 = \frac{1}{2}t^2 + c \qquad \text{or} \qquad y(t) = \left(\frac{1}{2}t^2 + c\right)^{1/3}$$

The notable point here is that the integration of each term is performed with respect to a different variable; it is this which makes the separable-variable equation comparatively easy to handle.

Example 2

Solve the equation $2t\,dy + y\,dt = 0$. At first glance, this differential equation does not seem to belong in this spot, because it fails to conform to the general form of (15.23). To be specific, the coefficients of dy and dt are seen to involve the "wrong" variables. However, a simple transformation—dividing through by $2yt$ ($\neq 0$)—will reduce the equation to the separable-variable form

$$\frac{1}{y}\,dy + \frac{1}{2t}\,dt = 0$$

From our experience with Example 1, we can work toward the solution (without first transposing a term) as follows:

$$\int \frac{1}{y}\,dy + \int \frac{1}{2t}\,dt = c$$

so

$$\ln y + \frac{1}{2}\ln t = c \qquad \text{or} \qquad \ln(yt^{1/2}) = c$$

Thus the solution is

$$yt^{1/2} = e^c = k \qquad \text{or} \qquad y(t) = kt^{-1/2}$$

where k is an arbitrary constant, as are the symbols c and A employed elsewhere.