5. Geometric probability distribution

An experiment is said to be geometric experiment if and only if it satisfied the following **properties** i.e.
i)The outcome of each trial may be divided into two categories success or failure.
ii) Probability of success "P" remain same for all trials.
iii) The successive trials are independent.
iv) The experiment is repeated until the first success is obtained.

$P(X=x)=Pq^{X-1}$ X=1,2,3,…

$P(X=x)=Pq^{X}$ X=0,1,2,3,…

Or

If a random variable "X" is denotes the number of failures faced to set the first success (X=0,1,2,…). Then "X" is called geometric random variable with probability function

$P(X=x)=Pq^{X}$ X=0,1,2,3,…

Or

If a random variable "X" denotes the number of trials to set first success (X=1,2,3,…), then "X" is called geometric random variable with probability function is given as

$P(X=x)=Pq^{X-1}$ X=1,2,3,…

Properties of Geometric distribution

Show that total probability of geometric probability distribution is one

Proof: Let by definition

$$Total\ \Pr obability=\sum_{x}^{\infty}P(X)$$

As $X\rightarrow Geometric\ dist(P)$

$P(X=x)=Pq^{X-1}$ X=1,2,3,…

$$Total\ \Pr obability=\sum_{x}^{\infty}Pq^{X-1}$$

$$Total\ \Pr obability=P\sum_{x}^{\infty}q^{X-1}$$

$$Total\ \Pr obability=P(q^{0}+q^{1}+q^{2}+q^{3}+...)$$

It is infinite geometric series $(q^{0}+q^{1}+q^{2}+q^{3}+...)$

$$S_{\infty}=\frac{a}{1-r}$$

Where

$a=Ist\ value=q^{0}=1$ and $r=ratio\,b/w\,two\,successive\,values=\frac{q^{1}}{q^{0}}=q$

$$Total\ \Pr obability=P(\frac{a}{1-r})=P(\frac{1}{1-q})=P(\frac{1}{p})=1$$ Hence prove

Derive moment generating function of geometric distribution

Solution:
Proof: Let by definition

$$m_0(t)=E(e^{tx})=\sum_{x}^{\infty}e^{tx}P(X)$$

As $X\rightarrow Geometric\ dist(P)$

$P(X=x)=Pq^{X-1}$ X=1,2,3,…

$$m_0(t)=E(e^{tx})=\sum_{x}^{\infty}e^{tx}Pq^{X-1}$$

$$m_0(t)=E(e^{tx})=p\sum_{x}^{\infty}e^{tx}q^{X-1}$$

$$m_0(t)=E(e^{tx})=p\sum_{x}^{\infty}(e^{t})^{x}q^{X-1}$$

$$m_0(t)=E(e^{tx})=p\sum_{x}^{\infty}(e^{t})^{x-1}(e^{t})q^{X-1}$$

$$m_0(t) = E(e^{tx}) = pe^t \sum_x^{\infty} (e^t)^{x-1} q^{X-1}$$

$$m_0(t) = E(e^{tx}) = pe^t \sum_x^{\infty} (e^t q)^{x-1}$$

$$m_0(t) = E(e^{tx}) = Pe^t \left[(e^t q)^0 + (e^t q)^1 + (e^t q)^2 + (e^t q)^3 + \ldots\right]$$

$$m_0(t) = Pe^t \left[1 + (e^t q)^1 + (e^t q)^2 + (e^t q)^3 + \ldots\right]$$

It is infinite geometric series $(1 + (e^t q)^1 + (e^t q)^2 + (e^t q)^3 + \ldots)$

$$S_\infty = \frac{a}{1-r}$$

Where

$a = Ist\ value = (e^t q)^0 = 1$ and $r = ratio\, b/w\, two\, successive\, values = \frac{(e^t q)^1}{(e^t q)^0} = (e^t q)$

$$m_0(t) = Pe^t \left(\frac{a}{1-r}\right) = Pe^t \left(\frac{1}{1-e^t q}\right) = Pe^t (1 - e^t q)^{-1}$$

$$m_0(t) = \frac{Pe^t}{e^t (e^{-t} - q)} = P(e^{-t} - q)^{-1}$$

Hence required result

Now we used it to find mean and variance

$$E(X) = Mean = \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0}$$

$$E(X) = Mean = \mu_1' = \left[\frac{d}{dt} P(e^{-t} - q)^{-1}\right]_{t=0} = p\left[\frac{d}{dt}(e^{-t} - q)^{-1} + \frac{d}{dt} e^t\right]_{t=0} = p\left[-1(e^{-t} - q)^{-2}(-e^{-t})\right]_{t=0}$$

$$E(X) = Mean = \mu_1' = p\left[(e^{-t} - q)^{-2} e^{-t}\right]_{t=0}$$

$$E(X) = Mean = \mu_1' = p\left[(e^0 - q)^{-2}(e^0)\right] = p\left[(1-q)^{-2}\right] = p\left[(P^{-2}\right] = \frac{1}{P}$$

$$E(X) = Mean = \mu_1' = \frac{1}{P}$$

Now for variance

$$Var(X) = \mu_2 = E(X^2) - [E(X)]^2 = \mu_2' - [\mu_1']^2$$

$$E(X^2) = \mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \frac{d}{dt}\left[\frac{d}{dt} M_0(t)\right]_{t=0}$$

$$E(X^2) = \mu_2' = pq \frac{d}{dt}\left[e^{-t}(e^{-t} - q)^{-2}\right]_{t=0} = p\left[(e^{-t} - q)^{-2} \frac{d}{dt} e^{-t} + e^{-t} \frac{d}{dt}(e^{-t} - q)^{-2}\right]_{t=0}$$

$$E(X^2) = \mu_2' = p\left[(e^{-t} - q)^{-2} e^{-t}(-1) + e^{-t}(-2)(e^{-t} - q)^{-3}(-e^{-t})\right]_{t=0} = p\left[-e^{-t}(e^{-t} - q)^{-2} e^t + 2e^{-2t}(e^{-t} - e^t)^{-3}\right]_{t=0}$$

$$E(X^2) = \mu_2' = p\left[-(e^0 - q)^{-2} e^0 + 2e^{-2(0)} q(e^0 - q)^{-3}\right] = p\left[-(1-q)^{-2} + 2(1-q)^{-3}\right] = p\left[-p^{-2} + 2p^{-3}\right]$$

$$E(X^2) = \mu_2' = p\left[-p^{-2} + 2p^{-2}\right] = p\left[\frac{2}{P^3} - \frac{1}{P^2}\right] = p\left[\frac{2-P}{P^3}\right] = \left[\frac{2-P}{P^2}\right]$$

$$Var(X) = \mu_2 = E(X^2) - [E(X)]^2 = \mu_2' - [\mu_1']^2$$

$$Var(X) = \mu_2 = \frac{2-P}{p^2} - \frac{1}{p^2} = \left(\frac{2-P-1}{p^2}\right) = \left(\frac{1-P}{p^2}\right) = \frac{q}{p^2}$$

Derive the general expression of rth mean moments of geometric distribution and find first four moments about mean.

Let we consider

$$\mu_r = E(X - mean)^r$$

$$\mu_r = \sum_X (X - mean)^r P(X)$$

As $X \to G(P)$

$P(X) = pq^{x-1}$ Where X=1,2,3,……

$$E(X) = Mean = \frac{1}{p}$$

$$\mu_r = \sum_{X=1}^{\infty} \left(X - \frac{1}{p}\right)^r Pq^{X-1}$$

$$\mu_r = \sum_{X=1}^{\infty}(X-\frac{1}{p})^r P(1-p)^{X-1}$$

Differentiate with respect to "P"

$$\frac{d}{dp}\mu_r = \frac{d}{dp}\left[\sum_{X=1}^{\infty}(X-\frac{1}{p})^r P(1-p)^{X-1}\right]$$

$$\frac{d}{dp}\mu_r = \sum_{X=1}^{\infty}P(1-p)^{X-1}\frac{d}{dp}(X-\frac{1}{p})^r + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r(1-p)^{X-1}\frac{d}{dp}P + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r P\frac{d}{dp}(1-p)^{X-1}$$

$$\frac{d}{dp}\mu_r = \sum_{X=1}^{\infty}P(1-p)^{X-1}r(X-\frac{1}{p})^{r-1}(\frac{-1}{p^2})(-1) + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r(1-p)^{X-1} + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r P(x-1)(1-p)^{X-2}(-1)$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\sum_{X=1}^{\infty}P(1-p)^{X-1}(X-\frac{1}{p})^{r-1} + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r q^{X-1} - \sum_{X=1}^{\infty}(X-\frac{1}{p})^r P(x-1)q^{X-2}$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\sum_{X=1}^{\infty}(X-\frac{1}{p})^{r-1}Pq^{X-1} + \frac{1}{p}\sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1} - \sum_{X=1}^{\infty}(X-\frac{1}{p})^r P(x-1)q^{X-2}$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\sum_{X=1}^{\infty}(X-\frac{1}{p})^{r-1}P(x) + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1}\left[\frac{1}{p}-(x-1)q^{-1}\right]$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\sum_{X=1}^{\infty}(X-mean)^{r-1}P(x) + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1}\left[\frac{1}{p}-\frac{(x-1)}{q}\right]$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1}\left[\frac{q-(x-1)p}{pq}\right]$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1}\left[\frac{1-p-px+p}{pq}\right]$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1}\left[\frac{1-px}{pq}\right]$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} + \sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1}\left[\frac{1}{pq}-\frac{x}{q}\right]$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} + \frac{-1}{q}\sum_{X=1}^{\infty}(X-\frac{1}{p})^r pq^{X-1}\left[X-\frac{1}{p}\right]$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\sum_{X=1}^{\infty}(X-\frac{1}{p})^{r+1} pq^{X-1}$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\sum_{X=1}^{\infty}(X-\frac{1}{p})^{r+1} p(x)$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\sum_{X=1}^{\infty}(X-Mean)^{r+1} p(x)$$

$$\frac{d}{dp}\mu_r = \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\mu_{r+1}$$

$$\frac{1}{q}\mu_{r+1} = \frac{r}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r$$

$$\mu_{r+1} = q\left[\frac{r}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r\right] \qquad (A)$$

Required result

Now we find first four moments about mean

Put "r=1" in eq (A)

$$\mu_{1+1} = q\left[\frac{1}{p^2}\mu_{1-1} - \frac{d}{dp}\mu_1\right]$$

$$\mu_2 = q\left[\frac{1}{p^2}\mu_0 - \frac{d}{dp}\mu_1\right]$$

Therefore $\mu_1 = 0 \quad and \quad \mu_0 = 1$

$$\mu_2 = \frac{q}{p^2}$$

Put "r=2" in eq (A)

$$\mu_{2+1} = q\left[\frac{2}{p^2}\mu_{2-1} - \frac{d}{dp}\mu_2\right]$$

$$\mu_3 = q\left[\frac{2}{p^2}\mu_1 - \frac{d}{dp}\mu_2\right]$$ Therefore $\mu_1 = 0$ *and* $\mu_0 = 1$ and $\mu_2 = \frac{q}{p^2}$

$$\mu_3 = q\left[0 - \frac{d}{dp}\frac{q}{p^2}\right] = -q\left[\frac{d}{dp}\frac{(1-p)}{p^2}\right] = -q\left[\frac{p^2\frac{d}{dp}(1-p) - (1-p)\frac{d}{dp}p^2}{(p^2)^2}\right] = -q\left[\frac{p^2(-1) - (1-p)2p}{(p^2)^2}\right]$$

$$\mu_3 = -q\left[\frac{-p^2 - 2p + 2p^2}{(p^2)^2}\right] = -q\left[\frac{p^2 - 2p}{(p^2)^2}\right] = qp\left[\frac{2-p}{p^4}\right] = q\left[\frac{1`+1-p}{p^3}\right] = q\left[\frac{1`+q}{p^3}\right]$$

Put "r=3" in eq (A)

$$\mu_{3+1} = q\left[\frac{3}{p^2}\mu_{3-1} - \frac{d}{dp}\mu_3\right]$$

$$\mu_4 = q\left[\frac{3}{p^2}\mu_2 - \frac{d}{dp}\mu_3\right]$$ Therefore $\mu_2 = \frac{q}{p^2}$ and $\mu_3 = (1-p)(2-p)p^{-3}$

$$\mu_4 = q\left[\frac{3q}{p^4} - \frac{d}{dp}(1-p)(2-p)p^{-3}\right] = q\left[\frac{3q}{p^4} - (2-p)p^{-3}\frac{d}{dp}(1-p) - (1-p)p^{-3}\frac{d}{dp}(2-p) - (2-p)(1-p)\frac{d}{dp}p^{-3}\right]$$

$$\mu_4 = q\left[\frac{3q}{p^4} - (2-p)p^{-3}(-1) - (1-p)p^{-3}(-1) - (1-p)(2-p)(-3)p^{-4}\right]$$

$$\mu_4 = q\left[\frac{3q}{p^4} + (2-p)p^{-3} + (1-p)p^{-3} + 3(1-p)(2-p)p^{-4}\right]$$

$$\mu_4 = q\left[\frac{3q}{p^4} + 2p^{-3} - p^{-2} + p^{-3} - p^{-2} + (3p^{-4} - 3p^{-3})(2-p)\right]$$

$$\mu_4 = q\left[\frac{3q}{p^4} + 2p^{-3} - p^{-2} + p^{-3} - p^{-2} + 6p^{-4} - 3p^{-3} - 6p^{-3} + 3p^{-2}\right]$$

$$\mu_4 = q\left[\frac{3q}{p^4} + p^{-2} + 6p^{-4} - 6p^{-3}\right]$$

$$\mu_4 = q\left[\frac{3(1-p)}{p^4} + \frac{1}{p^2} + \frac{6}{p^4} - \frac{6}{p^3}\right] = q\left[\frac{3(1-p)}{p^4} + \frac{p^2 + 6 - 6p}{p^4}\right] = q\left[\frac{3 - 3p + p^2 + 6 - 6p}{p^4}\right]$$

$$\mu_4 = q\left[\frac{9 + p^2 - 9p}{p^4}\right] = q\left[\frac{p^2 + 9 - 9p}{p^4}\right] = q\left[\frac{p^2 + 9(1-p)}{p^4}\right] = q\left[\frac{p^2 + 9q}{p^4}\right] = \left[\frac{qp^2 + 9q^2}{p^4}\right]$$

Derive Probability generating function

Solution:

Proof: Let by definition

$$G(\theta) = E(\theta^x) = \sum_x^{\infty} \theta^x P(X)$$

As $X \rightarrow Geometric\ dist(P)$

$P(X = x) = Pq^X$ X=0,1,2,3,…

$$G(\theta) = E(\theta^x) = \sum_x^{\infty} \theta^x Pq^X = P\sum_x^{\infty}(\theta q)^X = P\left[(\theta q)^0 + (\theta q)^1 + (\theta q)^2 + (\theta q)^3 + \ldots\right]$$

$$G(\theta) = P\left[1 + (\theta q)^1 + (\theta q)^2 + (\theta q)^3 + \ldots\right]$$

It is infinite geometric series $1 + (\theta q)^1 + (\theta q)^2 + (\theta q)^3 + \ldots$

$$S_\infty = \frac{a}{1-r}$$

Where

$a = Ist\ value = (\theta q)^0 = 1$ and $r = ratio\ b/w\ two\ successive\ values = \frac{(\theta q)^1}{(\theta q)^0} = (\theta q)$

$$G(\theta) = P(\frac{a}{1-r}) = P(\frac{1}{1-\theta q}) = P(1-\theta q)^{-1}$$

Hence required result

Derive Cummulent generating function

Let by definition

$$K(t) = \log m_X(t)$$

As

$$m_x(t) = P(1-e^t q)^{-1}$$

$$K(t) = \log P(1-e^t q)^{-1}$$

$$K(t) = \log P + \log(1-e^t q)^{-1}$$

$$K(t) = \log P - \log(1-e^t q)$$

$$K(t) = \log\left[\frac{P}{(1-e^t q)}\right] \qquad \text{Therefore} \qquad q+p=1$$

$$K(t) = \log\left[\frac{P}{q+p-e^t q}\right] \qquad \text{Therefore} \qquad e^t = 1+\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...$$

$$K(t) = \log\left[\frac{P}{q+p-q(1+\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)}\right]$$

$$K(t) = \log\left[\frac{P}{q+p-q-q(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)}\right]$$

$$K(t) = \log\left[\frac{P}{p-q(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)}\right]$$

$$K(t) = \log\left[\frac{1}{\frac{p}{p}-\frac{q}{p}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)}\right]$$

$$K(t) = \log\left[\frac{1}{1-\frac{q}{p}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)}\right]$$

$$K(t) = \log\left[1-\frac{q}{p}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)\right]^{-1}$$

$$K(t) = -\log\left[1-\frac{q}{p}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)\right]$$

$$\text{Therefore } \log(1-t) = -t-\frac{1}{2}t^2-\frac{1}{3}t^3-\frac{1}{4}t^4-...$$

$$K(t) = -\left[-\left\{\frac{q}{p}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)-\frac{q^2}{2p^2}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)^2-\frac{q^3}{3p^3}(\frac{t^1}{1!}+\frac{t^3}{3!}+...)^3-\frac{q^4}{4p^4}(\frac{t^1}{1!}+...)^4\right\}\right]$$

$$K(t) = \left\{\frac{q}{p}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)+\frac{q^2}{2p^2}(\frac{t^1}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...)^2+\frac{q^3}{3p^3}(\frac{t^1}{1!}+\frac{t^3}{3!}+...)^3+\frac{q^4}{4p^4}(\frac{t^1}{1!}+...)^4\right\}$$

$$K(t) = (\frac{q}{p})\frac{t^1}{1!}+(\frac{q}{p}+\frac{q^2}{p^2})\frac{t^2}{2!}+\frac{q}{p}\frac{t^3}{3!}+\frac{q^2}{2p^2}\frac{2t^3}{2!}+\frac{q^3}{3p^3}\frac{t^3}{1!}+\frac{q}{p}\frac{t^4}{4!}+\frac{q^2}{2p^2}(\frac{2t^4}{3!}+\frac{t^4}{4})+\frac{q^3}{3p^3}(\frac{3t^4}{2!}+\frac{q^4}{4p^4}(\frac{t^4}{1!})+...$$

$$K(t) = (\frac{q}{p})\frac{t^1}{1!}+(\frac{q}{p}+\frac{q^2}{p^2})\frac{t^2}{2!}+(\frac{q}{p}+\frac{3q^2}{p^2}+\frac{2q^3}{p^3})\frac{t^3}{3!}+(\frac{q}{p}+\frac{4q^2}{p^2}+\frac{3q^2}{p^2}+\frac{12q^3}{p^3}+\frac{6q^4}{p^4})\frac{t^4}{4!}+... \qquad \text{(A)}$$

As we know

$$K(t) = k_1 \frac{t^1}{1!} + k_1 \frac{t^2}{2!} + k_3 \frac{t^3}{3!} + k_4 \frac{t^4}{4!} + ... \qquad (B)$$

Comparing (A) and (B)

$$K_1 = \frac{q}{p} = Mean$$

$$K_2 = \frac{q}{p} + \frac{q^2}{p^2} = \frac{q}{p}(1 + \frac{q}{p}) = \frac{q}{p}(\frac{p+q}{p}) = \frac{q}{p^2} = V(x)$$

Similarly remaining

$$K_3 = (\frac{q}{p} + \frac{3q^2}{p^2} + \frac{2q^3}{p^3})$$

$$K_3 = \frac{qp^2 + 3q^2 p + 2q^3}{p^3}$$

$$K_3 = \frac{q}{p^3}(p^2 + 3qp + 2q^2)$$

$$K_3 = \frac{q}{p^3}(p^2 + 3(1-p)p + 2(1-p)^2)$$

$$K_3 = \frac{q}{p^3}(p^2 + 3(1-p)p + 2(1-p)^2)$$

$$K_3 = \frac{q}{p^3}(p^2 + 3p - 3p^2 + 2 + 2p^2 - 4p)$$

$$K_3 = \frac{q}{p^3}(2-p)$$

$$K_3 = \frac{q}{p^3}(1+1-p)$$

$$K_3 = \frac{q}{p^3}(1+q)$$

$$K_4 = (\frac{q}{p} + \frac{4q^2}{p^2} + \frac{3q^2}{p^2} + \frac{12q^3}{p^3} + \frac{6q^4}{p^4})$$

$$\mu_4 = K_4 + 3K_2^2 = (\frac{q}{p} + \frac{4q^2}{p^2} + \frac{3q^2}{p^2} + \frac{12q^3}{p^3} + \frac{6q^4}{p^4}) + 3\left(\frac{q}{p^2}\right)^2$$

Prepared by: Muhammad Riaz
M.Sc: Statistics (Silver medalist)
I.U.B (2006-08)
Lecturer: Statistics
NICAAS College (RYK)
03337368518