The function that assigns probability for a discrete random variable is called a probability mass function, because it shows how much probability (or mass), is given to each value of the random variables. Mass is thought of as weight in this case the total mass (or weight) for a probability distribution equals one. A continuous random variable doesn't actually assign probability or mass, it assigns density, which means it tells you how dense the probability is around x for any value of X. You find probabilities for intervals of X, not for particular values of X, when X is continuous. Continuous random variables have no probability at any single point because there is no area over a single point.

Mathematical Expectation OR Expected Value of a Discrete Random Variable

Historical Note

In 1657, Christiaan Huygens published the first book on probability theory. In that text, he introduced the idea of expected value.

A very important concept in probability is the idea of **expected values**. The expected value is the long-term mean or average value of a random variable. If the random variable is observed over a long period of time, we would expect that the expected value would be close to the average value of the observations generated by the random process. The larger the number of observations, the closer the expected value will be to the average value of the observations. Thus we define expected value as "*The theoretical average of a random variable is called expected value.*"

Let "X" be a discrete random variable which can take values as $x_1, x_2, \ldots, x_n$ and the associated probabilities be $f(x_1), f(x_2), \ldots, f(x_n)$ respectively; then the expectation of "X" (denoted by $E(X)$, μ_x or μ) is defined as:

$$\mu_x = E(X) = \sum_{\text{all x}} xf(x)$$

OR

$$\mu_x = E(X) = \sum_{\text{all x}} xP(x)$$

The expected value of X doesn't have to be equal to a possible value of X because it represents a long-term average value. It does, however, have to lie between the smallest and largest possible values of X, which is something to check after you have calculated E(X). Also, note that E(X) is not a probability, so it falls between zero and one only if all the possible values of X are between zero and one.

A Practical Example!!!

If we toss three coins let "X" represents the No. of heads so that x = 0, 1, 2, 3 then the probability distribution of the number of heads:

$$S=\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

Head *Tail*

X	0	1	2	3	$Total$
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$	1

Thus $E(X)=\sum xf(x)=(0\bullet 1/8)+(1\bullet 3/8)+(2\bullet 3/8)+(3\bullet 1/8)=1.5$

If toss three coins 50 times and the **number of heads** are recorded as given in the following table:

0	2	3	1	1	2	2	2	0	3
2	1	1	1	2	3	1	1	1	2
0	1	1	3	0	1	2	1	3	2
1	1	1	2	1	1	0	1	1	2
2	1	2	1	2	1	3	2	1	3

Now the Mean of this data is:

$$Mean=\frac{0+2+1+1+\cdots\cdots+1+3}{50}=1.48$$

Now if toss three coins 100 times and **the number of heads** are recorded as given in the following table:

0	3	0	1	0	2	2	2	0	3
2	2	1	2	2	3	2	1	1	2
0	1	1	3	0	2	2	1	3	2
1	2	0	2	1	2	0	1	1	2
1	2	2	3	1	1	1	2	1	3
2	1	1	2	0	2	3	1	1	2
1	2	2	1	3	1	1	2	1	2
1	2	1	2	1	0	2	3	2	1
2	1	2	1	2	1	2	1	2	1
1	0	1	1	2	1	3	2	1	2

Now the Mean of this data is:

$$Mean=\frac{0+3+0+1+\cdots\cdots+1+2}{100}=1.49\approx 1.5$$

It is clear that this mean is close to 1.5

Hence, ***"as the number of repetitions of the experiment increases, we expect that the actual mean get closer to the expected (theoretical) mean"***

Variance and Standard Deviation of a Discrete Random Variable

Let "X" be a discrete random variable which can take values as $x_1, x_2, \ldots, x_n$ and the associated probabilities be $f(x_1), f(x_2), \ldots, f(x_n)$ respectively; then the variance and S.D of "X" are defined as:

$$Var(X) = \sigma_x^2 = E(X^2) - \left(E(X)\right)^2$$

$$S.D(X) = \sigma_x = \sqrt{E(X^2) - \left(E(X)\right)^2}$$

Here $E(X^2) = \sum_{\text{all x}} x^2 f(x)$

Sigma

EXAMPLE 7.11

A random variable X has a probability distribution:

x	0	1	2
f(x)	1/4	2/4	1/4

Find Expected value, Variance and S.D of the random variable X.

Solution

x	$f(x)$	$xf(x)$	$x^2f(x)$
0	1/4	0	0
1	2/4	2/4	2/4
2	1/4	2/4	4/4
--	1	4/4	6/4

$$E(X) = \sum xf(x) = 4/4 = 1.0$$

And $$E(X^2) = \sum x^2 f(x) = 6/4 = 1.5$$

Therefore $$Var(X) = E(X^2) - \left(E(X)\right)^2$$
$$= 1.5 - (1.0)^2$$
$$= 1.5 - 1 = 0.5$$

$$S.D(X) = \sqrt{E(X^2) - \left(E(X)\right)^2}$$
$$= \sqrt{1.5 - (1.0)^2}$$
$$= \sqrt{1.5 - 1} = \sqrt{0.5} = 0.7$$

TIP

To compute E(X), round-off it to one more decimal place than the values of random variable x. This round-off rule is also used for the variance and S.D of a probability distribution.

EXAMPLE 7.12

Find "K" for the probability distribution given below:

x	0	1	2	3
f(x)	1/8	K	3/8	1/8

Also find the value of Mean and Variance of the random variable X.

Solution

To find the value of "K" we use:

$$\sum_{x=0}^{3} f(x) = 1$$
$$\Rightarrow 1/8 + K + 3/8 + 1/8 = 1$$
$$\Rightarrow K + 5/8 = 1$$
$$\Rightarrow K = 1 - 5/8$$
$$\Rightarrow K = 3/8$$

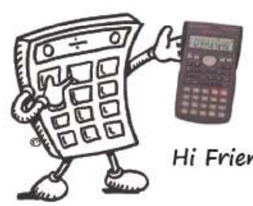
x	$f(x)$	$xf(x)$	$x^2 f(x)$
0	1/8	0	0
1	K = 3/8	3/8	3/8
2	3/8	6/8	12/8
3	1/8	3/8	9/8
--	1	12/8	24/8

$$Mean = E(X) = \sum xf(x) = 12/8 = 1.5$$

And $E(X^2) = \sum x^2 f(x) = 24/8 = 3$

Therefore $Var(X) = E(X^2) - (E(X))^2$

$$= 3 - (1.5)^2$$
$$= 3 - 2.25$$
$$= 0.75$$



Test Yourself

1) Find E(X), Var(X) and S.D(X) from the following Probability Distribution:

x	0	1	2	3
f(x)	1/4	1/6	2/6	1/4

2) Find the value of "K", E(X), Var(X) and S.D(X) from the following Probability Distribution:

x	1	2	3	4
f(x)	2/12	K	4/12	3/12

Properties of Expectation

- $E(a) = a$
- $E(aX) = aE(X)$
- $E(X \pm a) = E(X) \pm a$
- $E(aX \pm b) = aE(X) \pm b$
- $E(X \pm Y) = E(X) \pm E(Y)$
- $E(X \bullet Y) = E(X) \bullet E(Y)$ *(If X and Y are independent)*

EXAMPLE 7.13

Given the following Probability Distribution:

x	1	2	3	4
f(x)	1/8	1/4	1/2	1/8

Find (1) $E(X)$ (2) $E(X+10)$ (3) $E(2-X)$
(4) $E(3X)$ (5) $E(4X+100)$ (6) $E(20-5X)$

Solution

1) $E(X) = \sum xf(x) = 21/8 = 2.6$

2) $E(X+10) = E(X) + 10 = 2.6 + 10 = 12.6$

3) $E(2-X) = 2 - E(X) = 2 - 2.6 = -0.6$

4) $E(3X) = 3E(X) = 3(2.6) = 7.8$

5) $E(4X+100) = 4E(X) + 100 = 4(2.6) + 100 = 110.4$

6) $E(20-5X) = 20 - 5E(X) = 20 - 5(2.6) = 7.0$

x	*f(x)*	*xf(x)*
1	1/8	1/8
2	1/4	2/4
3	1/2	3/2
4	1/8	4/8
--	1	21/8

Test Yourself

Given the following Probability Distribution:

x	4	5	6	7
f(x)	1/8	1/4	1/2	1/8

Find (1) $E(X)$ (2) $E(X+9)$ (3) $E(12-X)$
(4) $E(8X)$ (5) $E(3X+70)$ (6) $E(17-3X)$

Properties of Variance and Standard Deviation

Variance

- $Var(c)=0$
- $Var(X \pm c)=Var(X)$
- $Var(c\,X)=c^2 Var(X)$
- $Var\left(\dfrac{X}{c}\right)=\left(\dfrac{1}{c^2}\right)Var(X)$
- *If X and Y are independent then* $Var(X \pm Y)=Var(X)+Var(Y)$

Standard Deviation

- $S.D(c)=0$
- $S.D(X \pm c)=S.D(X)$
- $S.D(c\,X)=|c|\,S.D(X)$
- $S.D\left(\dfrac{X}{c}\right)=\left|\dfrac{1}{c}\right|S.D(X)$
- *If X and Y are independent then* $S.D(X \pm Y)=S.D(X)+S.D(Y)$

EXAMPLE 7.14

Given the following Probability Distribution:

x	-50	-100	1500
f(x)	1/5	3/10	1/2

Find
(1) $E(X)$ (2) $E(X^2)$ (3) $Var(X)$
(4) $S.D(X)$ (5) $Var(X+3)$ (6) $S.D(2-3X)$
(7) $S.D(3X)$ (8) $Var\left(\dfrac{X}{5}\right)$ (9) $S.D\left(\dfrac{X}{5}\right)$

Solution

x	$f(x)$	$xf(x)$	$x^2f(x)$
-50	1/5	-10	500
-100	3/10	-30	3000
1500	1/2	75	11250
--	1	35	14750

1) $E(X)=\sum xf(x)=35.0$

2) $E(X^2)=\sum x^2 f(x)=14750$

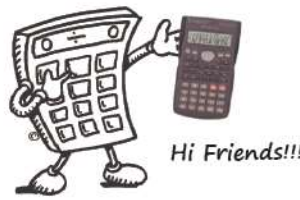
3) $Var(X)=E(X^2)-\left(E(X)\right)^2$
$=14750-(35)^2=13525.0$

4) $S.D(X)=\sqrt{E(X^2)-\left(E(X)\right)^2}$
$=\sqrt{14750-(35)^2}=\sqrt{13525}=116.3$

5) $Var(X+3) = Var(X) = 13525.0$

6) $S.D(2-3X) = 3S.D(X) = 3(116.3) = 348.9$

7) $S.D(3X) = 3S.D(X) = 3(116.3) = 348.9$



8) $Var\left(\dfrac{X}{5}\right) = \left(\dfrac{1}{25}\right)Var(X) = \left(\dfrac{1}{25}\right)(13525) = 541.0$

9) $S.D\left(\dfrac{X}{5}\right) = \left(\dfrac{1}{5}\right)S.D(X) = \left(\dfrac{1}{5}\right)(13525) = 2705.0$

Test Yourself

Given the following Probability Distribution:

x	-40	-900	1400
f(x)	1/5	3/10	1/2

Find (1) $E(X)$ (2) $E(X^2)$ (3) $Var(X)$ (4) $S.D(X)$ (5) $Var(X+4)$

(6) $S.D(5-2X)$ (7) $S.D(6X)$ (8) $Var\left(\dfrac{2X}{7}\right)$ (9) $S.D\left(\dfrac{3X}{8}\right)$

Important!!!

While reading probability problems, pay special attention to key phrases that translate into mathematical symbols. The following table lists various phrases and their corresponding mathematical equivalents:

Math Symbol	Phrases
$>$	"greater than" or "more than" or "exceed" or "better than" or "taller than" or "above"
$<$	"less than" or "smaller than" or "below" or "under" or "fewer than"
$\geq$	"at least" or "greater than or equal to" or "no less than"
$\leq$	"at most" or "less than or equal to" or "no more than"
$=$	"exactly" or "equal" or "is"

Amazing Histogram!!!

If two dice are rolled and "X" represents the sum of dots then:

$$S = \begin{Bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{Bmatrix}$$

x	2	3	4	5	6	7	8	9	10	11	12	Total
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	1