Lecture 3

Systems of Linear Equations

In this lecture we will discuss some ways in which systems of linear equations arise, how to solve them, and how their solutions can be interpreted geometrically.

Linear Equations

We know that the equation of a straight line is written as $y = mx + c$, where m is the slope of line(Tan of the angle of line with x-axis) and c is the y-intercept(the distance at which the straight line meets y-axis from origin).
Thus a line in $\boldsymbol{R^2}$ (2-dimensions) can be represented by an equation of the form $a_1x + a_2y = b$ (where a_1, a_2 not both zero). Similarly a plane in $\boldsymbol{R^3}$ (3-dimensional space) can be represented by an equation of the form $a_1x + a_2y + a_3z = b$ (where a_1, a_2, a_3 not all zero).

A linear equation in n variables $x_1, x_2, \cdots, x_n$ can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \text{(hyper plane in } \mathbb{R}^n \text{) --------(1)}$$

where $a_1, a_2, \cdots, a_n$ and b are constants and the *"a's"* are not all zero.

Homogeneous Linear equation

In the special case if $b = 0$, Equation (1) has the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ (2)
This equation is called homogeneous linear equation.

Note A linear equation does not involve any products or square roots of variables. All variables occur only to the first power and do not appear, as arguments of trigonometric, logarithmic, or exponential functions.

Examples of Linear Equations

(1) The equations

$$2x_1 + 3x_2 + 2 = x_3 \quad \textit{and} \quad x_2 = 2\left(\sqrt{5} + x_1\right) + 2x_3$$ are both linear

(2) The following equations are also linear

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + \ x_2 + \cdots + x_n \ = 1$$

(3) The equations $3x_1 - 2x_2 = x_1x_2$ *and* $x_2 = 4\sqrt{x_1} - 6$

are ***not linear*** because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

System of Linear Equations

A finite set of linear equations is called a system of linear equations or ***linear system***. The variables in a linear system are called the ***unknowns***.

For example,

$$\begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned}$$

is a linear system of two equations in three unknowns x_1, x_2, and x_3.

General System of Linear Equations

A general linear system of *m* equations in n-unknowns $x_1, x_2, \cdots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad & \;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (3)$$

Solution of a System of Linear Equations

A solution of a linear system in the unknowns $x_1, x_2, \cdots, x_n$ is a sequence of *n* numbers $s_1, s_2, \cdots, s_n$ such that when substituted for $x_1, x_2, \cdots, x_n$ respectively, makes every equation in the system a true statement. The set of all such solutions $\{s_1, s_2, \cdots, s_n\}$ of a linear system is called its ***solution set***.

Linear System with Two Unknowns

When two lines intersect in $\boldsymbol{R}^2$, we get system of linear equations with two unknowns

For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

The graphs of these equations are straight lines in the *xy-plane*, so a solution (*x, y*) of this system is infact a point of intersection of these lines.

Note that there are three possibilities for a pair of straight lines in xy-plane:

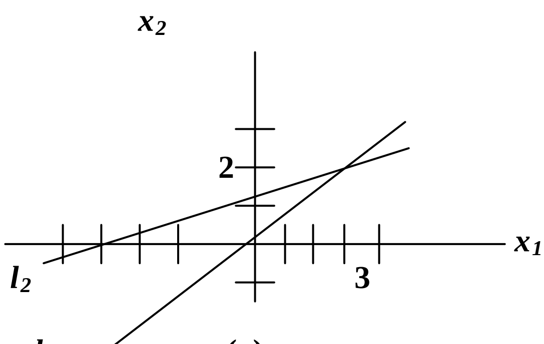
1. The lines may be parallel and distinct, in which case there is no intersection and consequently ***no solution***.
2. The lines may intersect at only one point, in which case the system has exactly ***one solution***.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently ***infinitely many solutions.***

Consistent and inconsistent system

A linear system is said to be **consistent** if it has at least one solution and it is called **inconsistent** if it has no solutions.

Thus, a consistent linear system of two equations in two unknowns has either one solution or infinitely many solutions – there is no other possibility.

Example consider the system of linear equations in two variables $x_1 - 2x_2 = -1,\ -x_1 + 3x_2 = 3$
Solve the equation simultaneously:
Adding both equations we get $x_2 = 2$, Put $x_2 = 2$ in any one of the above equation we get $x_1 = 3$. So the solution is the single point **(3, 2)**. See the graph of this linear system

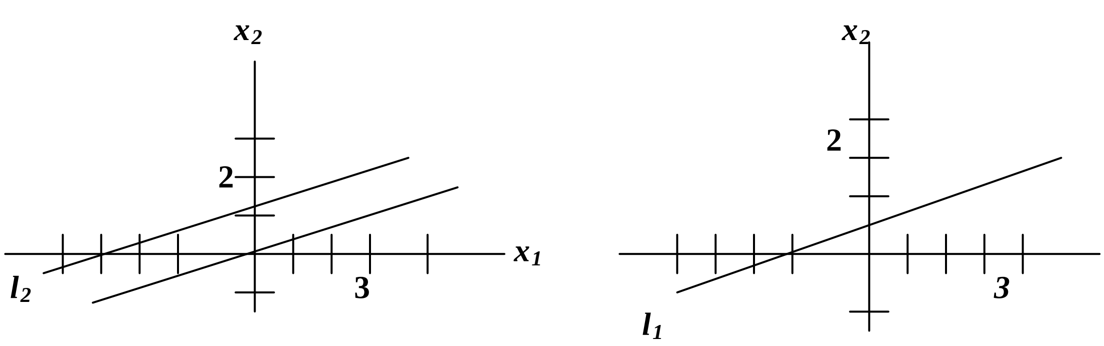


(a)

This system has exactly one solution

See the graphs to the following linear systems:

$$(a)\quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



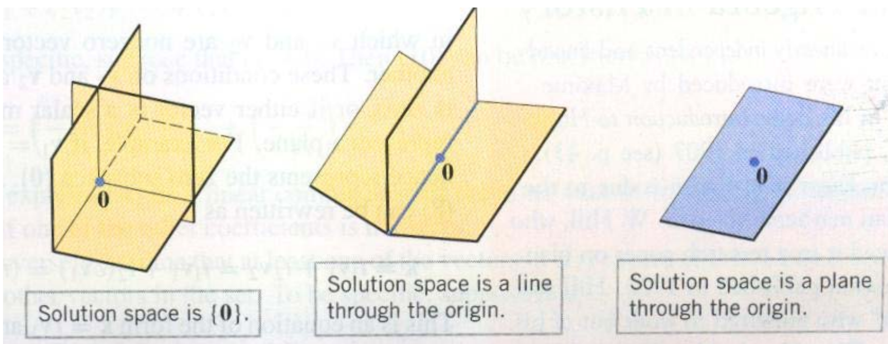
(a) No solution. (b) Infinitely many solutions.

Linear System with Three Unknowns

Consider r a linear system of three equations in three unknowns:

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_1z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

In this case, the graph of each equation is a plane, so the solutions of the system, If any correspond to points where all three planes intersect; and again we see that there are only three possibilities – no solutions, one solution, or infinitely many solutions as shown in figure.



Theorem 1 Every system of linear equations has zero, one or infinitely many solutions; there are no other possibilities.

Example 1 Solve the linear system

$$x-y=1$$
$$2x+y=6$$

Solution

Adding both equations, we get $x=\frac{7}{3}$. Putting this value of x in 1st equation, we get $y=\frac{4}{3}$. Thus, the system has the ***unique solution*** $x=\frac{7}{3}, y=\frac{4}{3}$.

Geometrically, this means that the lines represented by the equations in the system intersect at a single point $\left(\frac{7}{3},\frac{4}{3}\right)$ and thus has a unique solution.

Example 2 Solve the linear system

$$x+\ \ y=4$$
$$3x+3y=6$$

Solution

Multiply first equation by 3 and then subtract the second equation from this. We obtain

$$0=6$$

This equation is contradictory.

Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. So the given system has ***no solution.***

Example 3 Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Solution

Multiply the first equation by -4 and then add in second equation.

$$\begin{aligned} -16x + 8y &= -4 \\ 16x - 8y &= \ \ 4 \\ \hline 0 \qquad &= 0 \end{aligned}$$

Thus, the solutions of the system are those values of *x* and *y* that satisfy the single equation $4x - 2y = 1$

Geometrically, this means the lines corresponding to the two equations in the original system coincide and thus the system has infinitely many solutions.

Parametric Representation

It is very convenient to describe the solution set in this case is to ***express it parametrically.*** We can do this by letting *y* = *t* and solving for *x* in terms of *t*, or by letting *x* = *t* and solving for *y* in terms of *t*.

The first approach yields the following parametric equations (by taking *y=t in the equation* $4x - 2y = 1$)

$$4x - 2t = 1, \quad y = t$$

$$x = \frac{1}{4} + \frac{1}{2}t, \quad y = t$$

We can now obtain some solutions of the above system by substituting some numerical values for the parameter.

Example For *t* = *0* the solution is $(\frac{1}{4}, 0)$. For *t* = *1*, the solution is $(\frac{3}{4}, 1)$ and for $t = -1$ the solution is $(-\frac{1}{4}, -1)$ *etc.*

Example 4 Solve the linear system

$$\begin{aligned} x - \ \ y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution
Since the second and third equations are multiples of the first.

Geometrically, this means that the three planes coincide and those values of *x*, *y* and *z* that satisfy the equation $x - y + 2z = 5$ automatically satisfy all three equations.

We can express the solution set parametrically as

$$x = 5 + t_1 - 2t_2, y = t_1, \quad z = t_2$$

Some solutions can be obtained by choosing some numerical values for the parameters.

For example if we take $y = t_1 = 2$ and $z = t_2 = 3$ then

$$x = 5 + t_1 - 2t_2$$
$$= 5 + 2 - 2(3)$$
$$= 1$$

Put these values of x, y, and z in any equation of linear system to verify

$$x - y + 2z = 5$$
$$1 - 2 + 2(3) = 5$$
$$1 - 2 + 6 = 5$$
$$5 = 5$$

Hence x = 1, y = 2, z = 3 is the solution of the system. Verified.

Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.

Given the system
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

With the coefficients of each variable aligned in columns, the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

is called the coefficient matrix (or matrix of coefficients) of the system.

An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations. It is always denoted by A_b

$$A_b = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Solving a Linear System

In order to solve a linear system, we use a number of methods. 1st of them is given below.

Successive elimination method In this method the x_1 term in the first equation of a system is used to eliminate the x_1 terms in the other equations. Then we use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until we finally obtain a very simple equivalent system of equations.

Example 5 Solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Solution We perform the elimination procedure with and without matrix notation, and place the results side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

To eliminate the x_1 term from third equation add 4 times equation 1 to equation 3,

$$\begin{aligned} 4x_1 - 8x_2 + 4x_3 &= 0 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \\ \hline -3x_2 + 13x_3 &= -9 \end{aligned}$$

The result of the calculation is written in place of the original third equation:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Next, multiply equation 2 by ½ in order to obtain 1 as the coefficient for x_2

$$\begin{aligned} x_1 - 2x_2 + \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

To eliminate the x_2 term from third equation add 3 times equation 2 to equation 3,

The new system has a triangular form

$$\begin{aligned} & x_1 - 2x_2 + x_3 = 0 \\ & x_2 - 4x_3 = 4 \\ & x_3 = 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now using 3rd equation eliminate the x_3 term from first and second equation i.e. multiply 3rd equation with 4 and add in second equation. Then subtract the third equation from first equation we get

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Adding 2 times equation 2 to equation 1, we obtain the result

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This completes the solution.
Our work indicates that the only solution of the original system is (29, 16, 3).

To verify that (29, 16, 3) is a solution, substitute these values into the left side of the original system for x_1, x_2 and x_3 and after computing, we get

$$(29) - 2(16) + (3) = 29 - 32 + 3 = 0$$
$$2(16) - 8(3) = 32 - 24 = 8$$
$$-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9$$

The results agree with the right side of the original system, so (29, 16, 3) is a solution of the system.

This example illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a nonzero multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Row equivalent matrices

A matrix B is said to be row equivalent to a matrix A of the same order if B can be obtained from A by performing a finite sequence of elementary row operations of A.
If A and B are row equivalent matrices, then we write this expression mathematically as A $\sim$ B.

For example $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$ are row equivalent matrices because we add 4 times of 1^{st} row in 3^{rd} row in 1^{st} matrix.

Note If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Row operations are extremely easy to perform, but they have to be learnt and practice.

Two Fundamental Questions

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists is it the only one; that is, is the solution unique?

We try to answer these questions via row operations on the augmented matrix.

Example 6 Determine if the following system of linear equations is consistent

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Solution

First obtain the triangular matrix by removing x_1 and x_2 term from third equation and removing x_2 from second equation.

First divide the second equation by 2 we get

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= \ \ 4 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Now multiply equation 1 with 4 and add in equation 3 to eliminate x_1 from third equation.

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= \ \ 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Now multiply equation 2 with 3 and add in equation 3 to eliminate x_2 from third equation.

$$\begin{aligned} &x_1 - 2x_2 + x_3 = 0 \\ &x_2 - 4x_3 = 4 \\ &x_3 = 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Put value of x_3 in second equation we get

$x_2 - 4(3) = 4$

$x_2 = 16$

Now put these values of x_2 and x_3 in first equation we get

$x_1 - 2(16) + 3 = 0$

$x_1 = 29$

So a solution exists and the system is consistent and has a unique solution.

Example 7 Solve if the following system of linear equations is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

Solution The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To obtain x_1 in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To eliminate the $5x_1$ term in the third equation, add –5/2 times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

Next, use the x_2 term in the second equation to eliminate the –(1/2) x_2 term from the third equation. Add ½ times row 2 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The augmented matrix is in triangular form.
To interpret it correctly, go back to equation notation:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 2.5 \end{aligned}$$

There are no values of x_1, x_2, x_3 that will satisfy because the equation 0 = 2.5 is never true.
Hence original ***system is inconsistent (i.e., has no solution)***.

Exercises

1. State in words the next elementary "row" operation that should be performed on the system in order to solve it. (More than one answer is possible in (a).)

a.
$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7 \\ x_3 + 3x_4 &= -5 \end{aligned}$$

b.
$$\begin{aligned} x_1 - 3x_2 + 5x_3 - 2x_4 &= 0 \\ x_2 + 8x_3 &= -4 \\ 2x_3 &= 7 \\ x_4 &= 1 \end{aligned}$$

2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

3. Is (3, 4, –2) a solution of the following system?

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= -7 \end{aligned}$$

4. For what values of *h* and *k* is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

Solve the systems in the exercises given below;

5.
$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -1 \end{aligned}$$

6.
$$\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ 2x_1 - x_2 - 7x_3 &= 1 \end{aligned}$$

7.
$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - 3x_2 - 3x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

8.
$$\begin{aligned} 2x_1 - 4x_3 &= -10 \\ x_2 + 3x_3 &= 2 \\ 3x_1 + 5x_2 + 8x_3 &= -6 \end{aligned}$$

Determine the value(s) of h such that the matrix is augmented matrix of a consistent linear system.

9. $\begin{bmatrix} 1 & -3 & h \\ -2 & 6 & -5 \end{bmatrix}$

10. $\begin{bmatrix} 1 & h & -2 \\ -4 & 2 & 10 \end{bmatrix}$

Find an equation involving g, h, and that makes the augmented matrix correspond to a consistent system.

11. $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$

12. $\begin{bmatrix} 2 & 5 & -3 & g \\ 4 & 7 & -4 & h \\ -6 & -3 & 1 & k \end{bmatrix}$

Find the elementary row operations that transform the first matrix into the second, and then find the reverse row operation that transforms the second matrix into first.

13. $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -4 \\ 0 & -3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 0 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 1 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -2 \\ 0 & 5 & -3 \\ 2 & 1 & 8 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & 2 & -5 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 3 & -5 \end{bmatrix}$

Lecture 5

Vector Equations

This lecture is devoted to connect equations involving vectors to ordinary systems of equations. The term vector appears in a variety of mathematical and physical contexts, which we will study later, while studying "Vector Spaces". Until then, we will use vector to mean a list of numbers. This simple idea enables us to get interesting and important applications as quickly as possible.

Column Vector

"*A matrix with only one column is called column vector or simply a vector"*.

e.g. $u = \begin{bmatrix} 3 & -1 \end{bmatrix}^T = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}^T$ are all column vectors or simply vectors.

Vectors in R^2

If $\mathbb{R}$ is the set of all real numbers then the set of all vectors with two entries is denoted by $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

For example: the vector $u = \begin{bmatrix} 3 & -1 \end{bmatrix}^T = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \in \mathbb{R}^2$

Here real numbers are appeared as entries in the vectors, and the exponent **2** indicates that the vectors contain only two entries.

Similarly $\boldsymbol{R^3}$ and $\boldsymbol{R^4}$ contain all vectors with three and four entries respectively. The entries of the vectors are always taken from the set of real numbers ***R.*** The entries in vectors are assumed to be the elements of a set, called a ***Field***. It is denoted by F .

Algebra of Vectors

Equality of vectors in $\mathbb{R}^2$

Two vectors in $\boldsymbol{R^2}$ *are equal if and only if their corresponding entries are equal.*

If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ *then* $u = v$ *iff* $\boxed{u_1 = v_1} \wedge \boxed{u_2 = v_2}$

So $\begin{bmatrix} 4 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ as $4 = 4$ but $6 \neq 3$

Note In fact, vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\boldsymbol{R^2}$ are nothing but ordered pairs (x, y) of real numbers both representing the position of a point with respect to origin.

Addition of Vectors

Given two vectors ***u*** and ***v*** in $\boldsymbol{R^2}$, their sum is the *vector* ***u + v*** *obtained by adding corresponding entries of the vectors* ***u*** *and* ***v***, which is again a vector in $\mathbb{R}^2$

For $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ Then $u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \in \mathbb{R}^2$

For example, $\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Scalar Multiplication of a vector

Given a vector ***u*** and a real number *c*, the scalar multiple of ***u*** by *c* is the *vector c**u** obtained by multiplying each entry in **u** by c.*

For example, if $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ *and* $c = 5,$ *then* $cu = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$

Notations The number *c* in *c**u*** is a **scalar**; it is written in lightface type to distinguish it from the boldface vector ***u***.

Example 1 Given $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ *and* $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find 4***u***, (-3) ***v***, and 4u + (-3) ***v***

Solution $4u == 4\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4\times1 \\ 4\times(-2) \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix},$ $(-3)v = (-3)\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$

And $4u + (-3)v = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

Note: Sometimes for our convenience, we write a column vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the form (3, –1). In this case, we use *parentheses and a comma to distinguish the vector* (3, –1) *from the* 1×2 *row matrix* [3 –1], written with brackets and no comma.

Thus $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq [3 \quad -1]$ but $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1)$

Geometric Descriptions of R^2

Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$. So we may regard R^2 as the set of all points in the plane.

See Figure 1.

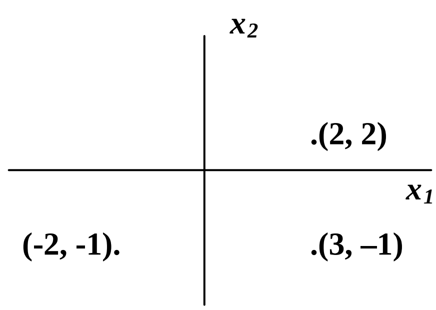


Figure 1 Vectors as points.

Vectors in R^3

Vectors in R^3 are 3×1 column matrices with three entries. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity.

Vectors in R^n

If **n** is a positive integer, R^n (read "r-n") denotes the collection of all lists (or ordered n- tuples) of **n** real numbers, usually written as $n\times 1$ column matrices, such as

$u = [u_1 \;\; u_2 \;\; \cdots u_n]^T$

The vector whose all entries are zero is called the **zero vector** and is denoted by **O.** (The number of entries in **O** will be clear from the context.)

Algebraic Properties of R^n

For all ***u, v, w*** in $\boldsymbol{R^n}$ and all scalars *c* and *d*:

(i) $\boldsymbol{u + v = v + u}$ (Commutative)

(ii) $\boldsymbol{(u + v) + w = u + (v + w)}$ (Associative)

(iii) $\boldsymbol{u + 0 = 0 + u = u}$ (Additive Identity)

(iv) $\boldsymbol{u + (-u) = (-u) + u = 0}$ (Additive Inverse)

where $\boldsymbol{-u}$ denotes $(-1)\boldsymbol{u}$

(v) $c(\boldsymbol{u + v}) = c\boldsymbol{u} + c\boldsymbol{v}$ (Scalar Distribution over Vector Addition)

(vi) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$ (Vector Distribution over Scalar Addition)

(vii) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$

(viii) $\boldsymbol{1u=u}$

Linear Combinations Given vectors $\boldsymbol{v_1, v_2, \ldots, v_p}$ in $\boldsymbol{R^n}$ and given scalars $c_1, c_2, \ldots, c_p$ the vector defined by

$$y = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

is called a **linear combination** of $\boldsymbol{v_1, \ldots, v_p}$ using weights $\boldsymbol{c_1, \ldots, c_p}$.

Property (ii) above permits us to omit parenthesis when forming such a linear combination. The weights in a linear combination can be any real numbers, including zero.

Example

For $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, if $w = \frac{5}{2}v_1 - \frac{1}{2}v_2$ then we say that w is a linear combination of v_1 and v_2.

Example As (3, 5 , 2) = 3(1, 0 , 0) + 5(0, 1 , 0) + 2(0, 0 , 1)

(3, 5 , 2) = $3v_1 + 5v_2 + 2v_3$ where v_1 = (1, 0 , 0) , v_2 = (0, 1 , 0) v_3 = (0, 0 , 1)

So (3, 5 , 2) is a vector which is linear combination of v_1, v_2, v_3

Example 5 Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, and \quad b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$

Determine whether ***b*** can be generated (or written) as a linear combination of $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$. That is, determine whether weights $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ exist such that

$$x_1 \boldsymbol{a_1} + x_2 \boldsymbol{a_2} = \boldsymbol{b} \qquad (1)$$

If the vector equation (1) has a solution, find it.

Solution Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$\underset{\boldsymbol{a_1}}{x_1\begin{bmatrix}1\\-2\\-5\end{bmatrix}} + \underset{\boldsymbol{a_2}}{x_2\begin{bmatrix}2\\5\\6\end{bmatrix}} = \underset{\boldsymbol{b}}{\begin{bmatrix}7\\4\\-3\end{bmatrix}}$$

$$\Rightarrow \quad \begin{bmatrix}x_1\\-2x_1\\-5x_1\end{bmatrix} + \begin{bmatrix}2x_2\\5x_2\\6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x_1+2x_2\\-2x_1+5x_2\\-5x_1+6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix} \qquad (2)$$

$$\Rightarrow \quad \begin{aligned}x_1+2x_2&=7\\-2x_1+5x_2&=4\\-5x_1+6x_2&=-3\end{aligned} \qquad (3)$$

We solve this system by row reducing the augmented matrix of the system as follows:

$$\begin{bmatrix}1&2&7\\-2&5&4\\-5&6&-3\end{bmatrix}$$

$$By\ R_2+2R_1\ ;R_3+5R_1$$

$$\sim\begin{bmatrix}1&2&7\\0&9&18\\0&16&32\end{bmatrix}$$

$$By\ \left(\frac{1}{9}\right)R_2;\left(\frac{1}{16}\right)R_3^{/}$$

$$\sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$By\, R_3 - R_2; R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence ***b*** is a linear combination of $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$**,** with weights $x_1 = 3$ and $x_2 = 2$.

Spanning Set

If $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ are in $\mathbf{R}^n$, then the set of all linear combinations of $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ is denoted by Span $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}\}$ and is called the **subset of $\mathbf{R}^n$ spanned** (or **generated**) by $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$. That is, Span $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}\}$ is the collection of all vectors that can be written in the form of $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_p\boldsymbol{v_p}$, with $c_1, \ldots, c_p$ scalars.

If we want to check whether a vector **b** is in Span $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}\}$ then we will see whether the vector equation

$x_1\boldsymbol{v_1} + x_2\boldsymbol{v_2} + \ldots + x_p\boldsymbol{v_p} = \boldsymbol{b}$ has a solution, or

Equivalently, whether the linear system with augmented matrix $[\,\boldsymbol{v_1}, \ldots, \boldsymbol{v_p} \quad \boldsymbol{b}]$ has a solution.

Note

(1) The set Span $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}\}$ contains every scalar multiple of $\boldsymbol{v_1}$

because $c\boldsymbol{v_1} = c\boldsymbol{v_1} + 0\boldsymbol{v_2} + \ldots + 0\boldsymbol{v_p}$ i.e every $c\boldsymbol{v_i}$ can be written as a linear combination of $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$

(2) Zero vector $= 0 \in Span\{v_1, v_2, \cdots v_n\}$ as 0 can be written as the linear combination of $v_1, v_2, \cdots v_n$ that is $0_v = 0_F v_1 + 0_F v_2 + \cdots + 0_F v_n$ here for the convenience it is mentioned that 0_v is the vector(zero vector) while 0_F is zero scalar (weight of all $v_1, v_2, \cdots v_n$) and in particular not to make confusion that 0_v and 0_F are same!

A Geometric Description of Span {*v*} and Span {*u, v*}

Let ***v*** be a nonzero vector in $\mathbf{R}^3$. Then Span **{*v*}** is the set of all linear combinations of ***v*** or in particular set of scalar multiples of ***v***, and we visualize it as the set of points on the line in $\mathbf{R}^3$ through ***v*** and ***0***.

If ***u*** and ***v*** are nonzero vectors in $\mathbf{R}^3$, *with **v** not a multiple of **u***, then Span **{*u, v*}** is the plane in $\mathbf{R}^3$ that contains ***u, v*** and **0**. In particular, Span **{*u, v*}** contains the line in $\mathbf{R}^3$ through ***u*** and **0** and the line through ***v*** and **0**.

Example 6 Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, and\ b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}.$

Then Span $\{\mathbf{a_1}, \mathbf{a_2}\}$ is a plane through the origin in $\mathbf{R}^3$. Does ***b*** lie in that plane?

Solution First we see the equation $x_1\mathbf{a_1} + x_2\mathbf{a_2} = \mathbf{b}$ has a solution?

To answer this, row-reduce the augmented matrix $[\mathbf{a_1}\ \ \mathbf{a_2}\ \ \mathbf{b}]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix}$$

$By\ R_2 + 2R_1$

$$\sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 18 & 10 \end{bmatrix}$$

$By\ R_3 + 6R_2$

$$\sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Last row $\Rightarrow 0x_2 = -2$ which can not be true for any value of $x_2 \in \mathbb{R}$

$\Rightarrow$ Given system has no solution

$\therefore b \notin Span\{a_1, a_2\}$ and

in geometrical meaning, vector b does not lie in the plane spanned by vectors $a_1\ and\ a_2$

Linear Combinations in Applications

The final example shows how scalar multiples and linear combinations can arise when a quantity such as "cost" is broken down into several categories. The basic principle for the example concerns the cost of producing several units of an item when the cost per unit is known:

$$\begin{Bmatrix} \text{number} \\ \text{of units} \end{Bmatrix} \cdot \begin{Bmatrix} \cos t \\ \text{per unit} \end{Bmatrix} = \begin{Bmatrix} \text{total} \\ \cos t \end{Bmatrix}$$

Example 7 A Company manufactures two products. For one dollar's worth of product B, the company spends \$0.45 on materials, \$0.25 on labor, and \$0.15 on overhead. For one dollar's worth of product C, the company spends \$0.40 on materials, \$0.30 on labor and \$0.15 on overhead.

Let $b = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix}$ *and* $c = \begin{bmatrix} .40 \\ .30 \\ .15 \end{bmatrix}$, then ***b*** and ***c*** represent the "costs per dollar of income" for the two products.

a) What economic interpretation can be given to the vector 100***b***?
b) Suppose the company wishes to manufacture x_1 dollars worth of product B and x_2 dollars worth of product C. Give a vector that describes the various costs the company will have (for materials, labor and overhead).

Solution

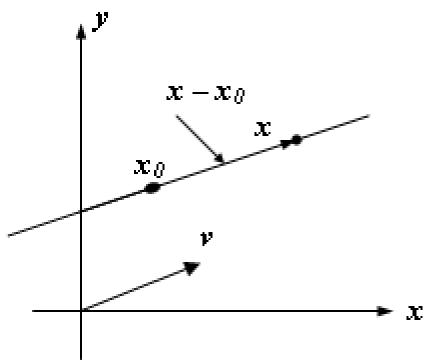
(a) We have $100b = 100\begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} = \begin{bmatrix} 45 \\ 25 \\ 15 \end{bmatrix}$

The vector 100***b*** represents a list of the various costs for producing \$100 worth of product B, namely, \$45 for materials, \$25 for labor, and \$15 for overhead.

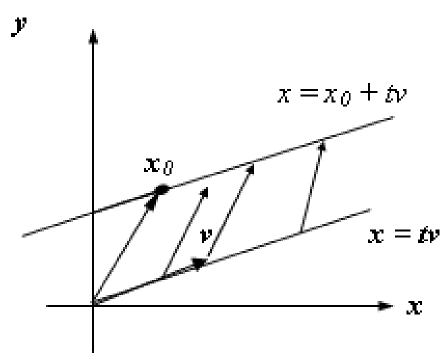
(b) The costs of manufacturing x_1 dollars worth of B are given by the vector $x_1\boldsymbol{b}$ and the costs of manufacturing x_2 dollars worth of C are given by $x_2\boldsymbol{c}$. Hence the total costs for both products are given by the vector $x_1\boldsymbol{b} + x_2\boldsymbol{c}$.

Vector Equation of a Line

Let $\mathbf{x_0}$ be a fixed point on the line and $\mathbf{v}$ be a nonzero vector that is parallel to the required line. Thus, if $\mathbf{x}$ is a variable point on the line through $\mathbf{x_0}$ that is parallel to $\mathbf{v}$, then the vector $\mathbf{x} - \mathbf{x_0}$ is a vector parallel to $\mathbf{v}$ as shown in fig below,



(b)



So by definition of parallel vectors $\boldsymbol{x} - \boldsymbol{x_0} = t\boldsymbol{v}$ *for some scalar t.*

it is also called a ***parameter*** which varies from $-\infty$ to $+\infty$. The variable point x traces out the line, so the line can be represented by the equation

$$\boldsymbol{x} - \boldsymbol{x_0} = t\boldsymbol{v} \text{ --------------(1)} \qquad (-\infty < t < +\infty)$$

This is a ***vector equation of the line*** through $\boldsymbol{x_0}$ and parallel to $\boldsymbol{v}$.

In the special case, where $\boldsymbol{x_0} = \mathbf{0}$, the line passes through the origin, it simplifies to

$$\boldsymbol{x} = t\boldsymbol{v} \qquad (-\infty < t < +\infty)$$

Parametric Equations of a Line in R^2

Let $\boldsymbol{x} = (x, y) \in R^2$ be a general point of the line through $\boldsymbol{x_0} = (x_0, y_0) \in R^2$ which is parallel to

$\boldsymbol{v} = (a, b) \in R^2$, then eq. 1 takes the form

$$(x, y) - (x_0, y_0) = t(a, b) \quad (-\infty < t < +\infty)$$

$$\Rightarrow (x - x_0 , y - y_0) = (ta , tb) \quad (-\infty < t < +\infty)$$

$$\Rightarrow x = x_0 + at, \quad y = y_0 + bt \quad (-\infty < t < +\infty)$$

These are called ***parametric equations*** of the line $in\ R^2$.

Parametric Equations of a Line in R^3

Similarly, if we let $\boldsymbol{x} = (x, y, z) \in R^3$ be a general point on the line through

$\boldsymbol{x_0} = (x_0, y_0 , z_0) \in R^3$ that is parallel to $\boldsymbol{v} = (a, b, c) \in R^3$, then again eq. 1 takes the form

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \ (-\infty < t < +\infty)$$

$$\Rightarrow \quad x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \quad (-\infty < t < +\infty)$$

These are the ***parametric equations*** of the line $in\ R^3$

Example 8

(a) Find a vector equation and parametric equations of the line in $\boldsymbol{R^2}$ that passes through the origin and is parallel to the vector $\boldsymbol{v}$ = (–2, 3).

(b) Find a vector equation and parametric equations of the line in $\boldsymbol{R^3}$ that passes through the point P_0(1, 2, –3) and is parallel to the vector $\boldsymbol{v}$ = (4, –5, 1).

(c) Use the vector equation obtained in part (b) to find two points on the line that are different from P_0.

Solution

(a) We know that a vector equation of the line passing through origin is $\boldsymbol{x} = t\boldsymbol{v}$.

Let $\boldsymbol{x} = (x, y)$. Then this equation can be expressed in component form as

$$(x, y) = t\,(-2, 3)$$

This is the vector equation of the line.

Equating corresponding components on the two sides of this equation yields the parametric equations $x = -2t, \quad y = 3t$

(b) The vector equation of the line is $\boldsymbol{x} = \boldsymbol{x_0} + t\boldsymbol{v}$.

Let $\boldsymbol{x} = (x, y, z)$, Here $\boldsymbol{x_0} = (1, 2, -3)$ and $\boldsymbol{v} = (4, -5, 1)$, then above equation can be expressed in component form as

$$(x, y, z) = (1, 2, -3) + t\,(4, -5, 1)$$

Equating corresponding components on the two sides of this equation yields the parametric equations

$$x = 1 + 4t, \quad y = 2 - 5t, \quad z = -3 + t$$

(c) Specific points on a line can be found by substituting numerical values for the parameter t.

For example, if we take $t = 0$ in part (b), we obtain the point $(x, y, z) = (1, 2, -3)$, which is the given point P_0.

$t = 1$ yields the point $(5, -3, -2)$ and

$t = -1$ yields the point $(-3, 7, -4)$.

Vector Equation of a Plane

Let x_0 be a fixed point on the required plane W and $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ be two nonzero vectors that are parallel to W and are not scalar multiples of one another. If x is any variable point in the plane W. Suppose $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ have their initial points at x_0, we can create a parallelogram with adjacent side's $t_1\boldsymbol{v_1}$ and $t_2\boldsymbol{v_2}$ in which $\boldsymbol{x} - \boldsymbol{x_0}$ is the diagonal given by the sum

$$\boldsymbol{x} - \boldsymbol{x_0} = t_1\boldsymbol{v_1} + t_2\boldsymbol{v_2}$$

or, equivalently, $$\boldsymbol{x} = \boldsymbol{x_0} + t_1\boldsymbol{v_1} + t_2\boldsymbol{v_2} \quad \text{---------------------(1)}$$

where t_1 and t_2 *are parameters* vary independently from $-\infty$ to $+\infty$,

This is a ***vector equation of the plane*** through $\boldsymbol{x_0}$ and parallel to the vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. In the special case where $x_0 = 0$, then vector equation of the plane passes through the origin takes the form

$$\boldsymbol{x} = t_1\boldsymbol{v}_1 + t_2\boldsymbol{v}_2 \qquad (-\infty < t_1 < +\infty, -\infty < t_2 < +\infty)$$

Parametric Equations of a Plane

Let $\boldsymbol{x} = (x, y, z)$ be a general or variable point in the plane passes through a fixed point $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ and parallel to the vectors $\boldsymbol{v}_1 = (a_1, b_1, c_1)$ and $\boldsymbol{v}_2 = (a_2, b_2, c_2)$, then the component form of eq. 1 will be

$$(x, y, z) = (x_0, y_0, z_0) + t_1(a_1, b_1, c_1) + t_2(a_2, b_2, c_2)$$

Equating corresponding components, we get

$$x = x_0 + a_1t_1 + a_2t_2$$
$$y = y_0 + b_1t_1 + b_2t_2 \qquad (-\infty < t_1 < +\infty, -\infty < t_2 < +\infty)$$
$$z = z_0 + c_1t_1 + c_2t_2$$

These are called the parametric equations for this plane.

Example 9 (Vector and Parametric Equations of Planes)

(a) Find vector and parametric equations of the plane that passes through the origin of $\boldsymbol{R}^3$ and is parallel to the vectors $\boldsymbol{v}_1 = (1, -2, 3)$ and $\boldsymbol{v}_2 = (4, 0, 5)$.

(b) Find three points in the plane obtained in part (a).

Solution

(a) As vector equation of the plane passing through origin is $x = t_1v_1 + t_2v_2$.

Let $\boldsymbol{x} = (x, y, z)$ then this equation can be expressed in component form as

$$(x, y, z) = t_1(1, -2, 3) + t_2 (4, 0, 5)$$

This is the vector equation of the plane.

Equating corresponding components, we get

$$x = t_1 + 4t_2, \quad y = -2t_1, \quad z = 3t_1 + 5t_2$$

These are the parametric equations of the plane.

(b) Points in the plane can be obtained by assigning some real values to the parameters t_1 and t_2:

$t_1 = 0$ and $t_2 = 0$ produces the point (0, 0, 0)

$t_1 = -2$ and $t_2 = 1$ produces the point (2, 4, –1)

$t_1 = ½$ and $t_2 = ½$ produces the point (5/2, –1, 4)

Vector equation of Plane through Three Points

If x_0, x_1 and x_2 are three non collinear points in the required plane, then, obviously, the vectors $\boldsymbol{v_1} = \boldsymbol{x_1} - \boldsymbol{x_0}$ and $\boldsymbol{v_2} = \boldsymbol{x_2} - \boldsymbol{x_0}$ are parallel to the plane. So, a vector equation of the plane is

$$\boldsymbol{x} = \boldsymbol{x_0} + t_1(\boldsymbol{x_1} - \boldsymbol{x_0}) + t_2(\boldsymbol{x_2} - \boldsymbol{x_0})$$

Example Find vector and parametric equations of the plane that passes through the points. $P(2, -4, 5)$, $Q(-1, 4, -3)$ and $R(1, 10, -7)$.

Solution

Let $\boldsymbol{x} = (x, y, z)$, and if we take $\boldsymbol{x_0}, \boldsymbol{x_1}$ and $\boldsymbol{x_2}$ to be the points P, Q and R respectively, then $x_1 - x_0 = \overrightarrow{PQ} = (-3, 8, -8)$ and $x_2 - x_0 = \overrightarrow{PR} = (-1, 14, -12)$

So the component form will be

$$(x, y, z) = (2, -4, 5) + t_1(-3, 8, -8) + t_2(-1, 14, -12)$$

This is the required vector equation of the plane.

By equating corresponding components, we get

$$x = 2 - 3t_1 - t_2, \quad y = -4 + 8t_1 + 14t_2, \quad z = 5 - 8t_1 - 12t_2$$

These are the parametric equations of the required plane.

Question: How can you tell that the points P, Q and R are not collinear?

Finding a Vector Equation from Parametric Equations

Example 11 Find a vector equation of the plane whose parametric equations are

$$x = 4 + 5t_1 - t_2, \quad y = 2 - t_1 + 8t_2, \quad z = t_1 + t_2$$

Solution First we rewrite the three equations as the single vector equation

$$(x, y, z) = (4 + 5t_1 - t_2, 2 - t_1 + 8t_2, t_1 + t_2)$$

$$\Rightarrow (x, y, z) = (4, 2, 0) + (5t_1, -t_1, t_1) + (-t_2, 8t_2, t_2)$$

$$\Rightarrow (x, y, z) = (4, 2, 0) + t_1(5, -1, 1) + t_2(-1, 8, 1)$$

This is a vector equation of the plane that passes through the point (4, 2, 0) and is parallel to the vectors $\boldsymbol{v_1}$ = (5, –1, 1) and $\boldsymbol{v_2}$ = (–1, 8, 1).

Finding Parametric Equations from a General Equation

Example 12 Find parametric equations of the plane $x - y + 2z = 5$.

Solution First we solve the given equation for x in terms of y and z

$$x = 5 + y - 2z$$

Now make y and z into parameters, and then express x in terms of these parameters.

Let y = t_1 and $z = t_2$

Then the parametric equations of the given plane are

$$x = 5 + t_1 - 2t_2, \quad y = t_1, \quad z = t_2$$

Exercises

1. Prove that $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{R}^n$.

2. For what value(s) of $\boldsymbol{h}$, $\boldsymbol{y}$ belongs to Span **{*v₁*, *v₂*, *v₃*}?** Where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \quad v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, and \quad y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

3. Determine whether $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, and $\boldsymbol{a_3}$.

i). $a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, a_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

ii). $a_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, a_2 = \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix}, a_3 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

4. Determine if ***b*** is a linear combination of the vectors formed from the columns of the matrix ***A***.

i). $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 5 & 0 \\ 2 & 5 & 8 \end{bmatrix}, b = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

ii). $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

In exercises 7-10, list seven vectors in Span **{$\boldsymbol{v_1}$, $\boldsymbol{v_2}$}.** For each vector, show that the weights on $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ used to generate the vector and list the three entries of the vector. Give also geometric description of the Span **{$\boldsymbol{v_1}$, $\boldsymbol{v_2}$}.**

7. $v_1 = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$

8. $v_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

9. $v_1 = \begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ -9 \\ 6 \end{pmatrix}$

10. $v_1 = \begin{pmatrix} -3.7 \\ -0.4 \\ 11.2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 5.8 \\ 2.1 \\ 5.3 \end{pmatrix}$

11. Let $a_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, a_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$. For what value(s) of h is ***b*** in the plane spanned by $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$?

12. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, and\ y = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$. For what value(s) of h is ***y*** in the plane generated by $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$?

13. Let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix} and \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in Span{u, v} for all h and k.

Lecture 6

Matrix Equations

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition will permit us to rephrase some of the earlier concepts in new ways.

Definition If ***A*** is an $m \times n$ matrix, with columns $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}$ and if ***x*** is in $\boldsymbol{R^n}$, then the product of ***A*** and ***x*** denoted by ***Ax***, is the linear combination of the columns of ***A*** using the corresponding entries in ***x*** as weights, that is,

$$Ax = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$$

Note that ***Ax*** is defined only if the number of columns of ***A*** equals the number of entries in ***x***.

Example 1

a) $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

b) $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

Example 2 For $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ in $\boldsymbol{R^m}$, write the linear combination $3\boldsymbol{v_1} - 5\boldsymbol{v_2} + 7\boldsymbol{v_3}$ as a matrix times a vector.

Solution Place $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ into the columns of a matrix ***A*** and place the weights 3, -5, and 7 into a vector ***x***.

That is, $$3v_1 - 5v_2 + 7v_3 = [v_1 \quad v_2 \quad v_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = Ax$$

We know how to write a system of linear equations as a vector equation involving a linear combination of vectors. For example, we know that the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \quad \text{is equivalent to} \quad x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}2\\-5\end{bmatrix} + x_3\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$$

Writing the linear combination on the left side as a matrix times a vector, we get

$$\begin{bmatrix}1 & 2 & -1\\0 & -5 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$$

Which has the form $\boldsymbol{Ax} = \boldsymbol{b}$, and we shall call such an equation a **matrix equation**, to distinguish it from a vector equation.

<u>Theorem</u> 1 If $\boldsymbol{A}$ is an $m \times n$ matrix, with columns $\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n}$ and if $\boldsymbol{b}$ is in $\boldsymbol{R^m}$, the matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$ has the same solution set as the vector equation

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \dots + x_n\boldsymbol{a_n} = \boldsymbol{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is $\begin{bmatrix}a_1 & a_2 & \dots & a_n & b\end{bmatrix}$

<u>Existence of Solutions</u> The equation $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is a linear combination of the columns of $\boldsymbol{A}$.

<u>Example 3</u> Let $A = \begin{bmatrix}1 & 3 & 4\\-4 & 2 & -6\\-3 & -2 & -7\end{bmatrix}$ and $b = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$.

Is the equation $\boldsymbol{Ax} = \boldsymbol{b}$ consistent for all possible b_1, b_2, b_3?

<u>Solution</u> Row reduce the augmented matrix for $\boldsymbol{Ax} = \boldsymbol{b}$:

$$\begin{bmatrix}1 & 3 & 4 & b_1\\-4 & 2 & -6 & b_2\\-3 & -2 & -7 & b_3\end{bmatrix} \overset{4R_1+R_2,\,3R_1+R_3}{\sim} \begin{bmatrix}1 & 3 & 4 & b_1\\0 & 14 & 10 & b_2+4b_1\\0 & 7 & 5 & b_3+3b_1\end{bmatrix}$$

$$R_3 - \frac{1}{2}R_2$$

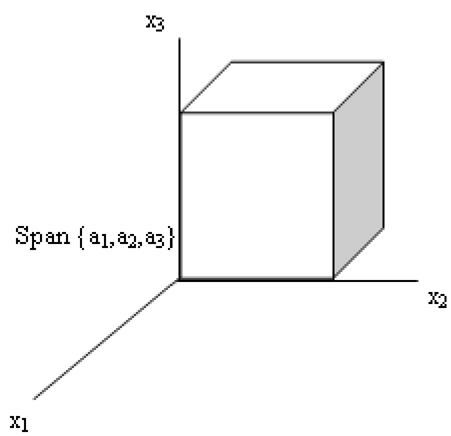
$$\sim\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & b_3+3b_1-\frac{1}{2}(b_2+4b_1) \end{bmatrix}$$

The third entry in the augmented column is $b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1)$

The equation ***Ax = b*** is not consistent for every ***b*** because some choices of ***b*** can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

The entries in ***b*** must satisfy $b_1 - \frac{1}{2}b_2 + b_3 = 0$

This is the equation of a plane through the origin in $\boldsymbol{R^3}$. The plane is the set of all linear combinations of the three columns of ***A***. See figure below.



The equation ***Ax = b*** fails to be consistent for all ***b*** because the echelon form of ***A*** has a row of zeros. If ***A*** had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as [0 0 0 1].

Example 4 Which of the following are linear combinations of

$$A=\begin{bmatrix}4 & 0\\ -2 & -2\end{bmatrix},\qquad B=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix},\qquad C=\begin{bmatrix}0 & 2\\ 1 & 4\end{bmatrix}$$

(a) $\begin{bmatrix}6 & -8\\ -1 & -8\end{bmatrix}$

(b) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ **(c)** $\begin{bmatrix}6 & 0\\ 3 & 8\end{bmatrix}$

Solution

(a) $\begin{bmatrix}6 & -8\\ -1 & -8\end{bmatrix}$ = a A + b B + c C

$$= a\begin{bmatrix}4 & 0\\ -2 & -2\end{bmatrix} + b\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix} + c\begin{bmatrix}0 & 2\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}4a+b & -b+2c\\ -2a+2b+c & -2a+3b+4c\end{bmatrix}$$

$\Rightarrow$ 4a + b = 6 (1)

-b + 2c = -8 (2)

-2a + 2b + c = -1 (3)

-2a + 3b + 4c = -8 (4)

Subtracting equation (4) from equation (3), we obtain

-b – 3c = 7 (5)

Subtracting equation (5) from equation (2):

5c = -15 $\Rightarrow$ c = -3

From (2), -b + 2(-3) = -8 $\Rightarrow$ b = 2

From (3), -2a + 2(2) – 3 = -1 $\Rightarrow$ a = 1

Now we check whether these values satisfy equation (1).

4(1) + 2 = 6

It means that $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is the linear combination of **A**, **B** and **C**.

Thus

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = 1A + 2B - 3C$$

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = a\,A + b\,B + c\,C$

$$= a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4a+b & -b+2c \\ -2a+2b+c & -2a+3b+4c \end{bmatrix}$$

$\Rightarrow$ 4a + b = 0 (1)

-b + 2c = 0 (2)

-2a + 2b + c = 0 (3)

-2a + 3b + 4c = 0 (4)

Subtracting equation (3) from equation (4) we get

b + 3c = 0 (5)

Adding equation (2) and equation (5), we get

5c = 0 $\Rightarrow$ c = 0

Put c = 0 in equation (5), we get b = 0

Put b = c = 0 in equation (3), we get a = 0

$\Rightarrow$ a = b = c =0

It means that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the linear combination of **A**, **B** and **C**.

Thus $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0A + 0B + 0C$

(c) $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = a\,A + b\,B + c\,C$

$$= a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4a+b & -b+2c \\ -2a+2b+c & -2a+3b+4c \end{bmatrix}$$

$\Rightarrow$ 4a + b = 6 (1)

-b + 2c = 0 (2)

-2a + 2b + c = 3 (3)

-2a + 3b + 4c = 8 (4)

Subtracting (4) from (3), we obtain

-b – 3c = -5 (5)

Subtracting (5) from (2):

5c = 5 $\Rightarrow$ c = 1

From (2), -b + 2(1) = 0 $\Rightarrow$ b = 2

From (3), -2a + 2(2) + 1 = 3 $\Rightarrow$ a = 1

Now we check whether these values satisfy (1).

4(1) + 2 = 6

It means that $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$ is the linear combination of **A**, **B** and **C**.

Thus $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$ = 1**A** + 2**B** +1**C**

Theorem 2 Let ***A*** be an $m \times n$ matrix. Then the following statements are logically equivalent.

(a) For each ***b*** in $\boldsymbol{R^m}$, the equation ***Ax = b*** has a solution.

(b) The columns of ***A*** Span $\boldsymbol{R^m}$.

(c) A has a pivot position in every row.

This theorem is one of the most useful theorems. It is about a coefficient matrix, not an augmented matrix. If an augmented matrix [***A*** ***b***] *has a pivot position in every row, then the equation* ***Ax = b*** *may or may not be consistent.*

Example 4 Compute ***Ax***, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ *and* $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Solution From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

Note

In above example the first entry in ***Ax*** is a sum of products (sometimes called a **dot product**), using the first row of ***A*** and the entries in ***x***.

That is $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \end{bmatrix}$

Examples

In each part determine whether the given vector span R^3

(a) $v_1 = (2, 2, 2), v_2 = (0, 0, 3),$
$v_3 = (0, 1, 1)$

(b) $v_1 = (3, 1, 4),\ v_2 = (2, -3, 5),$
$v_3 = (5, -2, 9),\ v_4 = (1, 4, -1)$

(c) $v_1 = (1, 2, 6),\ v_2 = (3, 4, 1),$
$v_3 = (4, 3, 1),\ v_4 = (3, 3, 1)$

Solutions

(a) We have to determine whether arbitrary vectors $b=(b_1,\, b_2,\, b_3)$ in R^3 can be expressed as a linear combination $b = k_1v_1 + k_2v_2 + k_3v_3$ of the vectors v_1, v_2, v_3

Expressing this in terms of components given by

$$(b_1,\, b_2,\, b_3) = k_1(2,2,2) + k_2(0,0,3) + k_3(0,1,1)$$
$$(b_1,\, b_2,\, b_3) = (2k_1 + 0k_2 + 0k_3, 2k_1 + 0k_2 + k_3, 2k_1 + 3k_2 + k_3)$$
$$2k_1 + 0k_2 + 0k_3 = b_1$$
$$2k_1 + 0k_2 + k_3 = b_2$$
$$2k_1 + 3k_2 + k_3 = b_3$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$ has a non zero determinant

Now

$$\det(A) = -6 \neq 0$$

Therefore v_1, v_2, v_3 span R^3

(b) The set S{ v_1, v_2, v_3, v_4 }of vectors in R^3 spans V= R^3 if

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = d_1w_1 + d_2w_2 + d_3w_3 \quad ....(1)$$

with

$$w_1 = (1,0,0)$$
$$w_2 = (0,1,0)$$
$$w_3 = (0,0,1)$$

With our vectors v_1, v_2, v_3, v_4 equation (1) becomes

$$c_1(3,1,4) + c_2(2,-3,5) + c_3(5,-2,9) + c_4(1,4,-1) = d_1(1,,0,0) + d_2(0,1,0) + d_3(0,0,1)$$

Rearranging the left hand side yields

$$3c_1 + 2c_2 + 5c_3 + 1c_4 = 1d_1 + 0d_2 + 0d_3$$

$$1c_1 - 3c_2 - 2c_3 + 4c_4 = 0d_1 + 1d_2 + 0d_3$$

$$4c_1 + 5c_2 + 9c_3 - 1c_4 = 0d_1 + 0d_2 + 1d_3$$

$$\begin{bmatrix} 3 & 2 & 5 & 1 & 1 & 0 & 0 \\ 1 & -3 & -2 & 4 & 0 & 1 & 0 \\ 4 & 5 & 9 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -1 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -3 & -2 \end{bmatrix}$$

The reduce row echelon form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & \frac{5}{17} & \frac{3}{17} \\ 0 & 1 & 1 & -1 & 0 & \frac{-4}{17} & \frac{1}{17} \\ 0 & 0 & 0 & 0 & 1 & \frac{-7}{17} & -\frac{11}{17} \end{bmatrix}$$ Corresponds to the system of equations

$$1c_1 + 1c_3 + 1c_4 = (\frac{5}{17})d_2 + (\frac{3}{17})d_3$$

$$1c_2 + 1c_3 + -1c_4 = (\frac{-4}{17})d_2 + (\frac{1}{17})d_3 \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

$$0 = 1d_1 + (\frac{-7}{17})d_2 + (-\frac{11}{17})d_3$$

So this system is inconsistent. The set S does not span the space V.

Similarly Part C can be solved by the same way.

<u>Exercise</u>

1. Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}, x = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}, and \quad b = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}.$

   It can be shown that ***Ax* = *b***. Use this fact to exhibit ***b*** as a specific linear combination of the columns of ***A***.

2. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}, u = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, and \quad v = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify ***A(u + v)* = *Au* +*Av***.

3. Solve the equation ***Ax* = *b***, with $A = \begin{bmatrix} 2 & 4 & -6 \\ 0 & 1 & 3 \\ -3 & -5 & 7 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$.

4. Let $u = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} and\ A = \begin{bmatrix} 3 & 5 \\ 1 & 1 \\ -2 & -8 \end{bmatrix}$. Is ***u*** belongs to the plane in R^3 spanned by the columns of A? Why or why not?

5. Let $u = \begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix} and\ A = \begin{bmatrix} 4 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$. Is ***u*** in the subset of R^3 spanned by the columns of A? Why or why not?

6. Let $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} and\ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation ***Ax* = *b*** is not consistent for all possible ***b***, and describe the set of all ***b*** for which ***Ax* = *b*** is consistent.

7. How many rows of $A = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & -1 & 5 \\ -1 & -2 & 1 & 7 \\ 1 & 1 & 0 & -6 \end{bmatrix}$ contain pivot positions?

In exercises 8 to 13 , explain how your calculations justify your answer, and mention an appropriate theorem.

8. Do the columns of the matrix $A = \begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & -6 \\ -5 & -1 & 8 \end{bmatrix}$ span R^3?

9. Do the columns of the matrix $A=\begin{bmatrix}1 & 3 & -2 & -2\\ 0 & 1 & -1 & 5\\ -1 & -2 & 1 & 7\\ 1 & 1 & 0 & -6\end{bmatrix}$ span R^4?

10. Do the columns of the matrix $A=\begin{bmatrix}0 & 0 & 2\\ 0 & -5 & 1\\ 4 & 6 & -3\end{bmatrix}$ span R^3?

11. Do the columns of the matrix $A=\begin{bmatrix}3 & 5\\ 1 & 1\\ -2 & -8\end{bmatrix}$ span R^3?

12. Let $v_1=\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, v_2=\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, v_3=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$. Does $\{v_1, v_2, v_3\}$ span R^4?

13. Let $v_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}, v_2=\begin{bmatrix}-1\\3\\7\end{bmatrix}, v_3=\begin{bmatrix}3\\-2\\-2\end{bmatrix}$. Does $\{v_1, v_2, v_3\}$ span R^3?

14. It can be shown that $\begin{bmatrix}4 & 1 & 2\\ -2 & 0 & 8\\ 3 & 5 & -6\end{bmatrix}\begin{bmatrix}-1\\4\\2\end{bmatrix}=\begin{bmatrix}4\\18\\5\end{bmatrix}$. Use this fact(and no row operations) to find scalars c_1, c_2, c_3 such that $\begin{bmatrix}4\\18\\5\end{bmatrix}=c_1\begin{bmatrix}4\\-2\\3\end{bmatrix}+c_2\begin{bmatrix}1\\0\\5\end{bmatrix}+c_3\begin{bmatrix}2\\8\\-6\end{bmatrix}$.

15. Let $u = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, and\ w = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. It can be shown that 2u – 5v – w = 0. Use this fact(and no row operations) to solve the equation $\begin{bmatrix} 3 & 1 \\ 8 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

Determine if the columns of the matrix span R^4.

16. $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$

17. $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$

Lecture 7

Solution Sets of Linear Systems

Solution Set

A solution of a linear system is an assignment of values to the variables $x_1, x_2, \ldots, x_n$ such that each of the equations in the linear system is satisfied. The set of all possible solutions is called the Solution Set

Homogeneous Linear System

A system of linear equations is said to be **homogeneous** if it can be written in the form ***Ax = 0***, where ***A*** is an $m \times n$ matrix and ***0*** is the zero vector in $\boldsymbol{R^m}$.

Trivial Solution

A homogeneous system ***Ax = 0*** always has at least one solution, namely, ***x = 0*** (the zero vector in $\boldsymbol{R^n}$). This zero solution is usually called the trivial solution of the homogeneous system.

Nontrivial solution

A solution of a linear system other than trivial is called its nontrivial solution.
i.e the solution of a homogenous equation ***Ax = 0*** such that $\boldsymbol{x} \neq \boldsymbol{0}$ is called **nontrivial solution**, that is, a nonzero vector ***x*** that satisfies ***Ax = 0***.

Existence and Uniqueness Theorem

*The homogeneous equation **Ax = 0** has a nontrivial solution if and only if the equation has at least one free variable.*

Example 1 Find the solution set of the following system

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$3x_1 + 2x_2 - 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

Solution

Let $A = \begin{bmatrix} 3 & 5 & -4 \\ 3 & 2 & -4 \\ 6 & 1 & -8 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The augmented matrix is

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ 3 & 2 & -4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

For solution set, row reduce to reduced echelon form

$$\sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \qquad -1R_1 + R_2, -2R_1 + R_3$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad -3R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 1/3R_1, 1/3R_2, 5/3R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (-1)R_2$$

$$\begin{aligned} x_1 \quad -\frac{4}{3}x_3 &= 0 \\ x_2 \quad &= 0 \\ 0 &= 0 \end{aligned}$$

It is clear that x_3 is a free variable, so $Ax = 0$ has nontrivial solutions (one for each choice of x_3). From above equations we have,

$$x_1 = \frac{4}{3}x_3, \quad x_2 = 0, \text{ with } x_3 \text{ free.}$$

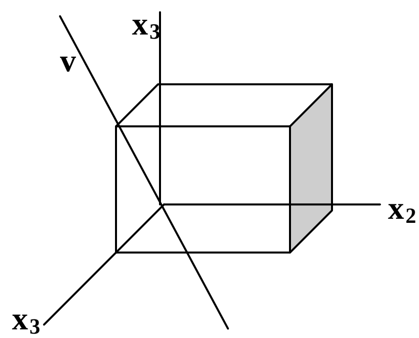
As a vector, the general solution of $Ax = 0$ is given by:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3 v, \quad where \quad v = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

This shows that every solution of $Ax = 0$ in this case is a scalar multiple of v (it means that v generate or spans the whole general solution).The trivial solution is obtained by choosing $x_3 = 0$.

Geometric Interpretation

Geometrically, the solution set is a line through 0 in R^3, as given in the **Figure below:**



Note: A nontrivial solution x can have some zero entries so long as not all of its entries are zero.

Example 2

Solve the following system

$$10x_1 - 3x_2 - 2x_3 = 0 \qquad (1)$$

Solution

Solving for the basic variable x_1 in terms of the free variables,
dividing eq. 1 by 10 and solve for x

$x_1 = 0.3x_2 + 0.2x_3$ *where* x_2 and x_3 free variables.

As a vector, the general solution is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

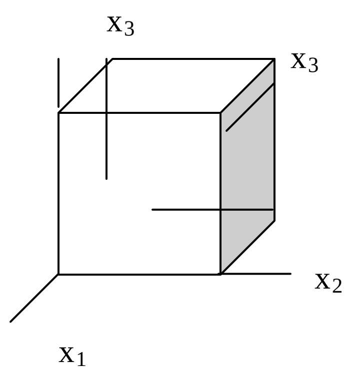
$$= x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} \qquad (2)$$

$$\quad\ \ \mathbf{u} \qquad\quad \mathbf{v}$$

This calculation shows that every solution of (1) is a linear combination of the vector ***u, v*** shown in (2). That is, the solution set is Span {***u, v***}
.

Geometric Interpretation

Since neither ***u*** nor ***v*** is a scalar multiple of the other, so these are not parallel, the solution set is a plane through the origin, see the Figure below:



Note:

Above examples illustrate the fact that the solution set of a homogeneous equation $Ax = 0$ can be expressed explicitly as Span $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ for suitable vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ (because solution sets can be written in the form of linear combination of these vectors). If the only solution is the zero-vector then the solution set is **Span {0}**.

Example 3 (For Practice) Find the solution set of the following homogenous system:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

Solution:

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The augmented matrix is:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \qquad 4R_1 + R_2,$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{1}{2}R_2 \ , \ (-3)R_2 + R_1$$

SO

$$\begin{aligned} x_1 \quad - 5x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

From above results, it is clear that x_3 is a free variable, so $\boldsymbol{Ax = 0}$ has nontrivial solutions (one for each choice of x_3).
From above equations we have,

$$x_1 = 5x_3, \quad x_2 = -2x_3, \text{ with } x_3 \text{ a free variable.}$$

As a vector, the general solution of $\boldsymbol{Ax = 0}$ is given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = x_3 v, \qquad where \quad v = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Parametric Vector Form of the solution

Whenever a solution set is described explicitly with vectors, we say that the solution is in **parametric vector form**

The equation

$$\boldsymbol{x} = s\boldsymbol{u} + t\boldsymbol{v} \qquad (s, t \text{ in } \boldsymbol{R})$$

is called a **parametric vector equation** of the plane. It is written in this form to emphasize that the parameters vary over all real numbers.

Similarly, the equation $\boldsymbol{x} = x_3\boldsymbol{v}$ (with x_3 free), or $\boldsymbol{x} = t\boldsymbol{v}$ (with t in $\boldsymbol{R}$), is a parametric vector equation of a line.

Solutions of Non-homogeneous Systems

When a non-homogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.
To clear this concept consider the following examples,

Example: 5 Describe all solutions of ***Ax = b***, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad and \quad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Solution
Row operations on **[A b]** produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \qquad R_1 + R_2, -2R_1 + R_3$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 3R_2 + R_3,\ \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad -5R_2 + R_1,\ \frac{1}{3}R_1$$

$$\begin{aligned} x_1 \qquad - \frac{4}{3}x_3 &= -1 \\ OR \qquad x_2 \qquad\qquad &= 2 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -1 + \frac{4}{3}x_3, \quad x_2 = 2$, and x_3 is free.
As a vector, the general solution of ***Ax = b*** has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+\frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{p} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{v}$$

The equation $\boldsymbol{x = p + x_3 v}$, or, writing $\boldsymbol{t}$ as a general parameter,

$$\boldsymbol{x = p + tv} \quad (\boldsymbol{t} \text{ in } \boldsymbol{R}) \qquad (3)$$

Note

We know that the solution set of this question when $\boldsymbol{Ax = 0}$ (example 1) has the parametric vector equation

$$\boldsymbol{x = tv} \ (\boldsymbol{t} \text{ in } \boldsymbol{R}) \qquad (4)$$

With the same $\boldsymbol{v}$ that appears in equation (3) in above example.
Thus the solutions of $\boldsymbol{Ax = b}$ are obtained by adding the vector $\boldsymbol{p}$ to the solutions of $\boldsymbol{Ax = 0}$. The vector $\boldsymbol{p}$ itself is just one particular solution of $\boldsymbol{Ax = b}$ (corresponding to $\boldsymbol{t = 0}$ in (3)).

The following theorem gives the precise statement.

Theorem

Suppose the equation *Ax = b* is consistent for some given *b*, and let *p* be a solution. Then the solution set of *Ax = b* is the set of all vectors of the form
$\boldsymbol{w = p + v_h}$, where $\boldsymbol{v_h}$ is any solution of the homogeneous equation Ax = 0.

Example 6: (For practice)

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

Solution

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix} \qquad 4R_1 + R_2,$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (-3)R_2 + R_1$$

SO

$$\begin{aligned} x_1 \qquad - 5x_3 &= -2 \\ x_2 + 2x_3 &= 1 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -2 + 5x_3, \quad x_2 = 1 - 2x_3$, and $\boldsymbol{x_3}$ is free.

As a vector, the general solution of ***Ax* = *b*** has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 5x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{p}} + x_3 \underbrace{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}_{\boldsymbol{v}}$$

So we can write solution set in parametric vector form as

$$x = p + x_3 v$$

Steps of Writing a Solution Set (of a Consistent System) in a Parametric Vector Form

Step 1:
Row reduces the augmented matrix to reduced echelon form.

Step 2:
Express each basic variable in terms of any free variables appearing in an equation.

Step 3:
Write a typical solution ***x*** as a vector whose entries depend on the free variables if any.

Step 4:
Decompose ***x*** into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Exercise

Determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1. $x_1 - 5x_2 + 9x_3 = 0$
$-x_1 + 4x_2 - 3x_3 = 0$
$2x_1 - 8x_2 + 9x_3 = 0$

2. $3x_1 + 6x_2 - 4x_3 - x_4 = 0$
$-5x_1 + 8x_3 + 3x_4 = 0$
$8x_1 - x_2 + 7x_4 = 0$

3. $5x_1 - x_2 + 3x_3 = 0$
$4x_1 - 3x_2 + 7x_3 = 0$

Write the solution set of the given homogeneous system in parametric vector form.

4. $x_1 - 3x_2 - 2x_3 = 0$
$x_2 - x_3 = 0$
$-2x_1 + 3x_2 + 7x_3 = 0$

5. $x_1 + 2x_2 - 7x_3 = 0$
$-2x_1 - 3x_2 + 9x_3 = 0$
$-2x_2 + 10x_3 = 0$

In exercises 6-8, describe all solutions of $\boldsymbol{Ax = 0}$ in parametric vector form where A is row equivalent to the matrix shown.

6. $\begin{bmatrix} 1 & -5 & 0 & 2 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -5 & 0 & 4 \end{bmatrix}$

9. Describe the solution set in R^3 of $x_1 - 4x_2 + 3x_3 = 0$, compare it with the solution set of $x_1 - 4x_2 + 3x_3 = 7$.

10. Find the parametric equation of the line through ***a*** parallel to ***b***.

$$a = \begin{bmatrix} 3 \\ -8 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

11. Find a parametric equation of the line *M* through ***p*** and ***q***.

$$p = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

12. Given $A = \begin{bmatrix} 5 & 10 \\ -8 & -16 \\ 7 & 14 \end{bmatrix}$, find one nontrivial solution of $\mathbf{Ax = 0}$ by inspection.

13. Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$, find one nontrivial solution of $\mathbf{Ax = 0}$ by inspection.