Equally likely Events

"Events are said to be equally likely if they have the same chances of occurrence"

EXAMPLE: If we toss a fair coin then "H" and "T" are equally likely; because they have the same chances of occurences.

Exhuastive Events

"Two or more events defined in the same sample sapce are said to be exhaustive if their union is equal to the sample space"

EXAMPLE:
If $S=\{1,2,3,4,5,6\}$
Let $A=\{1,3,5\}$ and $B=\{2,4,6\}$
Then $A\cup B=\{1,2,3,4,5,6\}=S$
Therefore "A" and "B" are exhaustive events.

REMEMBER: An event "A" and its compliment "$\bar{A}$" are always exhaustive i.e. $A\cup\bar{A}=S$

Counting Techniques

Sometimes it is very difficult **to list all the sample points** of a sample space; therefore we use some mathematical techniques for finding the number of sample points of the sample space. These techniques are called counting techniques i.e.

- Factorial
- Rule of Multiplication
- Permutation
- Combination

Factorial

"The product of first "n" natural numbers is called Factorial and is denoted by n!"

EXAMPLE:
$2!=2\times 1=2$
$5!=5\times 4\times 3\times 2\times 1=120$

In general $n!=n(n-1)(n-2)(n-3).....3..2.1$

0!=1
1!=1

Rule of Multiplication

"If a selection operation can be performed in "m" ways and a second selection operation can be performed in "n" ways; then the two operations can be performed together in " "$m \times n$" " ways"

- A coin is tossed and a die is rolled; here operation one i.e. the coin gives $\{H,T\}$ and the second operation i.e. the die gives $\{1,2,3,4,5,6\}$; hence the two operations can be performed in $2 \times 6 = 12$ ways.
- If a man has 3 suits and 5 ties; then he can wear a suit and a tie in $3 \times 5 = 15$ ways.

Historical Note

The first book on permutations and combinations is written by Swiss mathematician, Jacob Bernoulli in 1713 A.D

Permutation

"A permutation is an arrangement of "r" objects taken from "n" distinct objects in a particular order"

It is denoted by nPr and is given by: $nPr = \dfrac{n!}{(n-r)!}$

Instead of nPr we can also use ${}^nP_r$ or $P(n,r)$

EXAMPLE 6.01

How many different permutations can be formed from the letters A, B, C when two letters are taken at a time?

Solution

Here $n = 3$ and $r = 2$

Therefore $nPr = \dfrac{n!}{(n-r)!} \Rightarrow {}_3P_2 = \dfrac{3!}{(3-2)!} = 6$

ABC
ACB
BAC
BCA
CAB
CBA

EXAMPLE 6.02

In how many ways "3" persons can be seated on "4" chairs?

Solution Here $n = 4$ and $r = 3$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_4P_3 = \frac{4!}{(4-3)!} = 24$

EXAMPLE 6.03

In how many ways can president, vice-president, secretary and treasure be selected from nine members of a committee?

Solution Here $n = 9$ and $r = 4$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_9P_4 = \frac{9!}{(9-4)!} = 3024$

EXAMPLE 6.04

In how many ways 2 lottery tickets are drawn from 16 for the 1st and 2nd prizes?

Solution Here $n = 16$ and $r = 2$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_{16}P_2 = \frac{16!}{(16-2)!} = 240$

EXAMPLE 6.05

In how many ways can two different books out of 5 books be arranged on a shelf?

Solution Here $n = 5$ and $r = 2$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_5P_2 = \frac{5!}{(5-2)!} = 20$

EXAMPLE 6.06

In how many ways can 5 different books be arranged on a shelf?

Total number of permutation of "n" distinct objects taking all "n" at a time is equal to "n!"

Solution Here $n = 5$

Therefore $\quad Number\ of\ permutation = n! \Rightarrow 5! = 120$

EXAMPLE 6.07

In how many ways can four people be lined up to get on a bus?

Solution Here $n = 4$

Therefore $\quad Number\ of\ permutation = n! \Rightarrow 4! = 24$

EXAMPLE 6.08

How many different words can be formed from the letters of the word "BOXER" if:

1) All the letters are taken at a time
2) Three letters are taken at a time

Solution

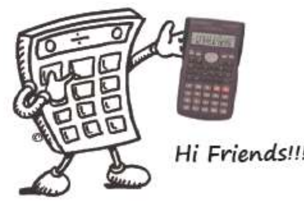
1) All the letters are taken at a time:

Here $n = 5$

Therefore $\quad Number\ of\ permutation = n! \Rightarrow 5! = 120$

2) Three letters are taken at a time

Here $n = 5$ and $r = 3$



Therefore $\quad nPr = \frac{n!}{(n-r)!} \Rightarrow {}_5P_3 = \frac{5!}{(5-3)!} = 60$

EXAMPLE 6.09

Find the number of arrangements of 8 distinct books on a shelf taken:

1) Taken all books at a time
2) Three books are taken at a time

Solution

1) All the letters are taken at a time:

Here $n = 8$
Therefore $\quad$ *Number of permutation* $= n! \Rightarrow 8! = 40320$

2) Three letters are taken at a time

Here $n = 8$ and $r = 3$

Therefore $\quad {}_nP_r = \dfrac{n!}{(n-r)!} \Rightarrow {}_8P_3 = \dfrac{8!}{(8-3)!} = 336$

If we arrange objects in a circle then there is no starting point to it, therefore we fixed one object and the remaining objects are arranged as in linear permutation. The formula for arranging "n" objects in a circle is (n-1)!

EXAMPLE 6.10

In how many ways can 4 people be seated at round table?

Solution Here $n = 4$

Therefore
Number of circular permutation $= (n-1)!$
$= (4-1)! = 3! = 6$

Group Permutation

The number of distinct permutations of "n" things when "n_1" are alike, "n_2" are alike but different from the first group; "n_3" are alike but different from the first and second group and so on; for "k" groups, is:

$$P = \frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!} \qquad \text{Where } n = \sum_{i=1}^{k} n_i$$

EXAMPLE 6.11

How many possible permutations can be formed from the letters of the word "STATISTICS"?

Solution Here $n = 10$

n_1 = *number of "S"* = 3
n_2 = *number of "T"* = 3
n_3 = *number of "A"* = 1
n_4 = *number of "I"* = 2
n_5 = *number of "C"* = 1

$$\text{Therefore } P = \frac{n!}{n_1! \times n_2! \times n_3! \times n_4 \times !n_5!} = \frac{10!}{3! \times 3! \times 1! \times 2! \times 1!} = 50400$$

EXAMPLE 6.12

How many different ways 3 red, 3 yellow and 3 blue balls are arranged in a string with 9 sockets?

Solution Here $n = 9$

n_1 = *number of red balls* = 3
n_2 = *number of yellow balls* = 3
n_3 = *number of blue balls* = 3

$$\text{Therefore } P = \frac{n!}{n_1! \times n_2! \times n_3!} = \frac{9!}{3! \times 3! \times 3!} = 1680$$

EXAMPLE 6.13

In how many possible orders can two boys and three girls be born to a family having five children?

Solution Here $n = 5$

n_1 = *number boys* = 2
n_2 = *number of girls* = 3

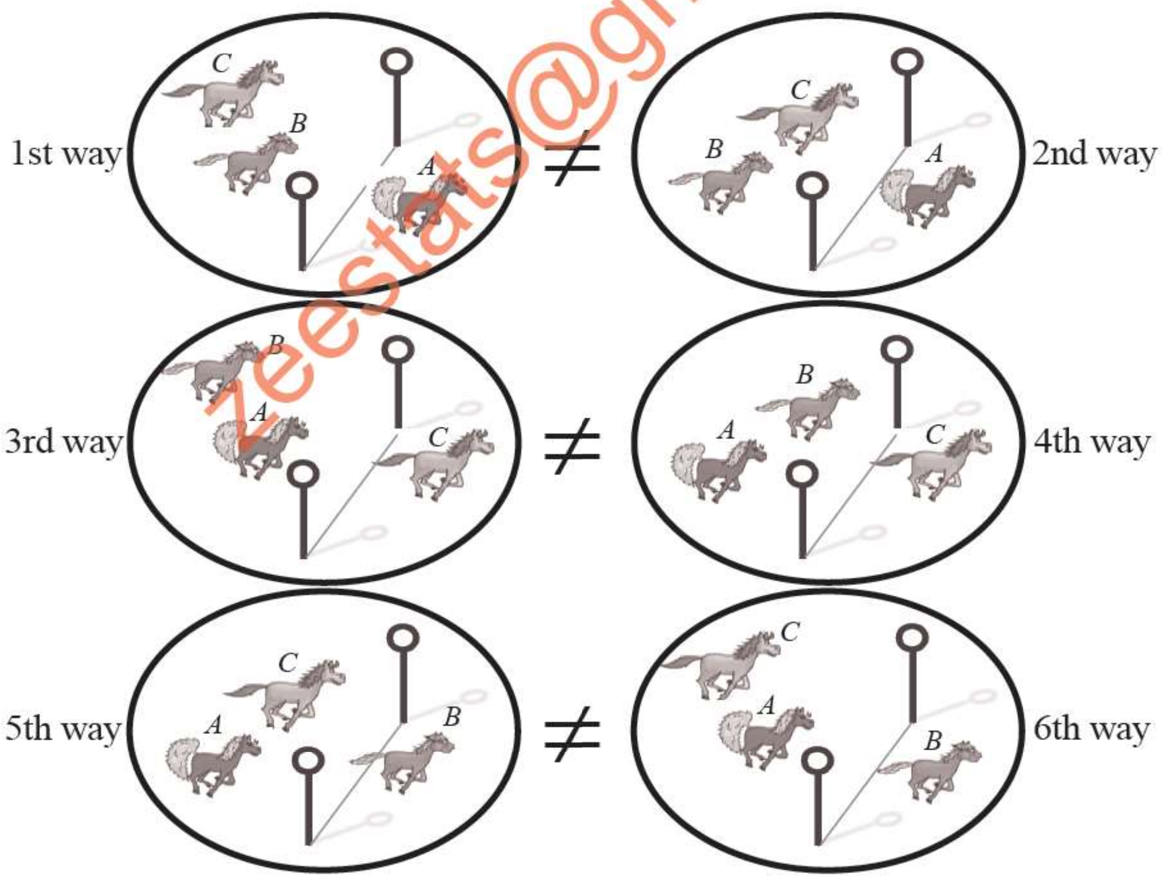
$$\text{Therefore } P = \frac{n!}{n_1! \times n_2!} = \frac{5!}{2! \times 3!} = 10$$

Test Yourself

1) How many permutations can be formed out of the letters of the word “MISSISSIPPI”?
2) Make permutations of A, B, C, D.
3) In how many ways can 4 people be seated at round table?
4) Fine ${}^{7}P_3$, ${}^{4}P_2$, ${}^{12}P_5$, ${}^{10}P_8$
5) Find the number of arrangements of 6 distinct books on a shelf taken:
   (i) Taken all books at a time (ii) Three books are taken at a time
6) In how many ways can 8 people be lined up to get on a bus?

The Order is important in Permutation!!!

There are **six different ways** in which three horses can finish a race as shown in the figure: (Assume that there are no ties and that every horse finishes)



Combination

"A combination is a selection of "r" objects taken from "n" distinct objects without regarding any order"

It is denoted by nCr and is given by:

$$nCr = \frac{n!}{n!(n-r)!}$$

Instead of nCr we can also use ${}^{n}C_{r}$, $C(n,r)$ or $\binom{n}{r}$

EXAMPLE 6.14

How many combinations of the letters A, B, C can be made if two letters are taken at a time?

Solution Here $n = 3$ and $r = 2$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{3}C_{2} = \frac{3!}{2!(3-2)!} = 3$$

ABC
BCA
CAB

EXAMPLE 6.15

In how many ways can a team of 11 players be chosen from a total of 15 players?

Solution Here $n = 15$ and $r = 11$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{15}C_{11} = \frac{15!}{11!(15-11)!} = 1365$$

EXAMPLE 6.16

In how many ways can we select a committee of 4 people from a group of 10 people?

Solution Here $n = 10$ and $r = 4$

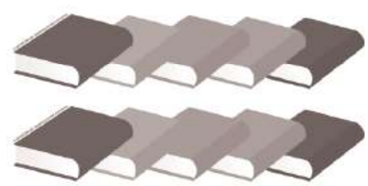
Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{10}C_4 = \frac{10!}{4!(10-4)!} = 210$$

EXAMPLE 6.17

In how many ways can we select a set of 6 books from 10 different books?

Solution Here $n = 10$ and $r = 6$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{10}C_6 = \frac{10!}{6!(10-6)!} = 210$$



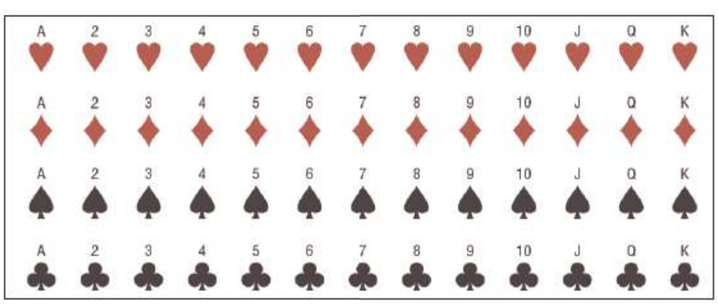
EXAMPLE 6.18

In how many ways can we select a card from a pack of 52 playing cards?

Solution Here $n = 52$ and $r = 1$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{52}C_1 = \frac{52!}{1!(52-1)!} = 52$$

The 52 ways shown in the following figure:



EXAMPLE 6.19

A bag contains 7 balls; in how many ways can we select 3 balls?

Solution Here $n = 7$ and $r = 3$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_7C_3 = \frac{7!}{3!(7-3)!} = 35$$

EXAMPLE 6.20

A basket contains 5 white and 4 black balls; in how many ways can we select 3 white and 2 black balls?

Solution Here

White	Black	Total
5	4	9

"3" white balls can be selected out of "5" in ${}_5C_3 = \frac{5!}{3!(5-3)!} = 10$ ways

"2" black balls can be selected out of "4" in ${}_4C_2 = \frac{4!}{2!(4-2)!} = 6$ ways

Hence the number ways in which "3" white and "2" black balls are selected = $10 \times 6 = 60$

EXAMPLE 6.21

In how many ways can a consonant and a vowel be chosen out of the letters of the word SCHOLAR?

Solution Here

SCHOLAR →

Consonants	Vowels	Total
5	2	7

A consonant can be selected out of "5" in ${}_5C_1 = \frac{5!}{1!(5-1)!} = 5$ ways

A vowel can be selected out of "2" in ${}_2C_1 = \frac{2!}{1!(2-1)!} = 2$ ways

Hence the number ways in which a consonant and a vowel is selected = $5 \times 2 = 10$

EXAMPLE 6.22

From 4 black, 5 white and 6 gray balls; how many selection of 9 balls are possible if 3 balls of each color are to be selected?

Solution Here

Black	White	Gray	Total
4	5	6	15

"3" black balls can be selected out of "4" in $_4C_3 = \frac{4!}{3!(4-3)!} = 4$ ways

"3" white balls can be selected out of "5" in $_5C_3 = \frac{5!}{3!(5-3)!} = 10$ ways

"3" gray balls can be selected out of "6" in $_6C_3 = \frac{6!}{3!(6-3)!} = 20$ ways

Hence the number ways in which 3 balls of each color are to be selected= 4x10x20=800

EXAMPLE 6.23

A committee of 5 persons is to be selected out of 6 men and 2 women. Fine the number of ways in which more men are selected than women?

Solution Here

Committee →

Men	Women	Total
6	2	8

Now here more men can be selected in "3" mutually exclusive ways i.e.

$$\begin{pmatrix} 5\,men \\ 0\,women \end{pmatrix} \text{or} \begin{pmatrix} 4\,men \\ 1\,women \end{pmatrix} \text{or} \begin{pmatrix} 3\,men \\ 2\,women \end{pmatrix}$$

$$\Rightarrow \binom{6}{5}\binom{2}{0} \text{ or } \binom{6}{4}\binom{2}{1} \text{ or } \binom{6}{3}\binom{2}{2}$$

Since the three ways are mutually exclusive; therefore the number of ways in which more men than women can be chosen are $= \binom{6}{5}\binom{2}{0} + \binom{6}{4}\binom{2}{1} + \binom{6}{3}\binom{2}{2} = 56$

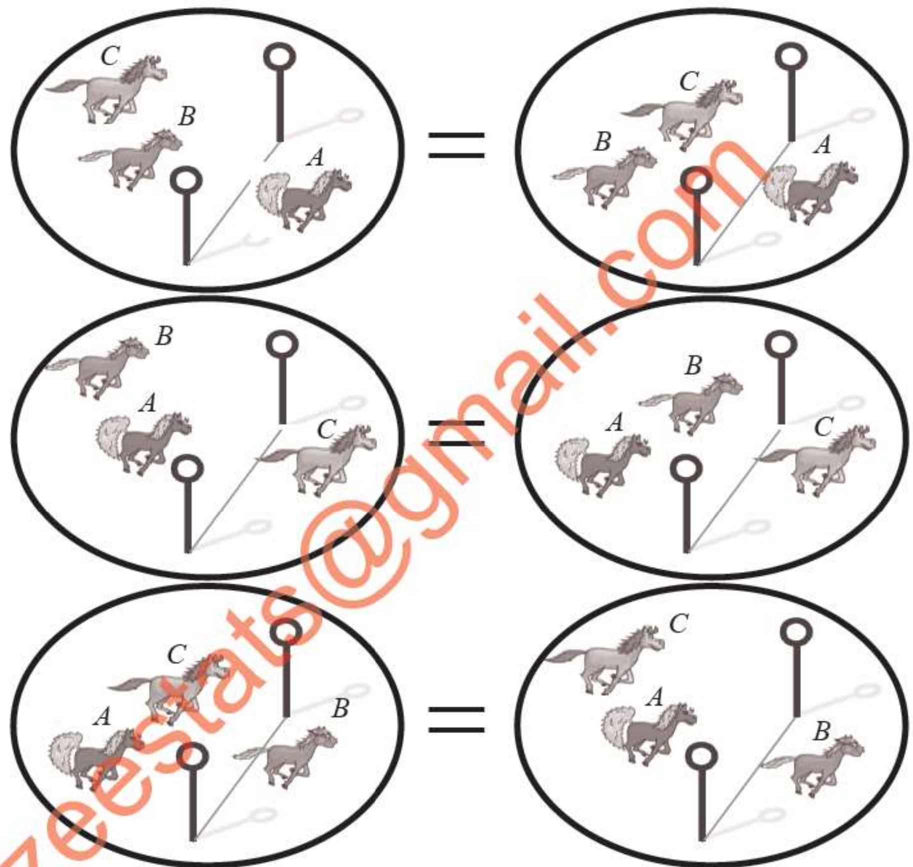
The Order Doesn't matter in Combination!!!

There are **three different combinations** in which three horses can finish a race as shown in the figure: (Assume that there are no ties and that every horse finishes)

1st combination

2nd combination

3rd combination



Test Yourself

1) In how many ways can we select a set of 3 tables from 9 different tables?
2) A bag contains 6 balls; in how many ways can we select 4 balls?
3) A bag contains 9 white and 8 black balls; in how many ways can we select 6 white and 4 black balls?
4) Fine ${}^{7}C_3$, ${}^{4}C_2$, ${}^{12}C_5$, ${}^{10}C_8$
5) In how many ways can a consonant and a vowel be chosen out of the letters of the word CHOSEN?
6) In how many ways can a team of 11 players be chosen from a total of 13 players?