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Central Limit Theorem:-

The Central limit theorem is perhaps the most important theorem in all of the statistical inference. It is Concerned with the mean of large samples and provided Solutions when the shape of the population distribution is unknown or highly skewed.

As the Sample size increases the sampling distribution of Sample mean approches to normal distribution with mean as population and standard deviation is the population Standard deviation by square root of Sample size (n)

CLT is used to established the normality of the results.

∵ In 1733 French Mathematician Abraham de moivre gave idea of CLT. Later on 1920 george polya give it particular name Central limit theorem.

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$$\bar{X} \sim N(\mu, S/\sqrt{n})$$

$$\bar{X} - \mu \sim N(0, S/\sqrt{n})$$

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N(0, 1)$$

CLT and Properties :- Let $X_1, X_2, \ldots, X_n$ be a Sequence of n independent and identically distribuled random variable having finite mean μ and variance σ^2. Then the CLT states that as sample size increase the distribution of sample statistic of the random variable approches to normal distribution with mean $\mu_{\bar{x}} = \mu$ and variance $\sigma^2_{\bar{x}} = \frac{\sigma^2}{n}$.
If we define $\bar{X}$ be sample mean or Sample average then

$$\bar{X}_n \sim N(\mu, \sigma^2/n)$$

In a similar way

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

Proof:- we know that moment generating Function of normal distribution.

$$M_{(x)}(t) = e^{\mu t + 1/2 \sigma^2 t^2}$$

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In case of standard Normal we have

$$M_{z_n}(t) = e^{t^2/2} \qquad z_n = \frac{\bar{x}_n - \mu}{\sigma/\sqrt{n}}$$

Here $M_{z_n}(t)$ shows the MGF of z_n and we have to show that when $n \to \infty$ $M_{z_n}(t) \to M_{(x)}(t) = e^{t^2/2}$

By Defination

$$M_{z_n}(t) = E\left[e^{t(z_n)}\right]$$

$$= E\left[e^{t\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)}\right] \Rightarrow E\left[e^{t\left(\frac{\bar{x}-\mu}{\sigma}\right)\sqrt{n}}\right]$$

$$= E\left[e^{t\left(\frac{\Sigma x}{n} - \mu\right)\frac{\sqrt{n}}{\sigma}}\right]$$

$$= E\left[e^{t \cdot \frac{1}{n}\Sigma(x-\mu) \cdot \frac{\sqrt{n}}{\sigma}}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{t \cdot \frac{1}{\sqrt{n}}\left(\frac{x-\mu}{\sigma}\right)}\right] \text{ ——— } (A)$$

Since x_i's are independent let $y_i = \frac{x_i - \mu}{\sigma}$ than $M_{y_i}(t)$ is the MGF of x_i and y_i has same distribution as x_i. Now (A) becomes as

$$M_{z_n}(t) = \prod_{i=1}^{n} E\left(e^{t \cdot y_i/\sqrt{n}}\right)$$

$$= \prod_{i=1}^{n} M_{y_i}(t/\sqrt{n})$$

$$= \left[M_{y_i}(t/\sqrt{n})\right]^n \text{ ——— } (B)$$

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By using property of MGF when variables are independent and identically distributed than products change into Power.

Taylor Series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{x=0}^{\infty} \frac{x^k}{k!}$$

This may be written as

$$e^{sx} = \sum_{x=0}^{\infty} \frac{(sx)^k}{k!} = \sum_{x=0}^{\infty} \frac{x^k s^k}{k!}$$

Thus, we have

$$M_X(s) = E(e^{sx}) = \sum_{k=0}^{\infty} E(x^k) \frac{s^k}{k!}$$

OR

$$M_X(t) = E(e^{tx}) = \sum_{k=0}^{\infty} \frac{E(x^k) t^k}{k!} \quad \text{— (C)}$$

From (B) Consider $M_{y_i}(\frac{t}{\sqrt{n}})$ and Solve this by using (C) we have

$$M_{y_i}(t/\sqrt{n}) = \sum_{k=0}^{\infty} \frac{E(y_i)^k (t/\sqrt{n})^k}{k!}$$

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$$= \frac{E(y_i)^0 (t/\sqrt{n})^0}{0!} + \frac{E(y_i)^1 (t/\sqrt{n})^1}{1!} + \frac{E(y_i)^2 (t/\sqrt{n})^2}{2!} + \dots$$

$$= 1 + \frac{E(x-\mu/\sigma)(t/\sqrt{n})^1}{1!} + \frac{E(x-\mu/\sigma)^2 (t/\sqrt{n})^2}{2!} + \dots$$

$$= 1 + \frac{1}{\sigma} E(x-\mu) \frac{t}{\sqrt{n}} + \frac{1}{\sigma^2} \cdot \frac{1}{2!} E(x-\mu)^2 \left(\frac{t}{\sqrt{n}}\right)^2 + \dots$$

$$= 1 + \frac{\mu_1}{\sigma} \left(\frac{t}{\sqrt{n}}\right) + \frac{1}{2\sigma^2} \mu_2 \left(\frac{t}{\sqrt{n}}\right)^2 + \dots$$

We know that odd order moments about mean are zero so

$$= 1 + \frac{0}{\sigma} \left(\frac{t}{\sqrt{n}}\right) + \frac{\sigma^2}{2\sigma^2} \left(\frac{t}{\sqrt{n}}\right)^2 + \dots$$

$$= 1 + \frac{t^2}{2n} + \dots$$

$$= 1 + P/n \qquad \because P = t^2/2$$

Now equation Ⓑ becomes As

$$M_{Z_n}(t) = (1 + P/n)^n$$

Applying limits on both sides we have

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$$\lim_{n\to\infty} M_{z_n}(t) = \lim_{n\to\infty} (1+P/n)^n$$

$$= e^{P} = e^{t^2/2}$$

$$= M_X(t)$$

This indicates that for z_n has the MGF as the standard normal distribution

$$M_{z_n}(t) \to M_X(t) \text{ when } n\to\infty$$