1. Binomial distribution

Discrete variable
A variable which takes some specific values in a given range is called discrete variable.
Usually; discrete variables take values as whole integers
Such as 0, 1, 2, 3,…, n
Example
Number of children in a family
Define Bernoulli trial
Ans: A trial that gives only two possible outcomes is called Bernoulli trials.
Example:
There are many trials they have only two possible outcomes such as
i) Head and tail ii) Success and failure III) Alive and dead
iv) Right and wrong v) Good and defective
What are the Properties of Bernoulli experiments?
Ans: There are four properties of Bernoulli experiment
i) Successive trials are Independent
ii) The probability of success remains same for all trials
iii) The experiment is repeated a single time
iv) Each trial classified into two categories such as success(S) or failure(F)
Define binomial experiment.
Ans: An experiment consisting of "n" Bernoulli trials is known as binomial experiment.
What are the Properties of Binomial experiment?
Ans: There are four properties of Binomial experiment
i) Successive trials are Independent
ii) The probability of success remains same for all trials
iii) The experiment is repeated a fixed number of times say "n"
iv) Each trial classified into two categories such as success(S) or failure(F)
Explain briefly binomial random variable?
Ans: The random variable which denotes the number of successes of binomial experiment is called binomial random variable. It is discrete variable and assume values as X=0, 1, 2… n.

Why discrete probability function is $b(x,n,p)=\begin{pmatrix} n \\ c \\ x \end{pmatrix} P^x q^{n-x}$ called the binomial distribution?

Ans: $b(x,n,p)=\begin{pmatrix} n \\ c \\ x \end{pmatrix} P^x q^{n-x}$ X=0, 1, 2… n. is called binomial distribution because its successive terms are equal to binomial expansion of $(q+p)^n$ and they give the probabilities of binomial random variable.
What is binomial probability function or mass function?
Ans: The formula used to find the probabilities of binomial random variable "X" is called binomial probability function or binomial probability distribution. It is given by

$b(x,n,p)=\begin{pmatrix} n \\ c \\ x \end{pmatrix} P^x q^{n-x}$ X=0, 1, 2… n.

Where $q=1-p$ P= probability of success of single trial.
n= number independent of trials
Define binomial probability distribution
Ans: Arrangement of all possible values of binomial random variable of "X" with their probabilities is called binomial probability distribution and given as

X	0	1	…..	n
P(X)	$\begin{pmatrix} n \\ c \\ 0 \end{pmatrix} P^0 q^{n-0}$	$\begin{pmatrix} n \\ c \\ 1 \end{pmatrix} P^1 q^{n-1}$	…	$\begin{pmatrix} n \\ c \\ n \end{pmatrix} P^n q^{n-n}$

Define binomial frequency distribution
Ans: If each probability of binomial distribution is multiplied by N then resulting distribution is called binomial frequency distribution as given below

X	0	1	…..	n
NP(X)	$N\begin{pmatrix} n \\ c \\ 0 \end{pmatrix} P^0 q^{n-0}$	$N\begin{pmatrix} n \\ c \\ 1 \end{pmatrix} P^1 q^{n-1}$	…	$N\begin{pmatrix} n \\ c \\ n \end{pmatrix} P^n q^{n-n}$

Describe the properties of binomial distribution
Ans: The properties of binomial distribution are given below

i) Total probability of binomial probability distribution is one.
ii) Mean of binomial distribution is "np"
iii) Variance of binomial distribution is "npq"
iv) Shape of binomial distribution $b(x,n,p)$ depends upon "n and p"

When $p = q = \frac{1}{2}$ or $p = q = 0.5$ The distribution is always symmetrical

When $p \neq q$ The distribution is said to be skewed

When $p > \frac{1}{2}$ or $p > 0.5$ The distribution is said to be negatively skewed

When $p < \frac{1}{2}$ or $p < 0.5$ The distribution is said to be positively skewed

How can the mean and variance of binomial probability distribution be calculated in terms of parameters?
Ans: Mean is calculated by multiplying by "n and p"   Mean=np
Variance is calculated by multiplying by "n, p and q"   Variance=npq

Why in binomial distribution variance is smaller than mean give reason?
Ans: We know that in binomial distribution mean is "np" and variance is "np(1-p). To calculate variance mean is further multiplied by a number which is less than one hence Mean> Variance

State the formula used to calculate binomial probabilities?
Ans: The formula used to find the probabilities of binomial random variable "X" is called binomial probability function or binomial probability distribution. It is given by

$b(x,n,p) = \binom{n}{c\ x} P^x q^{n-x}$ X=0, 1, 2… n.

Where $q = 1 - p$ P= probability of success of single trial.
n= number independent of trials

Which formulae's should be used to find the Mean, Variance, Standard deviation and Coefficient of variation of a binomial random variable?
Ans: To used the following formulae's to calculate Mean, Variance, Standard deviation and Coefficient of variation of a binomial random variable are given below

$E(X) = Mean = np$

$Variance(X) = np(1-p) \quad or \ npq$

$S.D(X) = \sqrt{np(1-p)} \quad or \ \sqrt{npq}$

$Coefficien\,of\ \text{var}iation = C.V = \frac{S.D(X)}{E(X)} \times 100$

Derivation of binomial distribution

Let the binomial probability experiment having the following four properties
i) Each trial classified as success and failure
ii) Probability of success remains same = Constant
iii) Successive trials = independent
iv) Experiment repetition= Fixed number of times
Then
Total trials = n
Total success = X
P(Success)=P(S)=P
P(Failure)=P(f)=q
Total possible outcomes $= 2^n$
Each outcome has a sequence = $\{a_1, a_2, ..., a_i, ..., a_n\}$
Where $a_i = Success\ and\ \ failure$
First we consider no success i.e. P(X=0)
$Then\ sequence\ of\ \ failure\ is = F.F.F...F$

$P(X = 0) = P(F.F.F...F) = P(F).P(F)...P(F) = q.q.q...q = q^n$

And number of mutually sequence $= {}^nC_0 \ ; P^0 = 1$

So, $P(X = 0) = {}^nC_0 \ P^0 q^n$

Now we consider one success then sequence of one success and n-1 failures are in form of

$S.F.F...F; F.S.F...F; ...; F.F.F...S$

$P(X=1) = P(S.F.F...F) + P(F.S.F...F) + ... + P(.F.F....S)$

$P(X=1) = Pq^{n-1} + Pq^{n-1} + ... + Pq^{n-1}$

And number of mutually sequence $= {}^nC_1$

So, $P(X=1) = {}^nC_1 \; Pq^{n-1}$

If we consider two success in "n" trials

$P(X = 2) = {}^nC_2 \; P^2 q^{n-2}$

In general if we consider x success and n-x failures in "n" trials

$P(X=x) = {}^nC_x \; P^x q^{n-x}$ Where $X = 0,1,2,3,...,n$

Properties of binomial distribution

i) Total probability of binomial distribution is one.

Proof: Let by definition

$Total \;\; probability = \sum P(X)$

As $X \to b(x,n,p)$ then its Probability function given as

$P(X) = {}^nC_x P^x q^{n-x}$ Where X=0, 1, 2,..., n

$\sum P(X) = {}^nC_0 P^0 q^{n-0} + {}^nC_1 P^1 q^{n-1} + {}^nC_2 P^2 q^{n-2} + ... + {}^nC_n P^n q^{n-n}$

$\sum P(X) = (q+p)^n = (1)^n = 1$

ii) Moment generating function

Proof: Let by definition

$m_0(t) = E(e^{tx}) = \sum e^{tx} P(X)$

As $X \to b(x,n,p)$ then its Probability function given as

$P(X) = {}^nC_x P^x q^{n-x}$ Where X=0, 1, 2,..., n

$m_0(t) = \sum e^{tx}\, {}^nC_x P^x q^{n-x} = \sum {}^nC_x (Pe^t)^x q^{n-x}$

$m_0(t) = {}^nC_0 (Pe^t)^0 q^{n-0} + {}^nC_1 (Pe^t)^1 q^{n-1} + {}^nC_2 (Pe^t)^2 q^{n-2} + ... + {}^nC_n (Pe^t)^n q^{n-n}$

$m_0(t) = (q + Pe^t)^n$ Required result

Use it to find mean and variance

$$\mu_1' = Mean = \frac{d}{dt}[m_0(t)]_{t=0}$$

$$\mu_1' = Mean = \frac{d}{dt}\left[(q+Pe^t)^n\right]_{t=0}$$

$$\mu_1' = Mean = \left[n(q+Pe^t)^{n-1}\frac{d}{dt}(q+Pe^t)\right]_{t=0}$$

$$\mu_1' = Mean = \left[n(q+Pe^t)^{n-1}Pe^t\right]_{t=0}$$

$$\mu_2' = \frac{d^2}{dt^2}[m_0(t)]_{t=0} = \frac{d}{dt}\left[\frac{d}{dt}m_0(t)\right]_{t=0} = \frac{d}{dt}\left[nPe^t\left[(q+Pe^t)^{n-1}\right]_{t=0}\right]_{t=0}$$

$$\mu_2' = \left[\left(nPe^t\frac{d}{dt}(q+Pe^t)^{n-1} + np(q+Pe^t)^{n-1}\frac{d}{dt}e^t\right)\right]_{t=0}$$

$$\mu_2' = \left[\left(nPe^t(n-1)(q+Pe^t)^{n-2}\frac{d}{dt}(q+Pe^t) + nP(q+Pe^t)^{n-1}e^t\right)\right]_{t=0}$$

$$\mu_2' = \left[\left(nPe^t(n-1)(q+Pe^t)^{n-2}Pe^t + nP(q+Pe^t)^{n-1}e^t\right)\right]_{t=0}$$

$$\mu_2' = \left[\left(n(n-1)P^2e^{2t}(q+Pe^t)^{n-2} + nP(q+Pe^t)^{n-1}e^t\right)\right]_{t=0}$$

$$\mu_2' = \left[\left(n(n-1)P^2e^{2(0)}(q+Pe^0)^{n-2} + nP(q+Pe^0)^{n-1}e^0\right)\right]$$

$$\mu_2' = \left[\left(n(n-1)P^2(q+P)^{n-2} + nP(q+P)^{n-1}\right)\right]$$

$$\mu_2' = \left[\left(n(n-1)P^2(1)^{n-2} + nP(1)^{n-1}\right)\right]$$

$$\mu_2' = \left(n(n-1)P^2 + nP\right) = n^2P^2 - nP^2 + nP$$

$$Var(x) = \mu_2' - (\mu_1')^2 = n^2P^2 - nP^2 + nP - (nP)^2$$

$Var(x) = n^2P^2 - nP^2 + nP - n^2P^2 = -nP^2 + nP$

$Var(x) = nP(1-P) = nPq$

Required result

iii) Characteristic function

Proof: Let by definition

$Q_x(t) = M_x(it)$

As we know

$m_x(t) = \left(q + Pe^t\right)^n$

$Q_x(t) = \left(q + Pe^{it}\right)^n$ Required result

iv) Probability generating function

Proof:

Let by definition

$G(\theta) = E(\theta)^X$

As $X \to b(x, n, p)$ then its Probability function given as

$P(X) = {}^nC_x P^x q^{n-x}$ Where X=0, 1, 2,..., n

$G(\theta) = \sum \theta^x P(x) = \sum \theta^x\, {}^nC_x P^x q^{n-x} = \sum {}^nC_x (P\theta)^x q^{n-x}$

$G(\theta) = {}^nC_0 (P\theta)^0 q^{n-0} + {}^nC_1 (P\theta)^1 q^{n-1} + {}^nC_2 (P\theta)^2 q^{n-2} + ... + {}^nC_n (P\theta)^n q^{n-n}$

$G(\theta) = (q + P\theta)^n$ Required result

Use it to find mean and variance

$\mu'_{(1)} = Mean = \frac{d}{d\theta}\left[G(\theta)\right]_{\theta=1}$

$\mu'_1 = Mean = \frac{d}{d\theta}\left[(q + P\theta)^n\right]_{\theta=1}$

$\mu'_{(1)} = Mean = \left[n(q + P\theta)^{n-1}\frac{d}{d\theta}(q + P\theta)\right]_{\theta=1}$

$\mu'_{(1)} = Mean = \left[n(q + P)^{n-1}P\right]$

$\mu'_{(1)} = Mean = np$

Similarly

$\mu'_2 = \mu'_{(2)} + \mu'_{(1)}$

$\mu'_3 = \mu'_{(1)} + \mu'_{(2)} + \mu'_{(3)}$

$\mu'_4 = \mu'_{(1)} + \mu'_{(2)} + \mu'_{(3)} + \mu'_{(4)}$

Example Find mean and variance of binomial distribution when "n=1"

Solution: Mean and variance of binomial distribution when "n" is one

X	P(x)	XP(X)	$X^2P(X)$
0	$\binom{1}{0}P^0q^1 = q$	0	0
1	$\binom{1}{1}P^1q^0 = P$	P	P
Total		$\sum XP(X) = p$	$\sum X^2P(X) = p$

$E(X) = \sum XP(X) = p$

$Var(X) = E(X^2) - (E(X))^2 = \sum X^2P(X) - (\sum XP(X))^2 = P - P^2 = P(1-P) = pq$

Example: Find mean and variance of binomial distribution when "n=2"
Solution:
Mean and variance of binomial distribution when "n" is two

X	P(x)	XP(X)	$X^2P(X)$
0	$\binom{2}{c\,0}P^0q^2 = q^2$	0	0
1	$\binom{2}{c\,1}P^1q^1 = 2pq$	$2pq$	$2pq$
2	$\binom{2}{c\,2}p^2q^0 = p^2$	$2p^2$	$4p^2$
Total		$\sum XP(X) = 2pq + 2p^2$	$\sum X^2P(X) = 2pq + 4p^2$

$$E(X) = \sum XP(X) = 2pq + 2p^2 = 2p(q+p) = 2p((q+p) = 2p(1) = 2p$$

$$Var(X) = E(X^2) - (E(X))^2 = \sum X^2P(X) - (\sum XP(X))^2 = 2pq + 4p^2 - (2p)^2 = 2pq + 4p^2 - 4p^2$$

$$Var(X) = 2pq + 4p^2 - 4p^2 = 2pq$$

$$S.D(X) = \sqrt{Var(X)} = \sqrt{2pq}$$

Example: Find mean and variance of binomial distribution when "n=3"
Solution: Mean and variance of binomial distribution when "n" is three

X	P(x)= $\binom{n}{c\,x}P^xq^{n-x}$	XP(X)	$X^2P(X)$
0	$\binom{3}{c\,0}P^0q^3 = q^3$	0	0
1	$\binom{3}{c\,1}P^1q^2 = 3pq^2$	$3pq^2$	$3pq^2$
2	$\binom{3}{c\,2}p^2q^1 = 3p^2q$	$6p^2q$	$12p^2q$
3	$\binom{3}{c\,3}p^3q^0 = p^3$	$3p^3$	$9p^3$
Total		$\sum XP(X)$	$\sum X^2P(X)$

$$E(X) = \sum XP(X) = 3pq^2 + 6p^2q + p^3 = 3p(q^2 + 2pq + p^2) = 3p(q+p)^2 = 3p(1)^2 = 3p$$

$$Var(X) = E(X^2) - (E(X))^2 = \sum X^2P(X) - (\sum XP(X))^{2\,2}$$

$$Var(X) = 3pq^2 + 12p^2q + 9p^3 - (3p)^2 = 3pq^2 + 12p^2q + 9p^3 - 9p^2$$

$$Var(X) = 3p(q^2 + 4pq + 3p^2 - 3p) = 3p(q^2 + 4pq + 3p(p-1))$$

$$Var(X) = 3p(q^2 + 4pq - 3p(1-p)) = 3p(q^2 + 4pq - 3pq)$$

$$Var(X) = 3pq(q + 4p - 3p) = 3pq(q+p) = 3pq(1) = 3pq$$

$$S.D(X) = \sqrt{Var(X)} = \sqrt{3pq}$$

Example: Find mean and variance of binomial distribution when "n=4"
Solution:
Mean and variance of binomial distribution when "n" is four

X	P(x)= $\binom{n}{c_x} P^x q^{n-x}$	XP(X)	$X^2P(X)$
0	$\binom{4}{c_0} P^0 q^4 = q^4$	0	0
1	$\binom{4}{c_1} P^1 q^3 = 4pq^3$	$4pq^3$	$4pq^3$
2	$\binom{4}{c_2} p^2 q^2 = 6p^2q^2$	$12p^2q^2$	$24p^2q^2$
3	$\binom{4}{c_3} p^3 q^1 = 4p^3q$	$12p^3q$	$36p^3q$
4	$\binom{4}{c_4} p^4 q^0 = p^4$	$4p^4$	$16p^4$
Total		$\sum XP(X)$	$\sum X^2P(X)$

$$E(X) = \sum XP(X) = 4pq^3 + 12p^2q^2 + 12p^3q + 12p^4$$
$$E(X) = 4p(q^3 + 3pq^2 + 3p^2q + 3p^3) = 4p(q+p)^3 = 4p(1)^3 = 4p$$
$$Var(X) = E(X^2) - (E(X))^2 = \sum X^2P(X) - (\sum XP(X))^{2\,2}$$
$$E(X^2) = \sum X^2P(X)$$
$$E(X^2) = 4pq^3 + 24p^2q^2 + 36p^3q + 16p^4$$
$$E(X^2) = 4p(q^3 + 6pq^2 + 9p^2q + 4p^3)$$
$$E(X^2) = 4p(q^3 + 3pq^2 + 3pq^2 + p^3 + 9p^2q + 3p^3)$$
$$E(X^2) = 4p\left[(q+p)^3 + 3p^2q + 6p^2q + 3p^3)\right]$$
$$E(X^2) = 4p\left[(q+p)^3 + 3p(pq + 2pq + p^2)\right]$$
$$E(X^2) = 4p\left[(q+p)^3 + 3p(p+q)^2\right]$$
$$E(X^2) = 4p\left[(1)^3 + 3p(1)^2\right]$$
$$E(X^2) = 4p[1 + 3p] = 4p + 12p^2$$
$$Var(X) = E(X^2) - (E(X))^2 = \sum X^2P(X) - \left(\sum XP(X)\right)^2$$
$$Var(X) = 4p + 12p^2 - (4p)^2 = 4p(1 + 3p - 4p) = 4p(1-p) = 4pq$$
$$S.D(X) = \sqrt{Var(X)} = \sqrt{4pq}$$

Example: Find mean and variance of binomial distribution when for "n"
Solution:
Mean and variance of binomial distribution when for "n"
The binomial random variable "X" With parameters "n and p" has the probability distribution

$$P(X = x) = b(x; n, p) = \overset{n}{\underset{x}{c}} p^x q^{n-x} \qquad x = 1,2,3,\ldots,n \qquad q + p = 1$$
$$E(X) = \sum_{x=0}^{n} xP(x) = \sum_{x=0}^{n} x \overset{n}{\underset{x}{c}} p^x q^{n-x} = \sum_{x=1}^{n} x \frac{n}{x} \overset{n-1}{\underset{x-1}{c}} p^{x-1} pq^{n-x} = np \sum_{x=1}^{n} \overset{n-1}{\underset{x-1}{c}} p^{x-1} q^{n-x}$$
$$E(X) = np\left[\overset{n-1}{\underset{0}{c}} p^0 q^{n-1} + \overset{n-1}{\underset{1}{c}} p^1 q^{n-2} + \overset{n-1}{\underset{2}{c}} p2q^{n-3} + \ldots + \overset{n-1}{\underset{n-1}{c}} p^{n-1} q^0\right]$$
$$E(X) = np\left[q^{n-1} + \overset{n-1}{\underset{1}{c}} p^1 q^{n-2} + \overset{n-1}{\underset{2}{c}} p2q^{n-3} + \ldots + p^{n-1}\right]$$
$$E(X) = np\left[(q+p)^{n-1}\right] = np(1)^{n-1} = np$$
$$E(X^2) = E[X(X-1) + X] = E[X(X-)] + E(X)$$

$E(X^2) = E[X(X-1)+X] = E[X(X-)] + np$ ………….. (A)

$E[X(X-)] = \sum_{x=0}^{n} X(X-1)P(X) = \sum_{x=0}^{n} X(X-1)\overset{n}{\underset{x}{c}} p^x q^{n-x} = \sum_{x=2}^{n} X(X-1)\frac{n(n-1)}{X(X-1)} \overset{n-2}{\underset{x-2}{c}} p^{x-2} p^2 q^{n-x}$

$E[X(X-)] = n(n-1)p^2 \sum_{x=2}^{n} \overset{n-2}{\underset{x-2}{c}} p^{x-2} q^{n-x}$

$E[X(X-1)] = n(n-1)p^2 \left[\overset{n-2}{\underset{0}{c}} p^0 q^{n-2} + \overset{n-2}{\underset{1}{c}} p^1 q^{n-3} + \overset{n-2}{\underset{2}{c}} p^2 q^{n-4} + ... + \overset{n-2}{\underset{n-2}{c}} p^{n-2} q^0 \right]$

$E[X(X-1)] = n(n-1)p^2 [(q+p)^{n-2}] = n(n-1)p^2 (1)^{n-2} = n(n-1)p^2$

Put in A

$E(X^2) = n(n-1)p^2 + np$

$Var(X) = E(X^2) - (E(X))^2 = \sum X^2 P(X) - (\sum XP(X))^{2\,2}$

$Var(X) = n(n-1)p^2 + np - (np)^2 = n^2p^2 - np^2 + np - n^2p^2 = -np^2 + np = np - np^2$

$Var(X) = np - np^2 = np(1-p) = npq$

$S.D(X) = \sqrt{Var(X)} = \sqrt{npq}$

The recurrence formula for binomial distribution

Proof: Let by definition

As $X \to b(x,n,p)$ then its Probability function given as

$P(X=x) = {}^nC_x P^x q^{n-x}$ (i)

$P(X=x-1) = {}^nC_{x-1} P^{x-1} q^{n-x+1}$ (i)

Dividing (i) by (ii)

$$\frac{P(X=x)}{P(X=x-1)} = \frac{{}^nC_x P^x q^{n-x}}{{}^nC_{x-1} P^{x-1} q^{n-x+1}}$$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{\frac{n!}{x!(n-x)!} P^x q^{n-x}}{\frac{n!}{(x-1)!(n-x+1)!} P^{x-1} q^{n-x+1}}$$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{n!(x-1)!(n-x+1)!}{n!x!(n-x)!} P^x q^{n-x} P^{-x+1} q^{-n+x-1}$$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{(x-1)!(n-x+1)(n-x)!}{x(x-1)!(n-x)!} P^1 q^{-1}$$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{(n-x+1)}{x}\frac{P}{q}$$

$$P(X=x) = \frac{(n-x+1)}{x}\frac{P}{q} P(X=x-1)$$

Required result

Assignment

$P(X=x+1) = \frac{(n-x)}{x+1}\frac{P}{q} P(X=x)$ It is also recurrence formula do yourself

Question: Show that recurrance relation in binomial distribution

$$\mu_{r+1} = Pq\left[nr\mu_{r-1} + \frac{d}{dP}\mu_r \right]$$

Use it to find first four moments about mean and moments of skewness and kurtosis

Proof: Let we consider

$\mu_r = E(X - Mean)^r$

$\mu_r = \sum (X - Mean)^r P(x)$

As $X \to b(x,n,p)$ then its Probability function given as

$P(X=x) = {}^nC_x P^x q^{n-x}$ And $Mean = nP$

$\mu_r = \sum (X - Mean)^r \, {}^nC_x P^x (1-P)^{n-x}$

Differentiate with respect to P

$$\frac{d}{dP}\mu_r = \frac{d}{dP}\left[\sum (X - Mean)^r \, {}^nC_x P^x (1-P)^{n-x}\right]$$

$$\frac{d}{dP}\mu_r = \begin{bmatrix} \sum {}^nC_x P^x (1-P)^{n-x} \frac{d}{dP}(X-nP)^r + \sum (X-nP)^r \, {}^nC_x P^x (1-P)^{n-x} \frac{d}{dP} P^x \\ + \sum (X-nP)^r \, {}^nC_x P^x \frac{d}{dP}(1-P)^x \end{bmatrix}$$

$$\frac{d}{dP}\mu_r = \begin{bmatrix} \sum {}^nC_x P^x (1-P)^{n-x} r(X-nP)^{r-1} \frac{d}{dP}(X-nP) + \sum (X-nP)^r \, {}^nC_x P^x (1-P)^{n-x} xP^{x-1} \frac{d}{dP} P \\ + \sum (X-nP)^r \, {}^nC_x P^x (n-x)(1-P)^{n-x-1} \frac{d}{dP}(1-P) \end{bmatrix}$$

$$\frac{d}{dP}\mu_r = \begin{bmatrix} \sum {}^nC_x P^x (1-P)^{n-x} r(X-nP)^{r-1}(-n) + \sum (X-nP)^r \, {}^nC_x P^x (1-P)^{n-x} xP^{x-1}(1) \\ + \sum (X-nP)^r \, {}^nC_x P^x (n-x)(1-P)^{n-x-1} \frac{d}{dP}(-1) \end{bmatrix}$$

$$\frac{d}{dP}\mu_r = \begin{bmatrix} -rn\sum {}^nC_x P^x (1-P)^{n-x} (X-nP)^{r-1} + \sum (X-nP)^r \, {}^nC_x P^x (1-P)^{n-x} xP^{x-1} \\ - \sum (X-nP)^r \, {}^nC_x P^x (n-x)(1-P)^{n-x-1} \frac{d}{dP} \end{bmatrix}$$

$$\frac{d}{dP}\mu_r = \begin{bmatrix} -rn\sum (X-nP)^{r-1} \, {}^nC_x P^x (1-P)^{n-x} + \sum (X-nP)^r \, {}^nC_x P^x (1-P)^{n-x} xP^{x-1} \\ - \sum (X-nP)^r \, {}^nC_x P^x (n-x)(1-P)^{n-x-1} \frac{d}{dP} \end{bmatrix}$$

$$\frac{d}{dP}\mu_r = -rn\sum (X-nP)^{r-1} P(x) + \sum (X-nP)^r \, {}^nC_x P^x (q)^{n-x} P^x \left[xP^{-1} - (n-x)q^{-1}\right]$$

$$\frac{d}{dP}\mu_r = -rnE(X-nP)^{r-1} + \sum (X-nP)^r \, {}^nC_x P^x (q)^{n-x} P^x \left[\frac{x}{P} - \frac{(n-x)}{q}\right]$$

$$\frac{d}{dP}\mu_r = -rnE(X-nP)^{r-1} + \sum (X-nP)^r \, {}^nC_x P^x (q)^{n-x} P^x \left[\frac{xq - P(n-x)}{Pq}\right]$$

$$\frac{d}{dP}\mu_r = -rnE(X-nP)^{r-1} + \sum (X-nP)^r \, {}^nC_x P^x (q)^{n-x} P^x \left[\frac{x(1-P) - nP + xP}{Pq}\right]$$

$$\frac{d}{dP}\mu_r = -rnE(X-nP)^{r-1} + \sum (X-nP)^r \, {}^nC_x P^x (q)^{n-x} P^x \left[\frac{x - xP - nP + xP}{Pq}\right]$$

$$\frac{d}{dP}\mu_r = -rn\mu_{r-1} + \frac{1}{Pq}\sum (X-nP)^r \, {}^nC_x P^x (q)^{n-x} P^x (x-nP)$$

$$\frac{d}{dP}\mu_r = -rn\mu_{r-1} + \frac{1}{Pq}\sum (X-nP)^{r+1} \, {}^nC_x P^x (q)^{n-x} P^x$$

$$\frac{d}{dP}\mu_r = -rn\mu_{r-1} + \frac{1}{Pq}\sum (X-nP)^{r+1} P(x)$$

$$\frac{d}{dP}\mu_r = -rn\mu_{r-1} + \frac{1}{Pq}\mu_{r+1}$$

$$\frac{d}{dP}\mu_r + rn\mu_{r-1} = \frac{1}{Pq}\mu_{r+1}$$

$$\mu_{r+1} = Pq\left(rn\mu_{r-1} + \frac{d}{dP}\mu_r\right)$$

Hence proved

Now first four moments about mean

Put $r = 0$

$$\mu_{0+1} = Pq\left(0(n\mu_{0-1}) + \frac{d}{dP}\mu_0\right) \qquad \text{Where } \mu_0 = 1$$

$$\mu_{0+1} = Pq\left(0 + \frac{d}{dP}1\right) = Pq(0) = 0$$

$\mu_1 = 0$

Put $r = 1$

$\mu_{1+1} = Pq\left(1(n\mu_{1-1}) + \frac{d}{dP}\mu_1\right)$ Where $\mu_0 = 1$ and $\mu_1 = 0$

$\mu_2 = Pq\left(n\mu_0 + \frac{d}{dP}1\right) = Pq(n+0) = nPq = Var(x)$

Put $r = 2$

$\mu_{2+1} = Pq\left(2(n\mu_{2-1}) + \frac{d}{dP}\mu_2\right)$ Where $\mu_2 = nP(1-P)$ and $\mu_1 = 0$

$\mu_3 = Pq\left(2(n\mu_1) + \frac{d}{dP}nP(1-P)\right)$

$\mu_3 = Pq\left(0 + n\frac{d}{dP}P(1-P)\right) = nPq\left(P\frac{d}{dP}(1-P) + (1-P)\frac{d}{dP}P\right)$

$\mu_3 = nPq(P(-1) + (1-P)1) = nPq(-P + (1-P))$

$\mu_3 = nPq(1-2P) = nPq(1-P-P) = nPq(q-P)$

Put $r = 3$

$\mu_{3+1} = Pq\left(3(n\mu_{3-1}) + \frac{d}{dP}\mu_3\right)$ Where $\mu_2 = nPq$ and $\mu_3 = nP - 3nP^2 + 2nP^3$

$\mu_4 = Pq\left(3(n\mu_2) + \frac{d}{dP}\left(nP - 3nP^2 + 2nP^3\right)\right)$

$\mu_4 = Pq\left(3n(nPq) + n\frac{d}{dP}P - 3n\frac{d}{dP}P^2 + 2n\frac{d}{dP}P^3\right)$

$\mu_4 = Pq\left(3n(nPq) + n(1) - 3n(2P) + 2n(3P^2)\right)$

$\mu_4 = Pq\left[3n^2Pq + n - 6nP + 6nP^2\right]$

$\mu_4 = 3n^2P^2q^2 + nPq\left[1 - 6P + 6P^2\right]$

$\mu_4 = 3n^2P^2q^2 + nPq\left[1 - 6P(1-P)\right]$

$\mu_4 = 3n^2P^2q^2 + nPq\left[1 - 6Pq\right]$

Moment of skewness $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(nPq(q-P))^2}{(nPq)^3} = \frac{(nPq)^2(q-P)^2}{(nPq)^3} = \frac{(q-P)^2}{nPq}$

Moment of kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2P^2q^2 + nPq[1-6Pq]}{(nPq)^2} = \frac{3n^2P^2q^2}{(nPq)^2} + \frac{nPq[1-6Pq]}{(nPq)^2} = 3 + \frac{[1-6Pq]}{nPq}$$

Q: Discuss the statement that in a binomial distribution mean=5 and Standard deviation=5.

Ans: Given that $\mu = 5$ and $\sigma = 5$ $\sigma^2 = 25$

$Mean < Variance$

It is wrong because in binomial distribution $Mean > Variance$

Example.8.3: Let "X" have a binomial distribution with $n = 4$ and $P = \frac{1}{3}$. Find

$P(X = 1), P(X = \frac{3}{2}), P(X = 6), P(X = -1), P(X = 0.5)$ and $P(X \le 2)$

Solution: Let "X" be binomial random variable and n=4 then "X=0, 1, 2, 3, 4"

$X \to b(x, 4, \frac{1}{3})$ And $P(X = x) = {}^nC_x P^x q^{n-x}$

i) $P(X = 1) = {}^4C_1 (\frac{1}{3})^1 (\frac{2}{3})^3 = 0.40$

ii) $P(X = \frac{3}{2}) = {}^4C_x (\frac{1}{3}) x (\frac{2}{3})^{4-x} = 0$ Because it is based on discrete variable which is assume whole integers?

iii) $P(X = 6) = {}^4C_x (\frac{1}{3}) x (\frac{2}{3})^{4-x} = 0$ Because it's out of range

iv) $P(X=-1)={}^4C_x(\frac{1}{3})x(\frac{2}{3})^{4-x}=0$ In binomial random variable we assume positive values it is also impossible

v) $P(X=0.5)={}^4C_x(\frac{1}{3})x(\frac{2}{3})^{4-x}=0$ Because it is based on discrete variable which is assume whole integers?

vi) $P(X\le 2)=P(X=2)+P(X=1)+P(X=0)=0.89$

Example:8.6: Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or a six?

Solution Given that $N=729$ and $n=6$

$$P(event)=\frac{Feverable\ cases}{Total\ possible\ outcomes}$$

When a single die throw possible outcomes

$$S=\{1,2,3,4,5,6\}$$

$$Feverable\ outcomes\ "5\ or\ 6"\ occur=\{5,6\}$$

$$P(5\ or\ 6)=\frac{2}{6}=\frac{1}{3}$$

$$P(X\ge 3)=P(X=3)+P(X=4)+P(X=5)+P(X=6)=\frac{233}{729}$$

The required expected numbers are

$$NP(X\ge 3)=729\left(\frac{233}{729}\right)=233$$

Example: 8.7: A certain event is believed to follow the binomial distribution. In 1024 samples of 5, the result was observed one 405 times and twice 270 times. Find "p and q".

Solution: Given that according to condition

$$NP(X=1)=1024\left({}^5C_1\right)(p)^1(q)^{5-1}=10254\left(5pq^4\right)$$

$$10254\left(5pq^4\right)=405 \qquad \text{(i)}$$

$$NP(X=2)=1024\left({}^5C_2\right)(p)^2(q)^{5-2}=10254\left(10p^2q^3\right)$$

$$10254\left(10p^2q^3\right)=270 \qquad \text{(ii)}$$

Now dividing eq (i) dividing by (ii)

$$\frac{10254\left(5pq^4\right)}{10254\left(10p^2q^3\right)}=\frac{405}{270}$$

$$\frac{(q)}{(2p)}=\frac{45}{30}$$

$$30q=90p$$

$$q=3p$$

$$1-p=3p$$

$$1=3p+p=4p$$

$$p=\frac{1}{4}=0.25 \qquad q=1-p=0.75=\frac{3}{4}$$

Q.8.3 (a): A die is rolled five times and a 5 or 6 is considered a success. Find the probability of i) no success ii) at least 2 success iii) At least one but not more than 3 success.

b) Using the binomial distribution, find the probability of

i) 3 success in 8 trials when "p==0.4" ii) 2 failures in 6 trials when "p=0.6"

iii) 2 or fewer success in 9 trials when "p=0.4"

Solution: a) given that $n=5$ $\quad p=\frac{1}{3}$ $\quad q=\frac{2}{3}$

i) $p(x=0)=?$ ii) $p(x\ge 2)=?$ iii) $p(1\le x\le 3)=P(x=1)+P9x=2)+P(x=3)$

b)

i)) Given that $P(x=3)=?$ $n=8$ $p=0.4$ $q=0.6$

i)) Given that $P(x=6-2=4\ Success)=?$ $n=6$ $p=0.6$ $q=0.4$

i)) Given that $P(x\le 2)=?$ $n=9$ $p=0.4$ $q=0.6$

Q.8.5 (a) if the probability of getting caught copying someone else's exam is 0.2, find the probability of not getting caught in 3 attempts. Assume independence

Solution: P=0.2 q=0.8 n=3 P(X=0) =0.512

Q.8.6 (b): If on the average rain falls on twelve days in every thirty, find the probability that i) The first three days of a given week will be fine and the remaining wet

ii) Rain will fall on just three days of a given week.

Solution: $P(rain) = \frac{12}{30} = 0.4 \quad q = 0.6$

i) $P(First\ three\ days\ \ fine\ and\ remaining\ wet) = q^3 P^4 = (0.4)^3 (0.6)^4 = 0.0055$

i) $P(First\ three\ days\ \ fine\ and\ remaining\ wet) = {}^7C_3 q^3 P^4 = 35(0.4)^3 (0.6)^4 = 0.2903$

Q.8.7: An insurance salesman sells policies to 5 men all of identical age and in good health. According to the actuarial tables, the probability that a men of this particular age will be alive 30 years hence 2/3. Find the probability that in 30 years i) All men ii) At least 3 men iii) Only two men iv) At most one man will be alive

Solution: $n = 5 \qquad P = \frac{2}{3} = 0.67 \qquad q = \frac{1}{3} = 0.33$

Q.8.10(b): An irregular six-faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times 10000 sets of 10 throws would you expect it to give no even number?

Solution: Given that $n = 10 \qquad N = 10000$

Expectation of 5 even numbers is $= \mathrm{N}\left({}^{10}\mathrm{C}_5\right)P^5 q^5$

Expectation of 4 even numbers is $= \mathrm{N}\left({}^{10}\mathrm{C}_4\right)P^4 q^6$

Now according to conditions

$\mathrm{N}P(X = 5) = 2NP(X = 4)$

$\mathrm{N}\left({}^{10}\mathrm{C}_5\right)P^5 q^5 = 2\mathrm{N}\left({}^{10}\mathrm{C}_4\right)P^4 q^6$

$252P^5 q^5 = 2(210)P^4 q^6$

$0.6 = \frac{P^4 q^6}{P^5 q^5} = \frac{q}{P}$

$0.6 = \frac{q}{P}$

$0.6P = q$

$0.6P = 1 - P$

$0.6P + P = 1$

$1.6P = 1$

$P = \frac{1}{1.6} = 0.625 \qquad q = 1 - P = 0.375$

Expectation of no even numbers is $= NP(X = 0) = 10000\left({}^{10}\mathrm{C}_0\right)(0.625)^0 (0.375)^{10} = 0.55 = 1$

Q.8.17: Let "X" be a random variable having a binomial distribution with Parameters $n = 25$ and $p = 0.2$. Evaluate $P(X < \mu - 2\sigma)$

Solution: As we know that

$mean = nP = 25(0.20) = 5$ And $Var(X) = nPq = 25(0.20)(0.8) = 4$ $S.D(X) = 2$

$P(X < \mu - 2\sigma) = P(X < 5 - 2(2)) = P(X < 5 - 4) = P(X < 1)$

b) Given binomial probability function $P(X = x) = \left({}^{10}\mathrm{C}_x\right)\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$ Find the median and mode of the distribution.

Solution: Given that $n = 10 \qquad P = \frac{1}{2} \qquad q = \frac{1}{2}$

As we know that when $p = q = \frac{1}{2}$ then binomial distribution will be symmetrical

So we get $Mean = Median = Mode = np = 10 \times \frac{1}{2} = 5$

Q.8.23 (b): If the m.g.f of "X" is $M_0(t)=\left[\frac{1}{4}+\left(\frac{3}{4}\right)e^t\right]^{12}$, find E(X), Var(X) and $P(X \geq 10)$

Solution: As we know that m.g.f of binomial distribution

$$M_0(t)=\left[q+pe^t\right]^n \qquad \text{(A)}$$

$$M_0(t)=\left[\frac{1}{4}+\left(\frac{3}{4}\right)e^t\right]^{12} \qquad \text{(B)}$$

Comparing eq (A) and (B) then we get

$n=10 \quad P=\frac{3}{4} \qquad q=\frac{1}{4}$

Question: Show that if two symmetrical binomial distributions $(P=q=\frac{1}{2})$ of degree "n" (and of the same number of observations) are so superposed, the rth term of the one coincide with rth term of the other, the distribution formed by adding superposed term is symmetrical binomial of degree (n+1).

Solution:

For symmetrical distribution

$$P=q=\frac{1}{2}$$

Binomial probability frequency distribution of degree "n" is $N\left(\frac{1}{2}+\frac{1}{2}\right)^n$

Let

$$T_r = rth\ term = N\left({}^{n}C_{r-1}\right)(p)^{r-1}(q)^{n-r+1}$$

$$T_r = rth\ term = N\left({}^{n}C_{r-1}\right)\left(\frac{1}{2}\right)^{r-1}\left(\frac{1}{2}\right)^{n-r+1}$$

$$T_r = rth\ term = N\left({}^{n}C_{r-1}\right)\left(\frac{1}{2}\right)^{r-1+n-r+1} = N\left({}^{n}C_{r-1}\right)\left(\frac{1}{2}\right)^{n}$$

Now

$$T_{r+1} = (r+1)th\ term = N\left({}^{n}C_{r}\right)(p)^{r}(q)^{n-r}$$

$$T_{r+1} = (r+1)th\ term = N\left({}^{n}C_{r}\right)\left(\frac{1}{2}\right)^{r}\left(\frac{1}{2}\right)^{n-r}$$

$$T_{r+1} = (r+1)th\ term = N\left({}^{n}C_{r}\right)\left(\frac{1}{2}\right)^{r+n-r} = N\left({}^{n}C_{r}\right)\left(\frac{1}{2}\right)^{n}$$

Then according to condition

$$T_r + T_{r+1} = N\left({}^{n}C_{r-1}\right)\left(\frac{1}{2}\right)^{n} + N\left({}^{n}C_{r}\right)\left(\frac{1}{2}\right)^{n}$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^{n}\left[\left({}^{n}C_{r-1}\right)+\left({}^{n}C_{r}\right)\right]$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^{n}\left[\frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!}\right]$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^{n}\left[\frac{n!}{(r-1)!(n-r+1)(n-r)!}+\frac{n!}{r(r-1)!(n-r)!}\right]$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^{n}\frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{(n-r+1)}+\frac{1}{r}\right]$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^{n}\frac{n!}{(r-1)!(n-r)!}\left[\frac{r+n-r+1}{(n-r+1)r}\right]$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^{n}\frac{n!}{(r-1)!(n-r)!}\left[\frac{n+1}{(n-r+1)r}\right]$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^{n}\frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!}$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^n \frac{(n+1)!}{r!(n-r+1)!}$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^n {}^{n+1}c_r$$

$$T_r + T_{r+1} = N\left(\frac{1}{2}\right)^n {}^{n+1}c_r\left(\frac{2}{2}\right)$$

$$T_r + T_{r+1} = 2N\left(\frac{1}{2}\right)^n {}^{n+1}c_r\left(\frac{1}{2}\right)$$

$$T_r + T_{r+1} = 2N\, {}^{n+1}c_r\left(\frac{1}{2}\right)^{n+1}$$

$$T_r + T_{r+1} = 2N\, {}^{n+1}c_r\left(\frac{1}{2}\right)^{rr}\left(\frac{1}{2}\right)^{n+1-r}$$

$$T_r + T_{r+1} = 2N\left(\frac{1}{2}+\frac{1}{2}\right)^{n+1}$$

$T_r + T_{r+1}$ =(r+1)th term of binomial distribution $2N\left(\frac{1}{2}+\frac{1}{2}\right)^{n+1}$ symmetrical frequency binomial probability distribution of degree (n+1).

Question: Find cummulent generating function of binomial distribution.

Proof:

Let by definition

$K(t) = \log M_x(t)$

As we know that

$M_x(t) = (q + Pe^t)^n$

$K(t) = \log(q + Pe^t)^n$

$K(t) = n\log(q + Pe^t)$

$K(t) = n\log(1 - p + Pe^t)$

We know that

$$e^t = \frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + ...$$

$$K(t) = n\log\left[1 - p + p(\frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + ...)\right]$$

$$K(t) = n\log\left[1 - p + P(1 + \frac{t^1}{1!} + \frac{t^2}{2!} + ...)\right]$$

$$K(t) = n\log\left[1 - p + p + P(\frac{t^1}{1!} + \frac{t^2}{2!} + ...)\right]$$

$$K(t) = n\log\left[1 + P(\frac{t^1}{1!} + \frac{t^2}{2!} + ...)\right]$$

Since

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + ...$$

$$K(t) = n\left[P(\frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ..) - \frac{p^2}{2}(\frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + ..)^2 + \frac{p^3}{3}(\frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + ...)^3 - \frac{p^4}{4}(\frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + ...)^4\right]$$

$$K(t) = n\left[P(\frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ..) - \frac{p^2}{2}(\frac{t^2}{1!} + \frac{t^4}{4!} + \frac{2t^3}{2!} + \frac{2t^4}{3!}..) + \frac{p^3}{3}(\frac{t^3}{1!} + \frac{3t^4}{2!} + ...) - \frac{p^4}{4}(\frac{t^4}{1!} + ...)\right]$$

$$K(t) = n\left[P\frac{t^1}{1!} + (p - p^2)\frac{t^2}{2!} + (p - 3p^2 + 2p^3)\frac{t^3}{3!} + (p - 7p^2 + 12p^3 - 6p^4)\frac{t^4}{4!}\right] \qquad (A)$$

As we know that general expression of cummulent

$$K(t) = K_1\frac{t^1}{1!} + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + K_4\frac{t^4}{4!} + ... \qquad (B)$$

Comparing (A) and (B), then we get

$K_1 = np = \mu_1' = Mean$

$K_2 = n(p - p^2) = np(1-p) = npq = \mu_2 = Var(x)$

$K_3 = n(p - 3p^2 + 2p^3) = np(1 - p - 2p(1-p)) = np(q - 2pq) = npq(1-2p) = npq(q-p) = \mu_3$

$K_4 = n(p - 7p^2 + 12p^3 - 6p^4) = np(1 - 7p + 12p^2 - 6p^3) = np(1 - p - 6p + 12p^2 - 6p^3)$

$K_4 = np(q - 6p(1 - 2p + p^2)) = np(q - 6p(1-p)^2) = np(q - 6pq^2) = npq(1 - 6pq)$

$\mu_4 = k_4 + 3k_2^2 = npq(1-6pq) + 3(npq)^2$

Theorem: If $n \to \infty$ then binomial distribution approaches to normal distribution.

Proof: We know in binomial distribution

$X \to b(x, n, p)$

$Mean = np = \mu'$

$Var(x) = np(1-p) = npq = \mu_2 = \sigma^2$

$S.D(x) = \sqrt{npq} = \sigma$

Now we consider standardized normal variate

$$Z = \frac{X - E(X)}{S.D(X)} = \frac{X - \mu}{\sigma}$$

$$Z = \frac{X - np}{\sqrt{npq}}$$

Let by definition of moment generating function

$M_z(t) = E(e^{tZ})$

$M_z(t) = E(e^{t(\frac{X-np}{\sqrt{npq}})})$

$M_z(t) = E(e^{t\frac{X}{\sqrt{npq}} - t(\frac{np}{\sqrt{npq}})})$

$M_z(t) = E(e^{\frac{tX}{\sqrt{npq}}} e^{-\frac{np}{\sqrt{npq}}t})$

$M_z(t) = e^{-\frac{np}{\sqrt{npq}}t} E(e^{\frac{tX}{\sqrt{npq}}})$ (A)

In binomial distribution "m.g.f" is

$M_x(t) = (q + Pe^t)^n$

Replacing "t" by $\frac{t}{\sqrt{npq}}$

$E(e^{\frac{tx}{\sqrt{npq}}}) = (q + Pe^{\frac{t}{\sqrt{npq}}})^n$ Put in (A)

$M_z(t) = e^{-\frac{np}{\sqrt{npq}}t} (q + Pe^{\frac{t}{\sqrt{npq}}})^n$

Taking log on both sides

$$\ln M_z(t) = \ln\left[e^{-\frac{np}{\sqrt{npq}}t} (q + Pe^{\frac{t}{\sqrt{npq}}})^n\right] = \frac{-npt}{\sqrt{npq}} \ln e + n\ln(q + Pe^{\frac{t}{\sqrt{npq}}})$$

$$\ln M_z(t) = \frac{-npt}{\sqrt{npq}} + n\ln(q + Pe^{\frac{t}{\sqrt{npq}}})$$

$$e^t = \frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + ...$$

$$\ln M_z(t) = \frac{-npt}{\sqrt{npq}} + n\ln(1 - p + p(\frac{(\frac{t}{\sqrt{npq}})^0}{0!} + \frac{(\frac{t}{\sqrt{npq}})^1}{1!} + \frac{(\frac{t}{\sqrt{npq}})^2}{2!} + ...))$$

$$\ln M_z(t) = \frac{-npt}{\sqrt{npq}} + n\ln(1 - p + p(1 + \frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + ...))$$

$$\ln M_z(t) = \frac{-npt}{\sqrt{npq}} + n\ln(1 - p + p + p(\frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + ...))$$

$$\ln M_z(t) = \frac{-npt}{\sqrt{npq}} + n\ln(1 + p(\frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + ...))$$

Since

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + ...$$

$$\ln M_z(t) = \frac{-npt}{\sqrt{npq}} + n\left[p(\frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + ...) - \frac{p^2}{2}(\frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + ...)^2 + \frac{p^3}{3}(...)^3 - ... \right]$$

$$\ln M_z(t) = \frac{-npt}{\sqrt{npq}} + \frac{npt}{\sqrt{npq}} + \frac{npt^2}{2npq} - \frac{np^2t^2}{2npq} + ...involving(\frac{1}{n})\ and\ higher\ term$$

$$\ln M_z(t) = \frac{t^2}{2q} - \frac{pt^2}{2q} + ...involving(\frac{1}{n})\ and\ higher\ term$$

$$\ln M_z(t) = \frac{t^2}{2}(\frac{1}{q} - \frac{p}{q}) + ...involving(\frac{1}{n})\ and\ higher\ term$$

$$\ln M_z(t) = \frac{t^2}{2}(\frac{1-p}{q}) + ...involving(\frac{1}{n})\ and\ higher\ term$$

$$\ln M_z(t) = \frac{t^2}{2}(\frac{q}{q}) + ...involving(\frac{1}{n})\ and\ higher\ term$$

$$\ln M_z(t) = \frac{t^2}{2} + ...involving(\frac{1}{n})\ and\ higher\ term$$

Applying limit $n \to \infty$ then all the term equal to zero $involving(\frac{1}{n})\ and\ higher\ term$

$$\ln M_z(t) = \frac{t^2}{2} + n \to \infty.....involving(\frac{1}{n})\ and\ higher\ term$$

$$\ln M_z(t) = \frac{t^2}{2} + 0 + 0 + 0 + ...$$

$$\ln M_z(t) = \frac{t^2}{2}$$

Taking antilog

$$M_z(t) = e^{\frac{t^2}{2}}$$

It is m.g.f of standard normal distribution. Hence its proved when $n \to \infty$ then binomial distribution approaches to normal distribution.

$b(X, n, p) \to N(np, npq) \quad When \quad n \to \infty$

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