Lecture 25

Rank

With the help of vector space concepts, for a matrix several interesting and useful relationships in matrix rows and columns have been discussed.
For instance, imagine placing 2000 random numbers into a 40 x 50 matrix ***A*** and then determining both the maximum number of linearly independent columns in ***A*** and the maximum number of linearly independent columns in A^T (rows in ***A***). Remarkably, the two numbers are the same. Their common value is called the rank of the matrix. To explain why, we need to examine the subspace spanned by the subspace spanned by the rows of ***A***.

The Row Space If ***A*** is an $m \times n$ matrix, each row of ***A*** has *n* entries and thus can be identified with a vector in R^n. The set of all linear combinations of the row vectors is called the row space of ***A*** and is denoted by Row ***A***. Each row has *n* entries, so Row ***A*** is a subspace of R^n. Since the rows of ***A*** are identified with the columns of A^T, we could also write Col A^T in place of Row ***A***.

Example 1 Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$ and $r_1 = (-2,-5,8,0,-17)$, $r_2 = (1,3,-5,1,5)$, $r_3 = (3,11,-19,7,1)$, $r_4 = (1,7,-13,5,-3)$

The row space of ***A*** is the subspace of R^5 spanned by $\{r_1, r_2, r_3, r_4\}$. That is, Row ***A*** = Span $\{r_1, r_2, r_3, r_4\}$. Naturally, we write row vectors horizontally; however, they could also be written as column vectors

Example Let

$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$ and $r_1 = (2,1,0)$, $r_2 = (3,-1,4)$

That is Row **A=**Span $\{r_1, r_2\}$.

We could use the Spanning Set Theorem to shrink the spanning set to a basis.
Some times row operation on a matrix will not give us the required information but row reducing certainly worthwhile, as the next theorem shows

Theorem 1 If two matrices ***A*** and ***B*** are row equivalent, then their row spaces are the same. If ***B*** is in echelon form, the nonzero rows of ***B*** form a basis for the row space of ***A*** as well as ***B***.

Theorem 2 If ***A*** and ***B*** are row equivalent matrices, then

(a) A given set of column vectors of ***A*** is linearly independent if and only if the corresponding column vectors of ***B*** are linearly independent.
(b) A given set of column vector of ***A*** forms a basis for the column space of ***A*** if and only if the corresponding column vector of ***B*** forms a basis for the column space of ***B***.

Example 2 **(Bases for Row and Column Spaces)**

Find the bases for the row and column spaces of $\boldsymbol{A} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$.

Solution We can find a basis for the row space of ***A*** by finding a basis for the row space of any row-echelon form of ***A***.

Now
$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -2R_1 + R_2 \\ -2R_1 + R_3 \\ R_1 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} -1R_2 + R_3$$

Row-echelon form of ***A***: $\boldsymbol{R} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Here Theorem 1 implies that that the non zero rows are the basis vectors of the matrix. So these bases vectors are

$$\boldsymbol{r_1} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix}$$
$$\boldsymbol{r_2} = \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix}$$
$$\boldsymbol{r_3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

A and ***R*** may have different column spaces, we cannot find a basis for the column space of ***A*** directly from the column vectors of ***R***. however, it follows from the theorem (2b) if we can find a set of column vectors of ***R*** that forms a basis for the column space of ***R***, then the corresponding column vectors of ***A*** will form a basis for the column space of ***A***.

The first, third, and fifth columns of ***R*** contains the leading 1's of the row vectors, so

$$\mathbf{c}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{c}'_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{c}'_5 = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of ***R***, thus the corresponding column vectors of ***A***

namely, $$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \qquad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of ***A***.

<u>Example</u>

The matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row-echelon form.

The vectors

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix}$$

$$\mathbf{r}_2 = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \end{bmatrix}$$

$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

form a basis for the row space of R, and the vectors

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; c_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; c_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

<u>Example 3</u> **(Basis for a Vector Space using Row Operation)**
Find bases for the space spanned by the vectors

$$\mathbf{v}_1 = (1,-2,0,0,3) \qquad \mathbf{v}_2 = (2,-5,-3,-2,6)$$

$$\mathbf{v}_3 = (0,5,15,10,0) \qquad \mathbf{v}_4 = (2,6,18,8,6)$$

<u>Solution</u> **T**he space spanned by these vectors is the row space of the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Transforming Matrix to Row Echelon Form:

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix} \begin{matrix} (-2)R_1 + R_2 \\ (-2)R_1 + R_4 \\ (-1)R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix} \begin{matrix} (-5)R_2 + R_3 \\ (-10)R_2 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & -12 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_{34}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (-1/12)R_3$$

Therefore, $\boldsymbol{R} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The non-zero row vectors in this matrix are

$$\boldsymbol{w_1} = (1,-2,0,0,3), \boldsymbol{w_2} = (0,1,3,2,0), \boldsymbol{w_3} = (0,0,1,1,0)$$

These vectors form a basis for the row space and consequently form a basis for the subspace of $\boldsymbol{R}^5$ spanned by $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$.

<u>Example 4</u> **(Basis for the Row Space of a Matrix)**

Find a basis for the row space of $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$ consisting entirely of row vectors from $\boldsymbol{A}$.

<u>Solution</u> We find $\boldsymbol{A}^T$; then we will use the method of example (2) to find a basis for the column space of $\boldsymbol{A}^T$; and then we will transpose again to convert column vectors back to row vectors. Transposing $\boldsymbol{A}$ yields

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

Transforming Matrix to Row Echelon Form:

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 5 & 10 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 2R_1 + R_2 \\ (-3)R_1 + R_5 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} (-1)R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (3)R_2 + R_3 \\ (2)R_2 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} (-1/12)R_3$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} 12\, R_3 + R_4$$

Now $$\boldsymbol{R} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second and fourth columns contain the leading 1's, so the corresponding column vectors in $\boldsymbol{A}^T$ form a basis for the column space of $\boldsymbol{A}^T$; these are

$$\boldsymbol{c_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \ \boldsymbol{c_2} = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix} \text{and}\, \boldsymbol{c_4} = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

Transposing again and adjusting the notation appropriately yields the basis vectors

$$\boldsymbol{r_1} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \end{bmatrix}, \boldsymbol{r_2} = \begin{bmatrix} 2 & -5 & -3 & -2 & 6 \end{bmatrix} \textit{and}\ \boldsymbol{r_4} = \begin{bmatrix} 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

for the row space of $\boldsymbol{A}$.

The following example shows how one sequence of row operations on ***A*** leads to bases for the three spaces: Row ***A***, Col ***A***, and Nul ***A***.

Example 5 Find bases for the row space, the column space and the null space of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Solution To find bases for the row space and the column space, row reduce ***A*** to an echelon form: $\boldsymbol{A} \sim \boldsymbol{B} = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

By Theorem (1), the first three rows of ***B*** form a basis for the row space of ***A*** (as well as the row space of ***B***). Thus Basis for Row ***A***:

$\{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

For the column space, observe from ***B*** that the pivots are in columns *1, 2* and *4*. Hence columns *1, 2* and *4* of ***A*** (not ***B***) form a basis for Col ***A***:

$$\text{Basis for Col } \boldsymbol{A} : \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

Any echelon form of ***A*** provides (in its nonzero rows) a basis for Row ***A*** and also identifies the pivot columns of ***A*** for Col ***A***. However, for Nul ***A***, we need the reduced echelon form. Further row operations on ***B*** yield

$$\boldsymbol{A} \sim \boldsymbol{B} \sim \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $\boldsymbol{Ax} = \boldsymbol{0}$ is equivalent to $\boldsymbol{Cx} = \boldsymbol{0}$, that is,

$$\begin{aligned} x_1 + \quad x_3 \quad + x_5 &= 0 \\ x_2 - 2x_3 \quad + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables. The usual calculations (discussed in lecture 21) show that

$$\text{Basis for Nul } \boldsymbol{A} : \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Observe that, unlike the bases for Col ***A***, the bases for Row ***A*** and Nul ***A*** have no simple connection with the entries in ***A*** itself.

Note

1. Although the first three rows of ***B*** in Example (5) are linearly independent, it is wrong to conclude that the first three rows of ***A*** are linearly independent. (In fact, the third row of A is 2 times the first row plus 7 times the second row).
2. Row operations do not preserve the linear dependence relations among the rows of a matrix.

Definition The **rank** of ***A*** is the dimension of the column space of ***A***.
Since Row ***A*** is the same as Col $\boldsymbol{A}^T$, the dimension of the row space of ***A*** is the rank of $\boldsymbol{A}^T$. The dimension of the null space is sometimes called the **nullity** of ***A***.

Theorem 3 (The Rank Theorem) The dimensions of the column space and the row space of an $m \times n$ matrix ***A*** are equal. This common dimension, the rank of ***A***, also equals the number of pivot positions in ***A*** and satisfies the equation

$$\text{rank } \boldsymbol{A} + \dim \text{Nul } \boldsymbol{A} = n$$

Example 6

(a) If ***A*** is a 7×9 matrix with a two – dimensional null space, what is the rank of ***A***?
(b). Could a 6×9 matrix have a two – dimensional null space?

Solution

(a) Since ***A*** has 9 columns, $(\text{rank } \boldsymbol{A}) + 2 = 9$ and hence rank $\boldsymbol{A} = 7$.
(b) No, If a 6×9 matrix, call it ***B***, had a two – dimensional null space, it would have to have rank 7, by the Rank Theorem. But the columns of ***B*** are vectors in $\boldsymbol{R}^6$ and so the dimension of Col ***B*** cannot exceed 6; that is, rank ***B*** cannot exceed 6.

The next example provides a nice way to visualize the subspaces we have been studying. Later on, we will learn that Row ***A*** and Nul ***A*** have only the zero vector in common and are actually "perpendicular" to each other. The same fact will apply to Row $\boldsymbol{A}^T$ (= Col ***A***) and Nul $\boldsymbol{A}^T$. So the figure in Example (7) creates a good mental image for the general case.

Example 7 Let $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$. It is readily checked that Nul $\boldsymbol{A}$ is the x_2 – axis, Row $\boldsymbol{A}$ is the x_1x_3 **–** plane, Col $\boldsymbol{A}$ is the plane whose equation is $x_1 - x_2 = 0$ and Nul $\boldsymbol{A}^T$ is the set of all multiples of *(1, -1, 0)*. Figure 1 shows Nul $\boldsymbol{A}$ and Row $\boldsymbol{A}$ in the domain of the linear transformation $\boldsymbol{x} \to \boldsymbol{Ax}$; the range of this mapping, Col $\boldsymbol{A}$, is shown in a separate copy of $\boldsymbol{R}^3$, along with Nul $\boldsymbol{A}^T$.

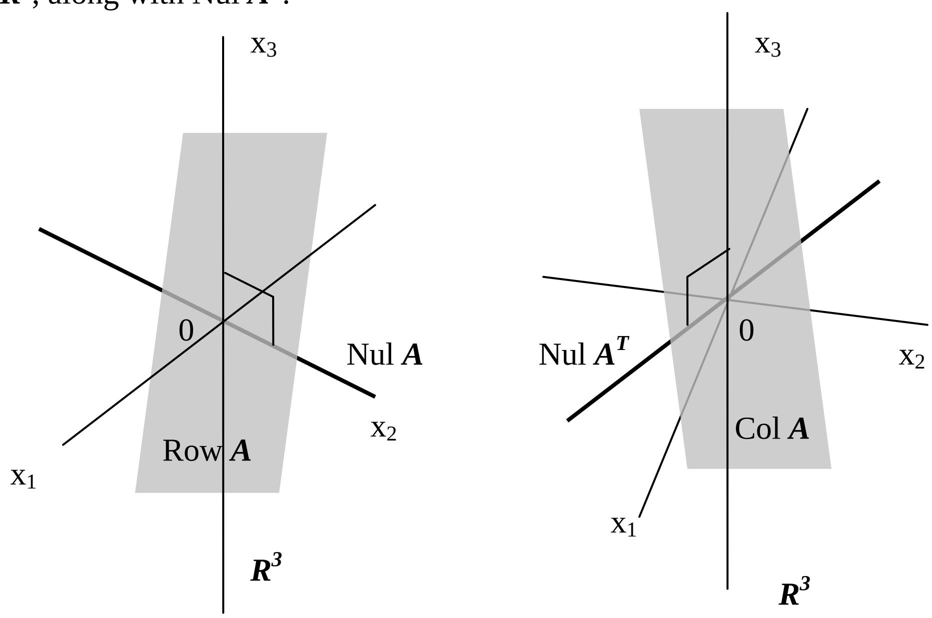

Figure 1 – Subspaces associated with a matrix *A*

Applications to Systems of Equations

The Rank Theorem is a powerful tool for processing information about systems of linear equations. The next example simulates the way a real-life problem using linear equations might be stated, without explicit mention of linear algebra terms such as matrix, subspace and dimension.

Example 8 A scientist has found two solutions to a homogeneous system of *40* equations in *42* variables. The two solutions are not multiples and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be certain that an associated non-homogeneous system (with the same coefficients) has a solution?

Solution Yes. Let $\boldsymbol{A}$ be the 40×42 coefficient matrix of the system. The given information implies that the two solutions are linearly independent and span Nul $\boldsymbol{A}$. So dim Nul $\boldsymbol{A} = 2$. By the Rank Theorem, dim Col $\boldsymbol{A} = 42 - 2 = 40$. Since $\boldsymbol{R}^{40}$ is the only subspace of $\boldsymbol{R}^{40}$ whose dimension is *40*, Col $\boldsymbol{A}$ must be all of $\boldsymbol{R}^{40}$. This means that every non-homogeneous equation $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution.

Example 9 Find the rank and nullity of the matrix $A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$.

Verify that values obtained verify the dimension theorem.

Solution

$$\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix} (-1)R_1$$

$$\begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix} \begin{matrix} (-3)R_1 + R_2 \\ (-2)R_1 + R_3 \\ (-4)R_1 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix} (-1)R_2$$

$$\begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 + R_3 \\ R_2 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} 2R_2 + R_1$$

The reduced row-echelon form of ***A*** is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (1)$$

The corresponding system of equations will be

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

or, on solving for the leading variables,

$$\begin{aligned} x_1 &= 4x_3 - 28x_4 + 37x_5 - 13x_6 \\ x_2 &= 2x_3 + 12x_4 + 16x_5 - 5x_6 \end{aligned} \qquad (2)$$

it follows that the general solution of the system is

$$x_1 = 4r + 28s + 37t - 13u$$
$$x_2 = 2r + 12s + 16t - 5u$$
$$x_3 = r$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = u$$

or equivalently, $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (3)$$

The four vectors on the right side of (3) form a basis for the solution space, so nullity $(\boldsymbol{A}) = 4$. The matrix $\boldsymbol{A} = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$ has 6 columns,

so rank$(\boldsymbol{A})$ + nullity$(\boldsymbol{A}) = 2 + 4 = 6 = n$

Example 10 Find the rank and nullity of the matrix; then verify that the values obtained satisfy the dimension theorem $\boldsymbol{A} = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

Solution Transforming Matrix to the Reduced Row Echelon Form:

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} \begin{matrix} (-2)R_1 + R_3 \\ (-3)R_1 + R_4 \\ 2R_1 + R_5 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} (1/3)R_2$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (-3)\,R_2 + R_3 \\ (-3)\,R_2 + R_4 \\ (-3)R_2 + R_5 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (-1/12)R_3$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} 12R_3 + R_4$$

$$\begin{bmatrix} 1 & -3 & 0 & 2 & 11/6 \\ 0 & 1 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (-2\ )\,R_3 + R_2 \\ (-2\ )\,R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 4/3 \\ 0 & 1 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (3)\,R_2 + R_1 \qquad (1)$$

Since there are three nonzero rows (or equivalently, three leading 1's) the row space and column space are both three dimensional so rank $(\boldsymbol{A}) = 3$.

To find the nullity of $\boldsymbol{A}$, we find the dimension of the solution space of the linear system $\boldsymbol{Ax} = \boldsymbol{0}$. The system can be solved by reducing the augmented matrix to reduced row echelon form. The resulting matrix will be identical to (1), except with an additional last column of zeros, and the corresponding system of equations will be

$$x_1 + 0x_2 + 0x_3 + 2x_4 + \frac{4}{3}x_5 = 0$$

$$0x_1 + x_2 + 0x_3 + 0x_4 - \frac{1}{6}x_5 = 0$$

$$0x_1 + 0x_2 + x_3 + 0x_4 - \frac{5}{12}x_5 = 0$$

The system has infinitely many solutions:

$x_1 = -2\,x_4 + (-4/3)\,x_5$ $\qquad$ $x_2 = (1/6)\,x_5$

$x_3 = (5/12)\,x_5$ $\qquad$ $x_4 = s$

$x_5 = t$

The solution can be written in the vector form:

$\boldsymbol{c_4} = (-2, 0, 0, 1, 0)$ $\qquad$ $\boldsymbol{c_5} = (-4/3, 1/6, 5/12, 0, 1)$

Therefore the **null space** has a basis formed by the set

$$\{(-2, 0, 0, 1, 0), (-4/3, 1/6, 5/12, 0, 1)\}$$

The nullity of the matrix is 2. Now Rank $(\boldsymbol{A})$ + nullity $(\boldsymbol{A}) = 3 + 2 = 5 = n$

Theorem 4 If $\boldsymbol{A}$ is an m x n, matrix, then
(a) rank ($\boldsymbol{A}$) = the number of leading variables in the solution of $\boldsymbol{Ax} = \boldsymbol{0}$
(b) nullity ($\boldsymbol{A}$) = the number of parameters in the general solution of $\boldsymbol{Ax} = \boldsymbol{0}$

Example 11 Find the number of parameters in the solution set of $\boldsymbol{Ax} = \boldsymbol{0}$ if $\boldsymbol{A}$ is a 5×7 matrix of rank 3.
Solution nullity $(\boldsymbol{A}) = n -$ rank $(\boldsymbol{A}) = 7-3 = 4$
Thus, there are four parameters.

Example Find the number of parameters in the solution set of $\boldsymbol{Ax} = \boldsymbol{0}$ if $\boldsymbol{A}$ is a 4×4 matrix of rank 0.
Solution nullity $(\boldsymbol{A}) = n -$ rank $(\boldsymbol{A}) = 4-0 = 4$
Thus, there are four parameters.

Theorem 5 If $\boldsymbol{A}$ is any matrix, then rank $(\boldsymbol{A})$ = rank $(\boldsymbol{A}^T)$

Four fundamental matrix spaces

If we consider a matrix $\boldsymbol{A}$ and its transpose $\boldsymbol{A}^T$ together, then there are six vectors spaces of interest:

Row space of $\boldsymbol{A}$	row space of $\boldsymbol{A}^T$
Column space of $\boldsymbol{A}$	column space of $\boldsymbol{A}^T$
Null space of $\boldsymbol{A}$	null space of $\boldsymbol{A}^T$

However, transposing a matrix converts row vectors into column vectors and column vectors into row vectors, so that, except for a difference in notation, the row space of $\boldsymbol{A}^T$ is the same as the column space of $\boldsymbol{A}$ and the column space of $\boldsymbol{A}^T$ is the same as row space of of $\boldsymbol{A}$.
This leaves four vector spaces of interest:

Row space of $\boldsymbol{A}$	column space of $\boldsymbol{A}$
Null space of $\boldsymbol{A}$	null space of $\boldsymbol{A}^T$

These are known as the **fundamental matrix spaces** associated with $\boldsymbol{A}$, if $\boldsymbol{A}$ is an m x n matrix, then the row space of $\boldsymbol{A}$ and null space of $\boldsymbol{A}$ are subspaces of $\boldsymbol{R}^n$ and the column space of $\boldsymbol{A}$ and the null space of $\boldsymbol{A}^T$ are subspaces of $\boldsymbol{R}^m$.

Suppose now that $\boldsymbol{A}$ is an m x n matrix of rank r, it follows from theorem (5) that $\boldsymbol{A}^T$ is an n x m matrix of rank r . Applying theorem (3) on $\boldsymbol{A}$ and $\boldsymbol{A}^T$ yields

$$\text{Nullity } (\boldsymbol{A}) = n-r, \text{ nullity } (\boldsymbol{A}^T) = m-r$$

From which we deduce the following table relating the dimensions of the four fundamental spaces of an m x n matrix $\boldsymbol{A}$ of rank r.

Fundamental space	**Dimension**
Row space of ***A***	*r*
Column space of ***A***	*r*
Null space of ***A***	*n-r*
Null space of A^T	*m-r*

Example 12 If ***A*** is a *7* x *4* matrix, then the rank of ***A*** is at most *4* and, consequently, the seven row vectors must be linearly dependent. If ***A*** is a *4* x *7* matrix, then again the rank of ***A*** is at most *4* and, consequently, the seven column vectors must be linearly dependent.

Rank and the Invertible Matrix Theorem The various vector space concepts associated with a matrix provide several more statements for the Invertible Matrix Theorem. We list only the new statements here, but we reference them so they follow the statements in the original Invertible Matrix Theorem in lecture 13.

Theorem 6 The Invertible Matrix Theorem (Continued)
Let ***A*** be an *n* x *n* matrix. Then the following statements are each equivalent to the statement that ***A*** is an invertible matrix.

m. The columns of ***A*** form a basis of R^n.
n. Col $A = R^n$.
o. dim Col $A = n$
p. rank $A = n$
q. Nul $A = \{0\}$
r. dim Nul $A = 0$

Proof Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning. The other statements above are linked into the theorem by the following chain of almost trivial implications:

$$(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$$

Only the implication (p) $\Rightarrow$ (r) bears comment. It follows from the Rank Theorem because ***A*** is $n \times n$. Statements (d) and (g) are already known to be equivalent, so the chain is a circle of implications.

We have refrained from adding to the Invertible Matrix Theorem obvious statements about the row space of ***A***, because the row space is the column space of A^T. Recall from (1) of the Invertible Matrix Theorem that ***A*** is invertible if and only if A^T is invertible. Hence every statement in the Invertible Matrix Theorem can also be stated for A^T.

Numerical Note

Many algorithms discussed in these lectures are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.
Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix.

For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as *7* in a computer, then the rank may be *1* or *2*, depending on whether the computer treats $x - 7$ as zero.
In practical applications, the effective rank of a matrix ***A*** is often determined from the singular value decomposition of ***A***.

Example 13 The matrices below are row equivalent

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank ***A*** and dim Nul ***A***.
2. Find bases for Col ***A*** and Row ***A***.
3. What is the next step to perform if one wants to find a basis for Nul ***A***?
4. How many pivot columns are in a row echelon form of A^T?

Solution

1. ***A*** has two pivot columns, so rank ***A*** = *2*. Since ***A*** has *5* columns altogether, dim Nul ***A*** = *5 – 2 = 3*.
2. The pivot columns of ***A*** are the first two columns. So a basis for Col ***A*** is

$$\{a_1, a_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$$

   The nonzero rows of ***B*** form a basis for Row ***A***, namely *{(1, –2, –4, 3, –2), (0, 3, 9, –12, 12)}*. In this particular example, it happens that any two rows of ***A*** form a basis for the row space, because the row space is two-dimensional and none of the rows of ***A*** is a multiple of another row. In general, the nonzero rows of an echelon form of ***A*** should be used as a basis for Row ***A***, not the rows of ***A*** itself.
3. For Nul ***A***, the next step is to perform row operations on ***B*** to obtain the reduced echelon form of ***A***.
4. Rank A^T = rank ***A***, by the Rank Theorem, because Col A^T = Row ***A***. So A^T has two pivot positions.

Exercises

In exercises 1 to 4, assume that the matrix ***A*** is row equivalent to ***B***. Without calculations, list rank ***A*** and dim Nul ***A***. Then find bases for Col ***A***, Row ***A***, and Nul ***A***.

1. $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. $\boldsymbol{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\boldsymbol{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

5. If a 3 x 8 matrix ***A*** has rank 3, find dim Nul ***A***, dim Row ***A***, and rank $\boldsymbol{A}^{\boldsymbol{T}}$.

6. If a 6 x 3 matrix ***A*** has rank 3, find dim Nul ***A***, dim Row ***A***, and rank $\boldsymbol{A}^{\boldsymbol{T}}$.

7. Suppose that a 4 x 7 matrix ***A*** has four pivot columns. Is Col ***A*** = $\boldsymbol{R}^{4}$? Is Nul ***A*** = $\boldsymbol{R}^{3}$? Explain your answers.

8. Suppose that a 5 x 6 matrix ***A*** has four pivot columns. What is dim Nul ***A***? Is Col ***A*** = $\boldsymbol{R}^{4}$? Why or why not?

9. If the null space of a 5 x 6 matrix ***A*** is 4-dimensional, what is the dimension of the column space of ***A***?

10. If the null space of a 7 x 6 matrix ***A*** is 5-dimensional, what is the dimension of the column space of ***A***?

11. If the null space of an 8 x 5 matrix ***A*** is 2-dimensional, what is the dimension of the row space of ***A***?

12. If the null space of a 5 x 6 matrix ***A*** is 4-dimensional, what is the dimension of the row space of ***A***?

13. If ***A*** is a 7 x 5 matrix, what is the largest possible rank of ***A***? If ***A*** is a 5 x 7 matrix, what is the largest possible rank of ***A***? Explain your answers.

14. If ***A*** is a 4 x 3 matrix, what is the largest possible dimension of the row space of ***A***? If ***A*** is a 3 x 4 matrix, what is the largest possible dimension of the row space of ***A***? Explain.

15. If ***A*** is a 6 x 8 matrix, what is the smallest possible dimension of Nul ***A***?

16. If ***A*** is a 6 x 4 matrix, what is the smallest possible dimension of Nul ***A***?