CHAPTER 07

Random Variables

Chapter Contents

You should read this chapter if you need to learn about:

Suppose your teacher asked you to write down your names on a slip distributed by him. You returned your slip, the teacher fold the slips in a uniform pattern and mixed them well. Then he asked one of the students to draw ten slips one by one or together. The teacher open the drawn slips one by one, read the names of the selected students and asked the following questions:

- What is your age?
- How many brothers and sisters are you?
- How many living rooms are available to your family?
- What is your height?

In this example, the selection of ten students by the method explained above is a **random experiment** and the procedure of selection is **random process**. The students selected in this way are the **outcomes** of the experiment and the questions asked from selected students are the characteristics in which we are interested. Since each characteristic can assume different values from outcome to outcome of the random experiment. So these characteristics may be considered not only as **variable** but are known as **random variables** or **chance variable** or **stochastic variables.**

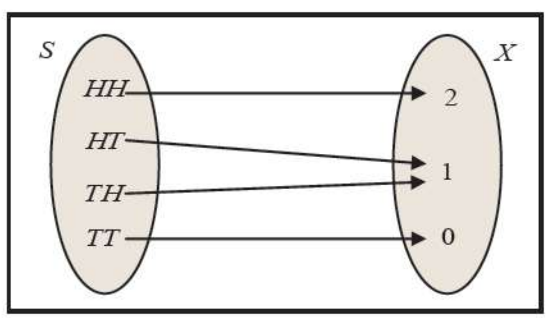
Random Variable

"A variable whose values are determined by the outcomes of a random experiment is called a random variable"

If we toss two coins then the sample space must be:

$$S=\{HH,HT,TH,TT\}$$

Let "X" is a variable denoting the "number of heads" then "X" have the values 0, 1, 2; since these values are determined from the results (outcomes) of the random experiment; therefore "X" is called as a random variable.



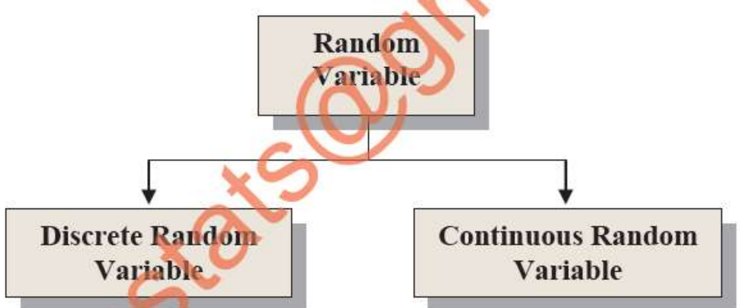
The following are some examples of random variables:

- The number of deaths in an accident.
- The number of heads in tossing two coins
- Temperature of a place
- The life time of a TV tube
- The number of daily admissions in a hospital
- The amount of rain falls at a certain place, etc.

Random variables are usually denoted by the last letters of alphabets e.g. X, Y or Z.

Types of Random Variable

There are two types of random variable:



Discrete Random Variable

"A random variable is called discrete random variable if it has counting phenomena and there can be certain jump or gap between two possible values of the random variable. Further it is free from the unit of measurement".

- The number of heads in tossing two coins
- No. of deaths in an accident
- No. of apples in a basket.
- No. of passengers carried by PIA in last ten years
- The number of daily admissions in a hospital, etc.

Continuous Random Variable

"A random variable is called continuous random variable if it has measuring phenomena and there can be infinite number of values between two possible values of the variable. Further it has the unit of measurement"

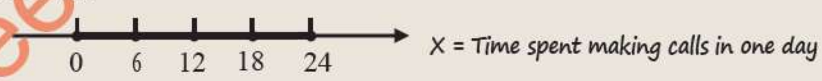
EXAMPLE

- Students heights, ages, weights
- Temperature of a place
- The amount of milk given by a cow
- The life time of a TV tube
- The amount of rain falls at a certain place, etc.

VS.

A discrete random variable has either a finite or countable infinite number of values that are usually integers or whole numbers. The values of a discrete random variable can be plotted on a number line with space between each point.

On the hand a continuous random variable has infinitely many values that are real numbers. The values of a continuous random variable can be plotted on a line in an uninterrupted fashion.



Discrete probability distribution

"A table listing all possible values that a discrete random variable can take on together with the associated probabilities is called discrete probability distribution"

Let "X" be a discrete random variable which can take values as $x_1, x_2, \ldots, x_n$ and the associated probabilities be $f(x_1), f(x_2), \ldots, f(x_n)$ respectively; then the discrete probability distribution is given as:

x	x_1	x_2	……………	x_n
$f(x)$ or $P(x)$	$f(x_1)$	$f(x_2)$	……………	$f(x_n)$

The function $f(x)$ *or* $P(x)$ that is used to assign the probabilities to different values of the random variable "X" is called probability function or probability **mass** function (p.m.f).

The discrete probability **mass** function may be defined as:

$$f(x) = P(X = x) = \begin{cases} \boxed{Function\ of\ "x"} & ; x = x_1, x_2, \ldots, x_n \\ 0 & ; otherwise \end{cases}$$

A p.m.f has the following two properties:

(i) $f(x) \geq 0$ *for all "x"*

(ii) $\sum_{all\ x} f(x) = 1$

TECHNICAL STUFF

Some writers do not make any distinction between the terms probability function and probability distribution but they use it interchangeably.

Graph of Discrete probability distribution

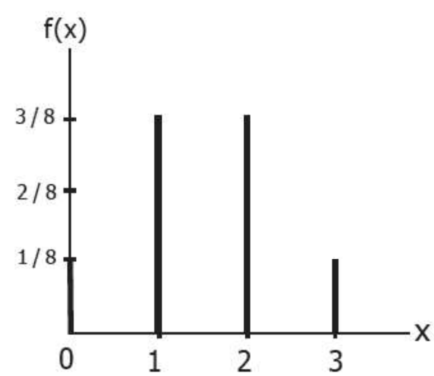
The discrete probability distribution is usually displayed by vertical lines graph or probability histogram. In both type of graphs we take the values of X on the X-axis and probabilities on the Y-axis as shown in the following figure:

x	0	1	2	3	Total
$f(x)$	1/8	3/8	3/8	1/8	1
Class Boundaries	-0.5 – 0.5	0.5 - 1.5	1.5 - 2.5	2.5 - 3.5	--

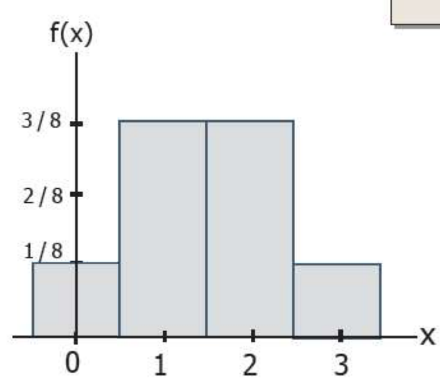
TECHNICAL STUFF

To make probability Histogram we first find class boundaries. These class boundaries are called factitious class boundaries because the discrete random variable cannot assume such values.

Vertical Lines Graph



Probability Histogram



EXAMPLE 7.01

Find the probability distribution of the number of heads when two coins are tossed?

Solution Since two coins are tossed therefore:

$$S = \{HH, HT, TH, TT\}$$

Let "X" is a random variable denoting the number of heads then x = 0, 1, 2

Now the probabilities are:

If X = 0 then $f(0) = \frac{1}{4}$, If X = 1 then $f(1) = \frac{2}{4}$, If X = 2 then $f(2) = \frac{1}{4}$

Hence the probability distribution of the number of heads (X) becomes:

x	*0*	*1*	*2*	*Total*
f(x)	*1/4*	*2/4*	*1/4*	*1*

EXAMPLE 7.02

Find the probability distribution of the number of heads when three coins are tossed?

Solution Since three coins are tossed therefore:

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

Let "X" is a random variable denoting the number of heads then x = 0, 1, 2, 3

Now the probabilities are:

If X = 0 then $f(0) = \frac{1}{8}$, If X = 1 then $f(1) = \frac{3}{8}$

If X = 2 then $f(2) = \frac{3}{8}$, If X = 3 then $f(3) = \frac{1}{8}$

Hence the probability distribution of the number of heads (X) becomes:

x	*0*	*1*	*2*	*3*	*Total*
f(x)	*1/8*	*3/8*	*3/8*	*1/8*	*1*

EXAMPLE 7.03

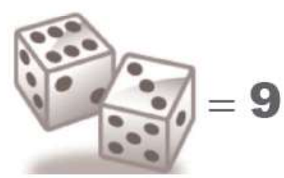
Find the probability distribution of the sum of dots when two dice are rolled?

Solution Since two dice are rolled then:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

For the sum of dots we may write the sample space as:

$$S = \begin{Bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{Bmatrix}$$



Let "X" is a random variable denoting the sum of dots then x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

Now the probabilities are:

If $X = 2$ then $f(2) = \frac{1}{36}$, If $X = 3$ then $f(3) = \frac{2}{36}$, If $X = 4$ then $f(4) = \frac{3}{36}$

If $X = 5$ then $f(5) = \frac{4}{36}$, If $X = 6$ then $f(6) = \frac{5}{36}$, If $X = 7$ then $f(7) = \frac{6}{36}$

If $X = 8$ then $f(8) = \frac{5}{36}$, If $X = 9$ then $f(9) = \frac{4}{36}$, If $X = 10$ then $f(10) = \frac{3}{36}$

If $X = 11$ then $f(11) = \frac{2}{36}$, If $X = 12$ then $f(12) = \frac{1}{36}$

Hence the probability distribution of the sum of dots (X) becomes:

x	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*Total*
f(x)	*1/36*	*2/36*	*3/36*	*4/36*	*5/36*	*6/36*	*5/36*	*4/36*	*3/36*	*2/36*	*1/36*	*1*

EXAMPLE 7.04

A basket contains 6 balls 2 white ball and 4 black balls. If three balls are selected at random then find the probability distribution for the number of black balls.

Solution Since "3" balls are selected out of "9"

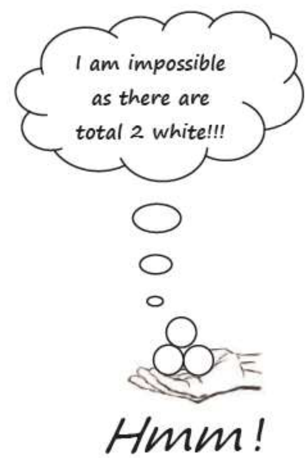
Black	White	Total
4	2	6

Therefore $n(S)=\binom{6}{3}=20$

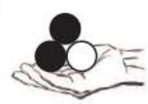
Let "X" is a random variable denoting the number of black balls then x = 1, 2, 3 (because 3 balls are selected)

Now the probabilities are:

If X = 1 then $f(1)=\dfrac{\binom{4}{1}\binom{2}{2}}{20}=\dfrac{4}{20}$,



If X = 2 then $f(2)=\dfrac{\binom{4}{2}\binom{2}{1}}{20}=\dfrac{12}{20}$,



If X = 3 then $f(3)=\dfrac{\binom{4}{3}\binom{2}{0}}{20}=\dfrac{4}{20}$

Hence the probability distribution of the number of black balls (X) becomes:

x	*0*	*1*	*2*	*Total*
f(x)	*4/20*	*12/20*	*4/20*	*1*

Test Yourself

1) Find the probability distribution of the number of tails when two coins are tossed?
2) Find the probability distribution of the number of tails when three coins are tossed?
3) Find the probability distribution of the difference of dots when two dice are rolled?
4) A basket contains 6 balls 2 white ball and 4 black balls. If three balls are selected at random then find the probability distribution for the number of white balls.

How to find the probabilities using a discrete probability distribution or a discrete probability density function

Let "X" be a discrete random variable then the discrete probability distribution is given as:

x	x_1	x_2	………………	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	………………	$f(x_n)$

Similarly the discrete probability mass function *f(x)* is given as:

$$f(x)=P(X=x)=\begin{cases}\boxed{\text{Function of "}x\text{"}} & ;x=x_1,x_2,\ldots,x_n\\ 0 & ;\text{otherwise}\end{cases}$$

Now to find the probabilities we have:

- $P(X=x_1)=f(x_1)$
- $P(x_1\le X\le x_3)=f(x_1)+f(x_2)+f(x_3)$
- $P(x_1< X\le x_3)=f(x_2)+f(x_3)$
- $P(X\le x_2)=f(x_1)+f(x_2)$
- $P(X<x_2)=f(x_1)$

EXAMPLE 7.05

Find the probability distribution of the sum of dots when two dice are rolled?
Also find the probability that:

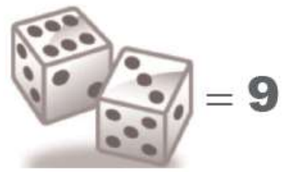
(i) The sum of dots is exactly 4
(ii) The sum of dots is greater than 10
(iii) The sum of dots is less than 6
(iv) The sum of dots is at least 9

Solution Since two dice are rolled then:

$$\Rightarrow S=\begin{Bmatrix}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{Bmatrix}$$

For the sum of dots we may write the sample space as:

$$S=\begin{Bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{Bmatrix}$$



Let "X" is a random variable denoting the sum of dots then x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

Now the probabilities are:

If X = 2 then $f(2)=\frac{1}{36}$, If X = 3 then $f(3)=\frac{2}{36}$, If X = 4 then $f(4)=\frac{3}{36}$

If X = 5 then $f(5)=\frac{4}{36}$, If X = 6 then $f(6)=\frac{5}{36}$, If X = 7 then $f(7)=\frac{6}{36}$

If X = 8 then $f(8)=\frac{5}{36}$, If X = 9 then $f(9)=\frac{4}{36}$, If X = 10 then $f(10)=\frac{3}{36}$

If X = 11 then $f(11)=\frac{2}{36}$, If X = 12 then $f(12)=\frac{1}{36}$

Hence the probability distribution of the sum of dots (X) becomes:

x	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*Total*
f(x)	*1/36*	*2/36*	*3/36*	*4/36*	*5/36*	*6/36*	*5/36*	*4/36*	*3/36*	*2/36*	*1/36*	*1*

(i) The sum of dots is exactly 4

$$\begin{aligned} P(\text{The sum of dots is exactly 4}) &= P(X=4) \\ &= f(4) \\ &= 3/36 \end{aligned}$$

(ii) The sum of dots is less than 6

$$\begin{aligned} P(\text{The sum of dots is less than 6}) &= P(X<6) \\ &= P(X=2 \text{ or } X=3 \text{ or } X=4 \text{ or } X=5) \\ &= P(X=2)+P(X=3)+P(X=4)+P(X=5) \\ &= f(2)+f(3)+f(4)+f(5) \\ &= 1/36+2/36+3/36+4/36 \\ &= 10/36 \end{aligned}$$

(iii) The sum of dots is greater than 10

$$
\begin{aligned}
P(\text{The sum of dots is greater than 10}) &= P(X > 10) \\
&= P(X = 11 \text{ or } X = 12) \\
&= P(X = 11) + P(X = 12) \\
&= f(11) + f(12) \\
&= 2/36 + 1/36 \\
&= 3/36
\end{aligned}
$$

(iv) The sum of dots is at least 9

$$
\begin{aligned}
P(\text{The sum of dots is at least 9}) &= P(X \geq 9) \\
&= P(X = 9 \text{ or } X = 10 \text{ or } X = 11 \text{ or } X = 12) \\
&= P(X = 9) + P(X = 10) + P(X = 11) + P(X = 12) \\
&= f(9) + f(10) + f(11) + f(12) \\
&= 4/36 + 3/36 + 2/36 + 1/36 \\
&= 10/36
\end{aligned}
$$

EXAMPLE 7.06

Given that: $f(x) = \frac{x^4}{98}$ x = 0, 1, 2, 3

Find (i) $P(1 \leq X \leq 3)$ (ii) $P(X < 2)$ (iii) $P(X = 3)$

Solution

(i) $P(1 \leq X \leq 3)$

$$
\begin{aligned}
P(1 \leq X \leq 3) &= P(X = 1 \text{ or } X = 2 \text{ or } X = 3) \\
&= P(X = 1) + P(X = 2) + P(X = 3) \\
&= f(1) + f(2) + f(3) \\
&= 1/98 + 16/98 + 81/98 = 1
\end{aligned}
$$

Given that $f(x) = \frac{x^4}{98}$

$$f(1) = \frac{1^4}{98} = \frac{1}{98}$$

$$f(2) = \frac{2^4}{98} = \frac{16}{98}$$

$$f(3) = \frac{3^4}{98} = \frac{81}{98}$$

(ii) $P(X < 2)$

$$
\begin{aligned}
P(X < 2) &= P(X = 0 \text{ or } X = 1) \\
&= P(X = 0) + P(X = 1) \\
&= f(0) + f(1) \\
&= 0/98 + 1/98 = 1/98
\end{aligned}
$$

(iii) $P(X = 3)$

$$P(X = 3) = f(3) = 81/98$$

EXAMPLE 7.07

What value of "k" makes the following function a density function?

$$f(x) = kx^4, \qquad x = 0, 1, 2, 3$$

Solution To find the value of "k" we use:

$$\sum_{x=0}^{3} f(x) = 1$$
$$\Rightarrow \sum_{x=0}^{3} kx^4 = 1$$
$$\Rightarrow k\left[(0)^4 + (1)^4 + (2)^4 + (3)^4\right] = 1$$
$$\Rightarrow k\left[0 + 1 + 16 + 81\right] = 1$$
$$\Rightarrow k\left[98\right] = 1$$
$$\Rightarrow k = \frac{1}{98}$$

EXAMPLE 7.08

Find "K" for the probability distribution given below:

x	0	1	2	3
f(x)	1/8	K	3/8	1/8

Solution To find the value of "k" we use:

$$\sum_{x=0}^{3} f(x) = 1$$
$$\Rightarrow 1/8 + K + 3/8 + 1/8 = 1$$
$$\Rightarrow K + 5/8 = 1$$
$$\Rightarrow K = 1 - 5/8$$
$$\Rightarrow K = 3/8$$

Test Yourself

1) Find the probability distribution of the sum of dots when two dice are rolled? Also find the probability that the sum of dots is exactly 6

2) Given that: $f(x) = \dfrac{x^4}{98}$ x = 0, 1, 2, 3

   Find (i) $P(1 \le X < 3)$ (ii) $P(X \le 1)$ (iii) $P(X = 0)$

3) What value of "k" makes the following function a density function?

$$f(x) = k\,{}^4C_x,\ \ x = 0, 1, 2, 3, 4$$

4) Find the value of "k" from the following probability distribution:

x	-2	-1	0	1	2	3
f(x)	0.1	0.1	0.2	2k	0.3	k

Continuous probability distribution

"Since a continuous random variable takes all possible values in a given range, therefore, we cannot obtain the probability of a continuous random variable at a particular point and also cannot express a probability distribution in tabular form. Hence the continuous probability distribution can only be expressed in the form of a mathematical equation which is known as probability function or probability density function"

Let "X" be a continuous random variable which can take values in the interval *(a, b)* or $(-\infty, +\infty)$ then The function *f(x)* is called probability function or probability density function (p.d.f) of the random variable "X".

The continuous probability density function may be defined as:

$$f(x) = \begin{cases} \boxed{\text{Function of "x"}} & ; a \le x \le b \\ 0 & ; \text{otherwise} \end{cases}$$

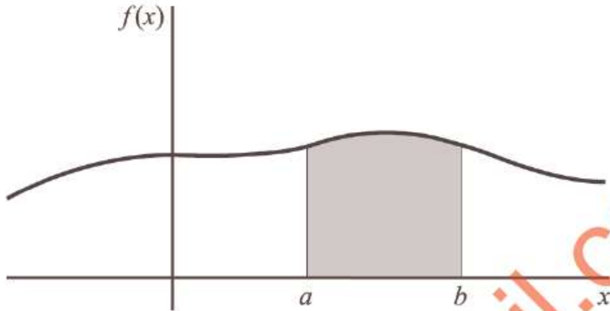
A p.d.f has the following two properties:

(i) $f(x) \ge 0$ *for all "x"*

(ii) $P(a \le X \le b) = \left[\dfrac{f(a) + f(b)}{2}\right](b - a) = 1$

Graph of Continuous probability distribution

The continuous probability distribution is usually displayed by a **continuous probability curves**. In this type of graphs we take the range of X on the X-axis and the probabilities on the Y-axis as shown in the following figure:



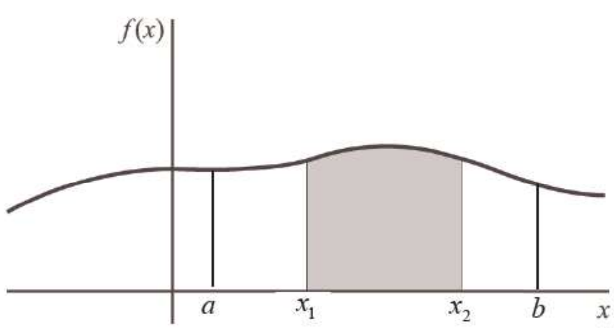
How to find the probabilities using a continuous probability distribution OR a continuous probability density function

Let "X" be a continuous random variable then continuous probability density function *f(x)* is given as:

$$f(x) = \begin{cases} \boxed{Function\ of\ "x"} & ; a \le x \le b \\ 0 & ; otherwise \end{cases}$$

Now to find the probabilities we have:

- $P(X = x_1) = 0$ (because there is no area over a single point)
- $P(x_1 \le X \le x_2) = \left[\dfrac{f(x_1) + f(x_2)}{2}\right](x_2 - x_1)$



In discrete case the probabilities:
$P(x_1 \le X \le x_3)$
$P(x_1 < X < x_3)$
have different meaning but in the case of continuous they are same.

EXAMPLE 7.09

Given that:

$$f(x) = \frac{x+1}{8}, \qquad 2 \le x \le 4$$

(i) Show that the area under the curve is equal to unity

(ii) Find $P(X < 3)$ (iii) Find $P(3 < X < 4)$ (iv) Find $P(X = 3)$

Solution

(i) Show that the area under the curve is equal to unity.

Here we use:

$$P(a \le X \le b) = \left[\frac{f(a)+f(b)}{2}\right](b-a)$$

$$P(2 \le X \le 4) = \left[\frac{f(2)+f(4)}{2}\right](4-2)$$

$$= \left[\frac{f(2)+f(4)}{2}\right](2)$$

$$= f(2)+f(4)$$
$$= 3/8 + 5/8$$
$$= 1$$

Given that $f(x) = \frac{x+1}{8}$

$$f(2) = \frac{2+1}{8} = \frac{3}{8}$$

$$f(3) = \frac{3+1}{8} = \frac{4}{8}$$

$$f(4) = \frac{4+1}{8} = \frac{5}{8}$$

Hence the area under the curve is equal to unity.

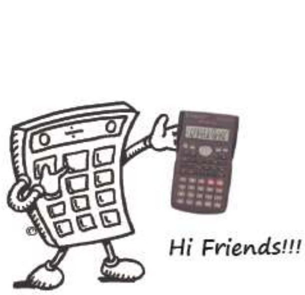
(ii) $P(X < 3)$

$$P(X < 3) = P(2 < X < 3)$$

$$= \left[\frac{f(2)+f(3)}{2}\right](3-2)$$

$$= \left[\frac{f(2)+f(3)}{2}\right](1)$$

$$= \left[\frac{3/8+4/8}{2}\right] = 7/16$$



(iii) $P(3 < X < 4)$

$$P(3 < X < 4) = \left[\frac{f(3)+f(4)}{2}\right](4-3)$$
$$= \left[\frac{4/8+5/8}{2}\right](1)$$
$$= 9/16$$

(iv) $P(X = 3)$

$P(X = 3) = 0$ (for continuous random variable the probability on a single point is zero)

EXAMPLE 7.10

Given that $f(x) = kx$, $\quad 0 \le x \le 2$

(i) Find the value of "k" (ii) Find $P(0.5 < X < 1.5)$
(iii) Find $P(X > 1)$

Solution

(i) Here we use:

$$P(a \le X \le b) = 1$$

$$\Rightarrow \left[\frac{f(a)+f(b)}{2}\right](b-a) = 1$$
$$\Rightarrow \left[\frac{f(0)+f(2)}{2}\right](2-0) = 1$$
$$\Rightarrow \left[\frac{f(0)+f(2)}{2}\right](2) = 1$$

Given that $f(x) = kx$
$f(0) = k(0) = 0$
$f(2) = k(2) = 2k$

$$\Rightarrow f(0) + f(2) = 1$$
$$\Rightarrow 0 + 2k = 1$$
$$\Rightarrow k = 1/2$$

Hence the p.d.f can be written as: $f(x) = kx \Rightarrow f(x) = \dfrac{x}{2}, \quad 0 \le x \le 2$

(ii) $P(0.5 < X < 1.5)$

$$P(0.5 < X < 1.5) = \left[\frac{f(0.5)+f(1.5)}{2}\right](1.5-0.5)$$

$$= \left[\frac{f(0.5)+f(1.5)}{2}\right](1.5-0.5)$$

$$= \left[\frac{f(0.5)+f(1.5)}{2}\right](1)$$

$$= \left[\frac{0.25+0.75}{2}\right] = 0.5$$

Since $f(x) = \frac{x}{2}$

$f(0.5) = \frac{0.5}{2} = 0.25$

$f(1.5) = \frac{1.5}{2} = 0.75$

(iii) $P(X > 1)$

$$P(X > 1) = P(1 < X < 2)$$

$$= \left[\frac{f(1)+f(2)}{2}\right](2-1)$$

$$= \left[\frac{f(1)+f(2)}{2}\right](1)$$

$$= \left[\frac{0.5+1}{2}\right] = 0.75$$

Since $f(x) = \frac{x}{2}$

$f(1) = \frac{1}{2} = 0.5$

$f(2) = \frac{2}{2} = 1$

Test Yourself

1) If $f(x) = \frac{1}{30}(5+2x)$, $1 \le x \le 4$

   (i) Show that f(x) is a density function
   (ii) Find $P(X \le 3)$ (iii) Find $P(X = 3)$

2) Given that $f(x) = kx$, $0 \le x \le 2$

   (i) Find the value of "k" (ii) Find $P(1 < X < 1.5)$
   (iii) Find $P(X > 0.5)$