Lecture 8

Linear Independence

Definition

An indexed set of vectors **{v_1, v_2, ... , v_p}** in R^n is said to be **linearly independent** if the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ has only the trivial solution. The set **{v_1, v_2, ... , v_p}** is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$ (1)

Equation (1) is called a **linear dependence relation** among $v_1, \ldots, v_p$, when the weights are not all zero.

Example 1

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(a) Determine whether the set of vectors **{v_1, v_2, v_3}** is linearly independent or not.
(b) If possible, find a linear dependence relation among v_1, v_2, v_3.

Solution

(a) Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \qquad (-2)R_1 + R_2,\ (-3)R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 + R_3 \qquad (2)$$

Clearly, x_1 and x_2 are basic variables and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution.
Hence v_1, v_2, v_3 are linearly dependent (and not linearly independent).

(b) To find a linear dependence relation among v_1, v_2, v_3, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{-1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 4R_2$$

$$\Rightarrow \begin{array}{lr} x_1 & -2x_3 = 0 \\ \quad x_2 & +x_3 = 0 \\ & 0 \quad = 0 \end{array}$$

Thus ***$x_1 = 2x_3$, $x_2 = -x_3$***, and ***x_3*** is free.

Choose any nonzero value for ***x_3***, say, ***$x_3 = 5$***, then ***$x_1 = 10$***, and ***$x_2 = -5$***.
Substitute these values into $x_1v_1 + x_2v_2 + x_3v_3 = 0$

$$\Rightarrow \; \boldsymbol{10v_1 - 5v_2 + 5v_3 = 0}$$

This is one (out of infinitely many) possible linear dependence relation among ***v_1, v_2, v_3***.

Example (for practice)

Check whether the vectors are linearly dependent or linearly independent.

$v_1 = (3,-1) \qquad v_2 = (-2,2)$

Solution

Consider two constants C_1 and C_2. Suppose:

$$c_1(3,-1) + c_2(-2,2) = 0$$
$$(3c_1 - 2c_2, -c_1 + 2c_2) = (0,0)$$

Now, set each of the components equal to zero to arrive at the following system of equations:

$$\begin{aligned} 3c_1 - 2c_2 &= 0 \\ -c_1 + 2c_2 &= 0 \end{aligned}$$

Solving this system gives the following solution,

$$c_1 = 0 \qquad c_2 = 0$$

The trivial solution is the only solution, so these two vectors are linearly independent.

Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [a_1 \quad \ldots \quad a_n]$ instead of a set of vectors. The matrix equation ***Ax = 0*** can be written as $x_1a_1 + x_2a_2 + \cdots + x_na_n = 0$

Each linear dependence relation among the columns of ***A*** corresponds to a nontrivial solution of ***Ax = 0***.

Thus we have the following important fact.

The columns of a matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.

Example 2 Determine whether the columns of $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

Solution To study ***Ax = 0,*** row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \qquad R_{12}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \qquad (-5)R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix} \qquad (2)R_2 + R_3$$

At this point, it is clear that there are three basic variables and no free variables. So the equation ***Ax = 0*** has only the trivial solution, and the columns of ***A*** are linearly independent.

Sets of One or Two Vectors

A set containing only one vector (say, ***v***) is linearly independent if and only if ***v*** is not the zero vector. This is because the vector equation $x_1 v = 0$ has only the trivial solution when $v \neq 0$. The zero vector is linearly dependent because $x_1 0 = 0$ has many nontrivial solutions.

Example 3
Check the following sets for linearly independence and dependence.

a. $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

b. $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Solution

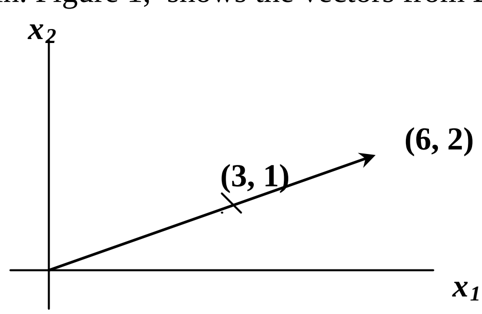
a) Notice that $\boldsymbol{v_2}$ is a multiple of $\boldsymbol{v_1}$, namely, $\boldsymbol{v_2} = \boldsymbol{2v_1}$.
Hence $\boldsymbol{-2v_1 + v_2 = 0}$, which shows that $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is linearly dependent.

b) $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are certainly not multiples of one another. Could they be linearly dependent?

Suppose $\boldsymbol{c}$ and $\boldsymbol{d}$ satisfy $\boldsymbol{cv_1 + dv_2 = 0}$

If $c \neq 0$, then we can solve for $\boldsymbol{v_1}$ in terms of $\boldsymbol{v_2}$, namely, $\boldsymbol{v_1 = (-d/c)\, v_2}$. This result is impossible because $\boldsymbol{v_1}$ is not a multiple of $\boldsymbol{v_2}$. So, $\boldsymbol{c}$ must be zero. Similarly, $\boldsymbol{d}$ must also be zero.
Thus $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is a linearly independent set.

Note A set of two vectors $\{v_1, v_2\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.
In geometric terms, two vectors are linearly dependent if and only if they lie on the same line through the origin. Figure 1, shows the vectors from Example 3.



Linearly dependent

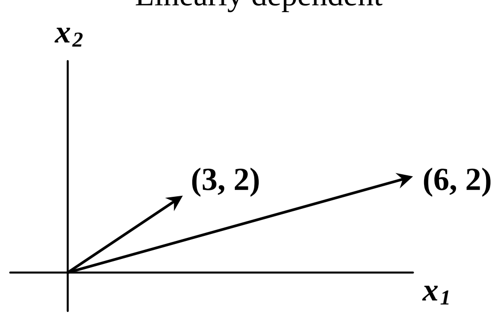


Figure 1 Linearly independent

Sets of Two or More Vectors

Theorem (Characterization of Linearly dependent Sets)

An indexed set $s = \{v_1, v_2, \cdots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in ***S*** is a linear combination of the others. In fact, if **S** is linearly dependent, and $v \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \cdots, v_{j-1}$.

Proof

If some v_j in ***S*** equals a linear combination of the other vectors, then v_j can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (–1) on v_j.

For instance, if $\boldsymbol{v_1 = c_2v_2 + c_3v_3}$, then $\boldsymbol{0 = (-1)v_1 + c_2v_2 + c_3v_3 + 0v_4 + \ldots + 0v_p}$.
Thus ***S*** is linearly dependent.

Conversely, suppose ***S*** is linearly dependent. If $\boldsymbol{v_1}$ is zero, then it is a (trivial) linear combination of the other vectors in ***S***.

If $v \neq 0$ and there exist weights $\boldsymbol{c_1, \ldots, c_p}$, not all zero(because vectors are linearly dependent), such that

$$\boldsymbol{c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0}$$

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $\boldsymbol{c_1v_1 = 0}$, which is impossible because $v_j \neq 0$.

So $j > 1$, and $c_1v_1 + \cdots + c_jv_j + 0v_{j+1} + \cdots + 0v_p = 0$

$$c_jv_j = -c_1v_1 - c_2v_2 - \cdots - c_{j-1}v_{j-1}$$

$$v_j = \left(-\frac{c_1}{c_j}\right)v_1 + \left(-\frac{c_2}{c_j}\right)v_2 + \cdots + \left(\frac{c_{j-1}}{c_j}\right)v_{j-1}$$

Note: This theorem does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

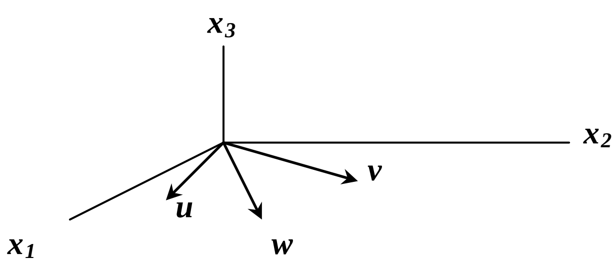
Example 4 Let $u = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by ***u*** and ***v***, and prove that a vector ***w*** is in Span **{*u, v*}** if and only if **{*u, v, w*}** is linearly dependent.

Solution

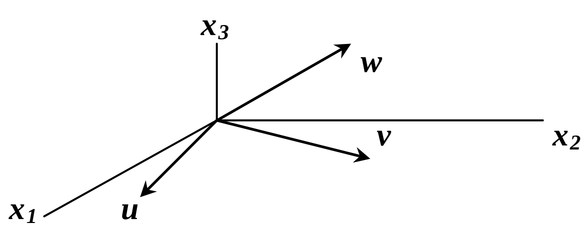
The vectors ***u*** and ***v*** are linearly independent because neither vector is a multiple of the other, nor so they span a plane in $\boldsymbol{R^3}$. In fact, Span **{*u, v*}** is the $\boldsymbol{x_1 x_2}$**-plane** (with $\boldsymbol{x_3 = 0}$). If ***w*** is a linear combination of ***u*** and ***v***, then **{*u, v, w*}** is linearly dependent.

Conversely, suppose that **{*u, v, w*}** is linearly dependent.

Some vector in **{*u, v, w*}** is a linear combination of the preceding vectors (since $u \neq 0$). That vector must be ***w***, since ***v*** is not a multiple of ***u***. So ***w*** is in Span **{*u, v*}**



Linearly dependent ***w*** in Span **{*u, v*}**.



Linearly independent ***w*** not in **Span {*u, v*}**

Figure 2: Linear dependence in $\boldsymbol{R^3}$.

This example generalizes to any set **{*u, v, w*}** in $\boldsymbol{R^3}$ with ***u*** and ***v*** linearly independent. The set **{*u, v, w*}** will be linearly dependent if and only if ***w*** is in the plane spanned by ***u*** and ***v***.

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, v_2, \dots, v_p\}$ in R^n is linearly dependent if $p > n$.

Example 5 The vectors $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\-1\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}$ are linearly dependent, because there are three vectors in the set and there are only two entries in each vector.

Notice, however, that none of the vectors is a multiple of one of the other vectors. See Figure 4.

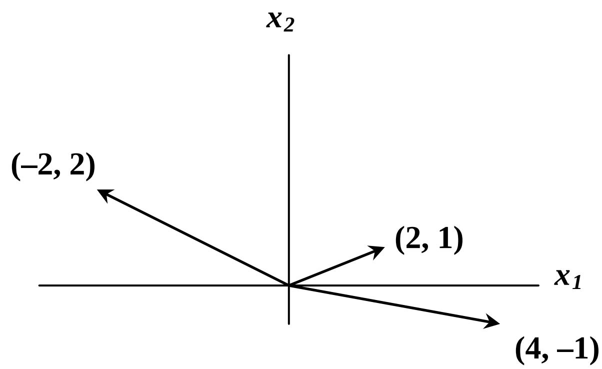


Figure 4 A linearly dependent set in $\boldsymbol{R^2}$.

Theorem

If a set $S = \{v_1, v_2, \ldots, v_p\}$ in R^n contains the zero vector, then the set is linearly dependent**.**

Proof

By renumbering the vectors, we may suppose that $\boldsymbol{v_1 = 0}$.

Then $(1)\boldsymbol{v_1} + \boldsymbol{0v_2} + \ldots + \boldsymbol{0v_p} = \boldsymbol{0}$ shows that $\boldsymbol{S}$ is linearly dependent(because in this relation coefficient of v_1 is non zero).

Example 6 Determine by inspection if the given set is linearly dependent.

a. $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$ **b.** $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$ **c.** $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$

Solution

a) The set contains four vectors that each has only three entries. So,the set is linearly dependent by the Theorem above.
b) The same theorem does not apply here because the number of vectors does not exceed the number of entries in each vector. Since the zero vector is in the set, the set is linearly dependent by the next theorem.

c) As we compare corresponding entries of the two vectors, the second vector seems to be –3/2 times the first vector. This relation holds for the first three pairs of entries, but fails for the fourth pair. Thus neither of the vectors is a multiple of the other, and hence they are linearly independent.

Exercise

1. Let $u = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, v = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, w = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, and \quad z = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

(i) Are the sets ***{u, v}, {u, w}, {u, z}, {v, w}, {v, z},*** and ***{w, z}*** each linearly independent? Why or why not?

(ii) Does the answer to Problem (i) imply that ***{u, v, w, z}*** is linearly independent?

(iii) To determine if ***{u, v, w, z}*** is linearly dependent, is it wise to check if, say ***w*** is a linear combination of ***u, v*** and ***z***?

(iv) Is ***{u, v, w, z}*** linear dependent?

Decide if the vectors are linearly independent. Give a reason for each answer.

2. $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}$

Determine if the columns of the given matrix form a linearly dependent set.

4. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 3 & 10 & -7 & 1 \\ -5 & -5 & 3 & 7 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 4 & 3 \\ -1 & -7 & 7 \\ 1 & 3 & -2 \\ 0 & 2 & -6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 0 & 4 \\ -1 & 0 & 3 & -1 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & 1 & 3 \end{bmatrix}$

For what values of h is v_3 in span $\{v_1, v_2\}$ and for what values of h is $\{v_1, v_2, v_3\}$ linearly dependent?

8. $v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ h \end{bmatrix}$

9. $v_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 9 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ -6 \\ h \end{bmatrix}$

Find the value(s) of h for which the vectors are linearly dependent.

10. $\begin{bmatrix}1\\3\\-3\end{bmatrix},\begin{bmatrix}-2\\-4\\1\end{bmatrix},\begin{bmatrix}-1\\1\\h\end{bmatrix}$

11. $\begin{bmatrix}1\\-5\\-2\end{bmatrix},\begin{bmatrix}-3\\8\\6\end{bmatrix},\begin{bmatrix}4\\h\\-8\end{bmatrix}$

Determine by inspection whether the vectors are linearly independent. Give reasons for your answers.

12. $\begin{bmatrix}5\\5\end{bmatrix},\begin{bmatrix}6\\1\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix},\begin{bmatrix}3\\-6\end{bmatrix}$

13. $\begin{bmatrix}2\\-5\\1\end{bmatrix},\begin{bmatrix}-6\\5\\3\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$

14. $\begin{bmatrix}6\\2\\-8\end{bmatrix},\begin{bmatrix}3\\1\\-2\end{bmatrix}$

15. Given $A=\begin{bmatrix}2 & 3 & 5\\-5 & 1 & -4\\-3 & -1 & -4\\1 & 0 & 1\end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $\boldsymbol{Ax = 0}$ without performing row operations.

Each statement in exercises 16-18 is either true(in all cases) or false(for at least one example). If false, construct a specific example to show that the statement is not always true. If true, give a justification.

16. If $v_1, \ldots, v_4$ are in R^4, and $v_3 = 2v_1 + v_2$, then $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

17. If v_1 and v_2 are in R^4, and v_1 is not a scalar multiple of v_2, then $\{v_1, v_2\}$ is linearly independent.

18. . If $v_1, \ldots, v_4$ are in R^4, and $\{v_1, v_2, v_3\}$ is linearly dependent, then $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent.

19. Use as many columns of $A=\begin{bmatrix}8 & -3 & 0 & -7 & 2\\-9 & 4 & 5 & 11 & -7\\6 & -2 & 2 & -4 & 4\\5 & -1 & 7 & 0 & 10\end{bmatrix}$ as possible to construct a matrix B with the property that equation $\boldsymbol{Bx = 0}$ has only the trivial solution. Solve

Bx = 0 to verify your work.

Lecture 9

Linear Transformations

Outlines

- Matrix Equation
- Transformation, Examples, Matrix as Transformations
- Linear Transformation, Examples, Some Properties

Matrix Equation

An equation ***Ax = b*** is called a matrix equation in which a matrix ***A*** acts on a vector ***x*** by multiplication to produce a new vector called ***b.***

For instance, the equations

$$\underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}_{b}$$

and

$$\underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{o}}$$

Solution of Matrix Equation

Solution of the ***Ax = b*** consists of those vectors ***x*** in ***the domain*** that are transformed into the vector ***b*** in ***range***.

Matrix equation ***Ax = b*** is an important example of transformation we would see later in the lecture.

Transformation or Function or Mapping

A transformation (or **function** or **mapping**) $\boldsymbol{T}$ from $\boldsymbol{R}^{\boldsymbol{n}}$ to $\boldsymbol{R}^{\boldsymbol{m}}$ is a rule that assigns to each vector $\boldsymbol{x}$ in $\boldsymbol{R}^{\boldsymbol{n}}$, an image vector $\boldsymbol{T(x)}$ in $\boldsymbol{R}^{\boldsymbol{m}}$.

$$T : R^n \longrightarrow R^m$$

The set $\boldsymbol{R}^{\boldsymbol{n}}$ is called the **domain** of $\boldsymbol{T}$, and $\boldsymbol{R}^{\boldsymbol{m}}$ is called the **co-domain** of $\boldsymbol{T}$. For $\boldsymbol{x}$ in $\boldsymbol{R}^{\boldsymbol{n}}$ the set of all images $\boldsymbol{T(x)}$ is called the **range** of $\boldsymbol{T}$.

Note

To define a mapping or function, domain and co-domain are the ordinary sets. However to define a linear transformation, the domain and co-domain has to be $\mathbb{R}^m (or\ \mathbb{R}^n)$. Moreover a map $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if for any two vectors say $u, v \in \mathbb{R}^m$ and the scalars c_1, c_2, the following equation is satisfied

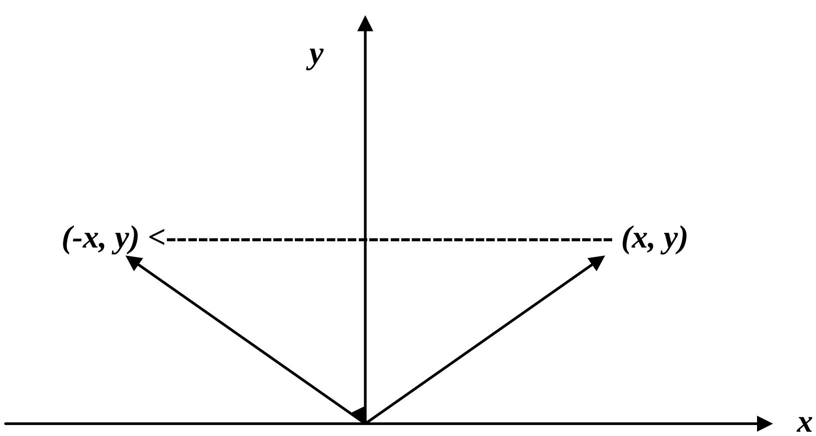
$$T\left(c_1 u + c_2 v\right) = c_1 T(u) + c_2 T(v)$$

Example1 Consider a mapping $T : R^2 \to R^2$ defined by $T(x, y) = (-x, y)$. This transformation is a reflection about y-axis in xy plane.

Here $T(1,2) = (-1,2)$. T has transformed vector (1,2) into another vector (-1,2).

In matrix form:

$$Tv = Av = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



Further the projection transformation $T : R^2 \to R^2$ defined by $T(x, y) = (x, 0)$ is given as:

$$Tv = Av = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Example 2 Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : R^2 \to R^3$ by ***T(x) = Ax***, so that

$$T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a) Find ***T (u)***, the image of ***u*** under the transformation ***T***.
b) Find an ***x*** in $\boldsymbol{R^2}$, whose image under ***T*** is ***b***.
c) Is there more than one ***x*** whose image under ***T*** is ***b***?
d) Determine if ***c*** is in the range of the transformation ***T***.

<u>Solution</u> (a)

$$T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

Here $\quad T(u) = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

Here the matrix transformation has transformed $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ into another vector $\begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

(b) We have to find an x such that **T (x) = b or Ax = b**

i. e $\quad \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \qquad (1)$

Now row reduced augmented matrix will be:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} -3R_1 + R_2,\ R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \frac{1}{14} R_2,\ -4R_2 + R_3,$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} 3R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $x_1 = 1.5$, $x_2 = -0.5$, and $x = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$.

The image of this ***x*** under ***T*** is the given vector ***b***.

(c) From (2) it is clear that equation
(1) has a unique solution. So, there is exactly one ***x*** whose image is ***b***.

(d) The vector ***c*** is in the range of ***T*** if ***c*** is the image of some ***x*** in $\boldsymbol{R}^2$, that is, if $c = T(x)$ for some ***x***. This is just another way of asking if the system ***Ax = c*** is consistent. To find the answer, we will row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} -3R_1 + R_2,\ R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \frac{1}{4} R_3, R_{23}$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} -14R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

$$x_1 - 3x_2 = 3$$
$$0x_1 + x_2 = 2$$
$$0x_1 + 0x_2 = -35 \Rightarrow 0 = 35 \; but \; 0 \neq 35$$

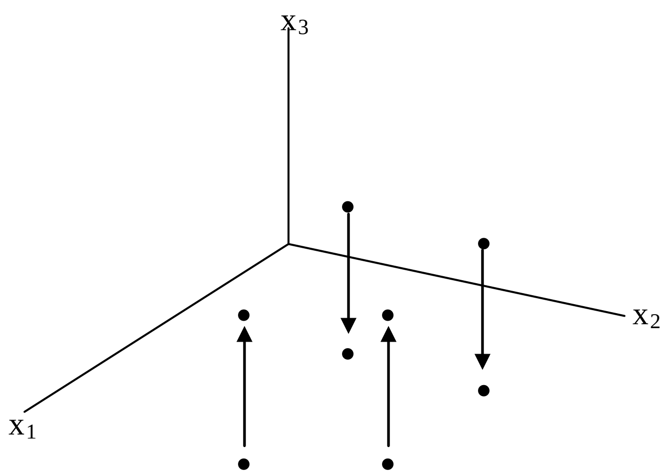
Hence the system is inconsistent. So **c** is not in the range of **T**.

So from the above example we can view a transformation in the form of a matrix. We'll see that a transformation $T : R^n \to R^m$ can be transformed into a matrix of order $m \times n$ and every matrix of order $m \times n$ can be viewed as a linear transformation.

The next two matrix transformations can be viewed geometrically. They reinforce the dynamic view of a matrix as something that transforms vectors into other vectors.

Example 3 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $x \to Ax$ projects points in $\boldsymbol{R^3}$

onto the $x_1x_2 - coordinate \;\; plane$ because $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$

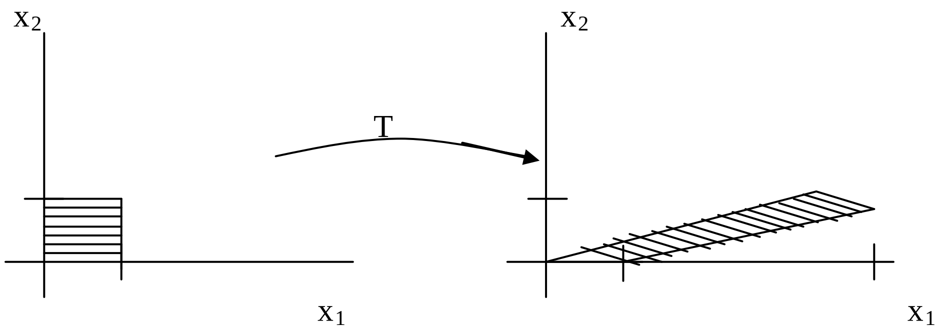


A projection transformation

Example 4 Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$., the transformation $T : R^2 \to R^2$ defined by ***T (x) = Ax*** is called a **shear** transformation.

The image of the point $u = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ *is* $T(u) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$,
and the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ *is* $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

Here, ***T*** deforms the square as if the top of the square was pushed to the right while the base is held fixed. Shear transformations appear in physics, geology and crystallography.



A shear transformation

Linear Transformations

We know that if ***A*** is $m \times n$ matrix, then the transformation $x \to Ax$ has the properties $A(u+v) = Au + Av$ *and* $A(cu) = cAu$ for all ***u, v*** in $\boldsymbol{R^n}$ and all scalars ***c***.
These properties for a transformation identify the most important class of transformations in linear algebra.

Definition A transformation (or mapping) ***T*** is linear if:

1. ***T(u + v) = T(u) + T(v)*** for all ***u, v*** in the domain of ***T***;
2. ***T(cu) = cT(u)*** for all ***u*** and all scalars ***c***.

Example 5 Every matrix transformation is a linear transformation.

Example 6 Let $L: R^3 \to R^2$ be defined by $L(x, y, z) = (x, y)$.

we let $\boldsymbol{u} = (x_1, y_1, z_1)$ and $\boldsymbol{v} = (x_2, y_2, z_2)$.

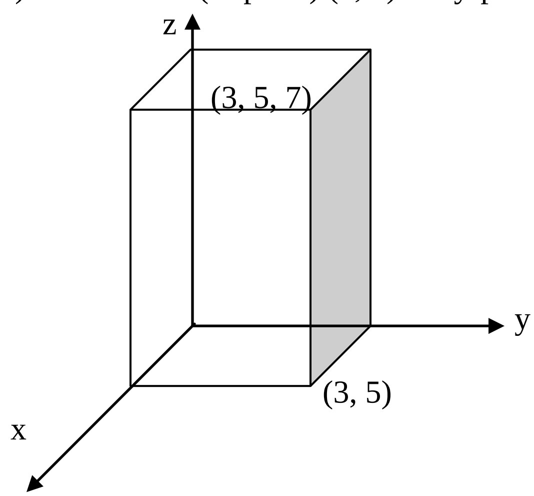
$$\begin{aligned} L(\boldsymbol{u}+\boldsymbol{v}) &= L((x_1, y_1, z_1) + (x_2, y_2, z_2)) \\ &= L(x_1 + x_2, y_1 + y_2, z_1 + z_2) \end{aligned}$$

$$= (x_1 + x_2, y_1 + y_2)$$
$$= (x_1, y_1) + (x_2, y_2) = L(\boldsymbol{u}) + L(\boldsymbol{v})$$

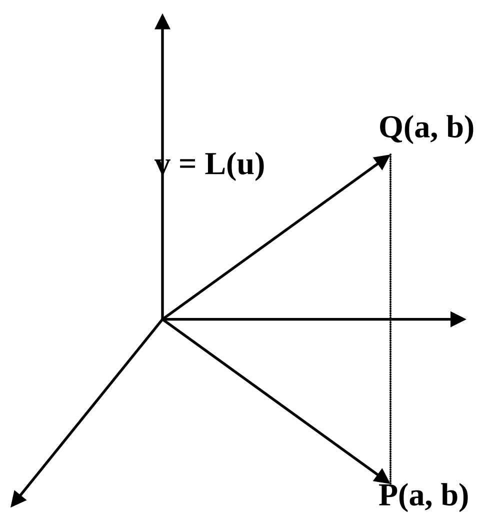
Also, if k is a real number, then

$$L(k\boldsymbol{u}) = L(kx_1, ky_1, kz_1) = (kx_1, ky_1) = kL(\boldsymbol{u})$$

Hence, ***L*** is a linear transformation, which is called a projection. The image of the vector (or point) (3, 5, 7) is the vector (or point) (3, 5) in xy-plane. See figure below:



Geometrically the image under ***L*** of a vector (a, b, c) in $\boldsymbol{R}^3$ is (a, b) in $\boldsymbol{R}^2$ can be found by drawing a line through the end point P(a, b, c) of ***u*** and perpendicular to $\boldsymbol{R}^2$, the *xy*-plane. The intersection Q(a, b)of this line with the *xy*-plane will give the image under ***L***. See the figure below:



Example 7 Let $L : R \rightarrow R$ be defined by $L(x) = x^2$

Let x and y in R and

$$L(x+y) = (x+y)^2 = x^2 + y^2 + 2xy \neq x^2 + y^2 = L(x) + L(y)$$
$$\Rightarrow L(x+y) \neq L(x) + L(y)$$

So we conclude that the function ***L*** is not a linear transformation.

Linear transformations preserve the operations of vector addition and scalar multiplication.

Properties

If ***T*** is a linear transformation, then

1. $\boldsymbol{T(0) = 0}$
2. $\boldsymbol{T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})}$
3. $\boldsymbol{T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p)}$

for all vectors ***u, v*** in the domain of ***T*** and all scalars *c*, *d*.

Proof

1. By the definition of Linear Transformation we have ***T(cu) = cT(u)*** for all ***u*** and all scalars ***c***. Put $c = 0$ we'll get ***T(0u)= 0T(u)*** This implies ***T(0) = 0***

2. Just apply the definition of linear transformation. i. e

$$\boldsymbol{T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})}$$

Property (3) follows from (ii), because ***T(0) = T (0u) =0T(u) = 0***.
Property (4) requires both (i) and (ii):

OBSERVATION Observe that if a transformation satisfies property 2 for all ***u, v*** and *c*, *d*, it must be linear (Take *c* = *d* = *1* for preservation of addition, and take *d* = *0*)

3. Generalizing Property 2 we'll get 3

$$T(c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p}) = c_1T(\mathbf{v_1}) + \ldots + c_pT(\mathbf{v_p})$$

Applications in Engineering

In engineering and physics, property 3 is referred to as a superposition principle.
Think of $\mathbf{v_1}, \ldots, \mathbf{v_p}$ as signals that go into a system or process and $T(\mathbf{v_1}), \ldots, T(\mathbf{v_p})$ as the responses of that system to the signals. The system satisfies the superposition principle if an input is expressed as a linear combination of such signals, the system's response is the same linear combination of the responses to the individual signals.

Example 8 Given a scalar *r*, define $T : R \to R$ by

$$T(x) = x+1.$$

T is not a linear transformation (why!) because $T(0) \neq 0$ (by property 3)

Example 9 Given a scalar *r*, define $T : R^2 \to R^2$ by $T(\mathbf{x}) = r\mathbf{x}$.
T is called a **contraction** when $0 \leq r < 1$
and a **dilation** when $r \geq 1$.

Let *r* = 3 and show that ***T*** is a linear transformation.

Solution Let ***u, v*** be in $\mathbf{R^2}$ and let *c*, *d* be scalars, then

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= 3(c\mathbf{u} + d\mathbf{v}) && \text{Definition of } \mathbf{T} \\ &\left.\begin{aligned} &= 3c\mathbf{u} + 3d\mathbf{v} \\ &= c(3\mathbf{u}) + d(3\mathbf{v}) \\ &= c\mathbf{T}(\mathbf{u}) + d\mathbf{T}(\mathbf{v}) \end{aligned}\right\} && \text{Vector arithmetic} \end{aligned}$$

Thus ***T*** is a linear transformation because it satisfies (4).

Example 10 Define a linear transformation $T : R^2 \to R^2$ by

$$T(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the images under ***T*** of $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad and \quad u + v = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$

Solution $T(u)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix}=\begin{bmatrix}-1\\ 4\end{bmatrix},\quad T(v)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\ 3\end{bmatrix}=\begin{bmatrix}-3\\ 2\end{bmatrix}$

$$T(u+v)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}6\\ 4\end{bmatrix}=\begin{bmatrix}-4\\ 6\end{bmatrix}$$

In the above example, ***T*** rotates ***u, v*** and ***u + v*** counterclockwise through 90^0.
In fact, ***T*** transforms the entire parallelogram determined by ***u*** and ***v*** into the one determined by ***T (u)*** and ***T (v)***

Example 11 Let L: $R^3 \rightarrow R^2$ be a linear transformation for which:
L (1, 0, 0) = (2, -1),
L (0, 1, 0) = (3, 1), and
L (0, 0, 1) = (-1, 2).
Then find L (-3, 4, 2).

Solution Since (-3, 4, 2) = -3**i** + 4**j** + 2**k**,

$$L(-3,4,2)=L(-3\boldsymbol{i}+4\boldsymbol{j}+2\boldsymbol{k})=-3L(\boldsymbol{i})+4L(\boldsymbol{j})+2L(\boldsymbol{k})$$
$$=-3(2,-1)+4(3,1)+2(-1,2)=(4,11)$$

Exercise

1. Suppose that $T:R^5\rightarrow R^2$ and ***T(x) = Ax*** for some matrix ***A*** and each ***x*** in $\boldsymbol{R^5}$. How many rows and columns do ***A*** have?

2. Let $A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Give a geometric description of the transformation $x\rightarrow Ax$.

3. The line segment from **0** to a vector ***u*** is the set of points of the form ***tu***, where $0\le t\le 1$. Show that a linear transformation ***T*** maps this segment into the segment between **0** and ***T(u)***.

4. Let $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}, u=\begin{bmatrix}1\\ 0\\ -3\end{bmatrix}, and \begin{bmatrix}5\\ -1\\ 4\end{bmatrix}$. Define T:$R^3\rightarrow R^3$ by T (x) = Ax. Find T (u) and T (v).

In exercises 5 and 6, with T defined by T (x) = Ax, find an x whose image under T is b, and determine if x is unique.

5. $A=\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -5 \\ -4 & 2 & 1 \end{bmatrix}, b=\begin{bmatrix} 0 \\ -5 \\ -6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 3 & 2 & 1 \\ -2 & -1 & -2 \end{bmatrix}, b=\begin{bmatrix} 1 \\ -5 \\ -7 \\ 3 \end{bmatrix}$

Find all x in R^4 that are mapped into the zero vector by the transformation $x \to Ax$.

7. $A=\begin{bmatrix} 1 & 2 & -7 & 5 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 1 & 6 \\ 2 & -1 & 6 & 8 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & 4 & -3 \\ 0 & 1 & 3 & -2 \\ 3 & 7 & 6 & -5 \end{bmatrix}$

9. Let $b=\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ and let A be the matrix in exercise 8. Is b in the range of the linear transformation $x \to Ax$?

10. Let $b=\begin{bmatrix} 9 \\ 5 \\ 0 \\ -9 \end{bmatrix}$ and let A be the matrix in exercise 7. Is b in the range of the linear transformation $x \to Ax$?

Let T (x) = Ax for x in R^2.

(a) On a rectangular coordinate system, plot the vectors u, v, T (u) and T (v).
(b) Give a geometric description of what T does to a vector x in R^2.

11. $A=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, u=\begin{bmatrix} 5 \\ 2 \end{bmatrix}, and\ v=\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

12. $A=\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}, u=\begin{bmatrix} 4 \\ 2 \end{bmatrix}, and\ v=\begin{bmatrix} -5 \\ -2 \end{bmatrix}$

13. Let T: $R^2 \to R^2$ be a linear transformation that maps $u=\begin{bmatrix} 1 \\ 5 \end{bmatrix} into \begin{bmatrix} 2 \\ 0 \end{bmatrix} and\ maps\ v=\begin{bmatrix} 3 \\ 1 \end{bmatrix} into \begin{bmatrix} 1 \\ -4 \end{bmatrix}$. Use the fact that T is linear find the images under T of 2u, 3v, and 2u + 3v.

14. Let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, y_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, and\ y_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$. Let T: $R^2 \rightarrow R^2$ be a linear transformation that maps e_1 into y_1 and maps e_2 into y_2. Find the images of $\begin{bmatrix} 7 \\ 6 \end{bmatrix} and \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

15. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, v_1 = \begin{bmatrix} -7 \\ 4 \end{bmatrix} and\ v_2 = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$. Let T: $R^2 \rightarrow R^2$ be a linear transformation that maps x into $x_1 v_1 + x_2 v_2$. Find a matrix A such that T (x) is Ax for each x.

Lecture 10

The Matrix of a Linear Transformation

Outline

- Matrix of a Linear Transformation.
- Examples, Geometry of Transformation, Reflection and Rotation
- Existence and Uniqueness of solution of ***T(x)=0***

In the last lecture we discussed that every linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$ is actually a matrix transformation $x \to Ax$, where A is a matrix of order $m \times n$. First see an example

Example 1 The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose $\boldsymbol{T}$ is a linear transformation from $\boldsymbol{R}^2$ into $\boldsymbol{R}^3$ such that

$$T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad and \quad T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

with no additional information, find a formula for the image of an arbitrary $\boldsymbol{x}$ in $\boldsymbol{R}^2$.

Solution Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2$

Since $\boldsymbol{T}$ is a linear transformation, $T(x) = x_1 T(e_1) + x_2 T(e_2)$

$$T(x) = x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix}$$

$$Hence \quad T(x) = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix}$$

Theorem Let $T : R^n \to R^m$ be a linear transformation. Then there exists a unique matrix $\boldsymbol{A}$ such that $T(x) = Ax$ for all $\boldsymbol{x}$ in $\boldsymbol{R}^n$

In fact, ***A*** is the $m \times n$ matrix whose jth column is the vector $\boldsymbol{T(e_j)}$, where e_j is the jth column of the identity matrix in $\boldsymbol{R^n}$.

$$A = \left[T(e_1) \quad \ldots \quad T(e_n) \right]$$

Proof Write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x_1 e_1 + \ldots + x_n e_n = \left[e_1 \quad \ldots \quad e_n \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left[e_1 \quad \ldots \quad e_n \right] x$$

Since T is Linear, So

$$T(x) = T(x_1 e_1 + \ldots + x_n e_n) = x_1 T(e_1) + \ldots + x_n T(e_n)$$

$$= \left[T(e_1) \quad \ldots \quad T(e_n) \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax \qquad (1)$$

The matrix ***A*** in (1) is called the **standard matrix for the linear transformation *T***. We know that every linear transformation from $\boldsymbol{R^n}$ to $\boldsymbol{R^m}$ is a matrix transformation and vice versa.

The term linear transformation focuses on a property of a mapping, while matrix transformation describes how such a mapping is implemented, as the next three examples illustrate.

Example 2 Find the standard matrix ***A*** for the dilation transformation $\boldsymbol{T(x) = 3x}$, $x \in R^2$.

Solution Write

$$T(e_1) = 3e_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad and \quad T(e_2) = 3e_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Example 3 Let $L: R^3 \to R^3$ is the linear operator defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix}$.

Find the standard matrix representing ***L*** and verify ***L (x) = Ax***.

Solution

The standard matrix ***A*** representing ***L*** is the 3 x 3 matrix whose columns are ***L*** (e_1), ***L*** (e_2), and ***L*** (e_3) respectively. Thus

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 0-0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = col_1(A)$$

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 1-0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = col_2(A)$$

$$L(e_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+0 \\ 0-1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = col_3(A)$$

Hence
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Now
$$\mathbf{Ax} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix} = L(x)$$

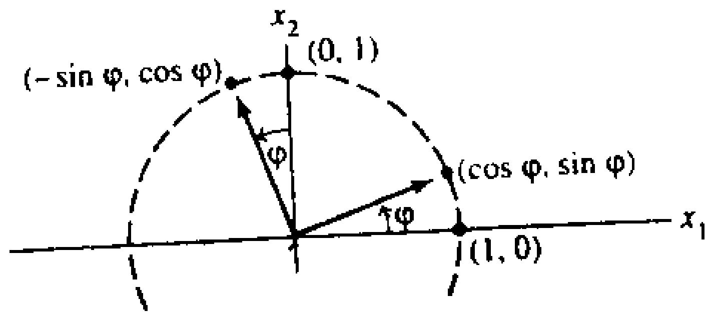
Hence verified.

Example 4 Let $T: R^2 \to R^2$ be the transformation that rotates each point in $\boldsymbol{R^2}$ through an angle φ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. Find the standard matrix ***A*** of this transformation.

Solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$.

See figure below.

By above theorem $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$



A rotation transformation

Example 5 A reflection with respect to the ***x***-axis of a vector ***u*** in $\boldsymbol{R^2}$ is defined by the

linear operator $L(u) = L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$.

Then $L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Hence the standard matrix representing ***L*** is $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Thus we have $L(u) = Au = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$

To illustrate a reflection with respect to the x-axis in computer graphics, let the triangle ***T*** have vertices (-1, 4), (3, 1), and (2, 6).

To reflect ***T*** with respect to ***x***-axis, we let $u_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and compute the images ***L (u₁), L (u₂),*** and ***L (u₃)*** by forming the products

$$Au_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix},$$

$$Au_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

$$Au_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

Thus the image of ***T*** has vertices (-1, -4), (3, -1), and (2, -6).

Geometric Linear Transformations of R^2

Examples 3-5 illustrate linear transformations that are described geometrically. In example 4 transformations is a rotation in the plane. It rotates each point in the plane through an angle φ. Example 5 is reflection in the plane.

Existence and Uniqueness of the solution of *T(x)=b*

The concept of a linear transformation provides a new way to understand existence and uniqueness questions asked earlier. The following two definitions give the appropriate terminology for transformations.

Definition A mapping $T : R^n \to R^m$ is said to be onto $\boldsymbol{R^m}$ if each ***b*** in $\boldsymbol{R^m}$ is the image of at least one ***x*** in $\boldsymbol{R^n}$.

OR

Equivalently, ***T*** is onto $\boldsymbol{R^m}$ if for each ***b*** in $\boldsymbol{R^m}$ there exists at least one solution of ***T (x) = b***. "Does ***T*** map $\boldsymbol{R^n}$ onto $\boldsymbol{R^m}$?" is an existence question.

The mapping ***T*** is not onto when there is some ***b*** in $\boldsymbol{R^m}$ such that the equation ***T (x) = b*** has no solution.

Definition A mapping $T : R^n \to R^m$ is said to be one-to-one (or 1:1) if each ***b*** in $\boldsymbol{R^m}$ is the image of at most one ***x*** in $\boldsymbol{R^n}$.

OR

Equivalently, **T** is one-to-one if for each $\boldsymbol{b}$ in $\boldsymbol{R^m}$ the equation $\boldsymbol{T(x) = b}$ has either a unique solution or none at all, "Is $\boldsymbol{T}$ one-to-one?" is a uniqueness question.

The mapping $\boldsymbol{T}$ is not one-to-one when some $\boldsymbol{b}$ in $\boldsymbol{R^m}$ is the image of more than one vector in $\boldsymbol{R^n}$. If there is no such $\boldsymbol{b}$, then $\boldsymbol{T}$ is one-to-one.

<u>Example 6</u> Let $\boldsymbol{T}$ be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does $\boldsymbol{T}$ map $\boldsymbol{R^4}$ onto $\boldsymbol{R^3}$? Is $\boldsymbol{T}$ a one-to-one mapping?

<u>Solution</u> Since $\boldsymbol{A}$ happens to be in echelon form, we can see at once that $\boldsymbol{A}$ has a pivot position in each row.

We know that for each $\boldsymbol{b}$ in $\boldsymbol{R^3}$, the equation $\boldsymbol{Ax = b}$ is consistent. In other words, the linear transformation $\boldsymbol{T}$ maps $\boldsymbol{R^4}$ (its domain) onto $\boldsymbol{R^3}$.

However, since the equation $\boldsymbol{Ax = b}$ has a free variable (because there are four variables and only three basic variables), each $\boldsymbol{b}$ is the image of more than one $\boldsymbol{x}$. That is, $\boldsymbol{T}$ is not one-to-one.

<u>Theorem</u> Let $T : R^n \to R^m$ be a linear transformation. Then $\boldsymbol{T}$ is one-to-one if and only if the equation $\boldsymbol{T(x) = 0}$ has only the trivial solution.

<u>Proof:</u> Since $\boldsymbol{T}$ is linear, $\boldsymbol{T(0) = 0}$ if $\boldsymbol{T}$ is one-to-one, then the equation $\boldsymbol{T(x) = 0}$ has at most one solution and hence only the trivial solution. If $\boldsymbol{T}$ is not one-to-one, then there is a $\boldsymbol{b}$ that is the image of at least two different vectors in $\boldsymbol{R^n}$ (say, $\boldsymbol{u}$ and $\boldsymbol{v}$).
That is, $\boldsymbol{T(u) = b}$ and $\boldsymbol{T(v) = b}$.

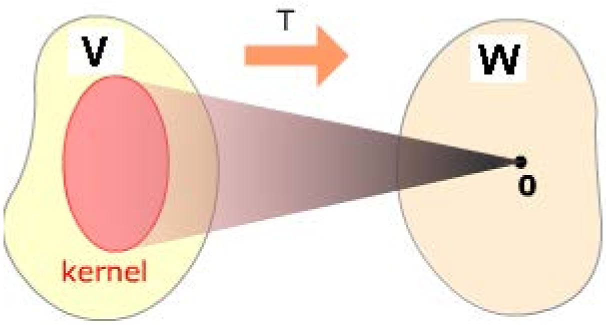
But then, since $\boldsymbol{T}$ is linear $T(u-v) = T(u) = b - b = 0$

The vector $\boldsymbol{u - v}$ is not zero, since $u \neq v$. Hence the equation $\boldsymbol{T(x) = 0}$ has more than one solution. So either the two conditions in the theorem are both true or they are both false.

<u>Kernel of a Linear Transformation</u>

Let $T : V \to W$ be a linear transformation. Then kernel of T (usually written as Ker$T$), is the set of those elements in V which maps onto the zero vector in W .Mathematically:

$$KerT = \{ v \in V \mid T(v) = 0 \ in \ W \}$$



Remarks

i) Ker$T$ is subspace of V

ii) T is one-one iff $KerT = 0$ in V

One-One Linear Transformation

Let $T : V \to W$ be a linear transformation. Then T is said to be one-one if for any $u, v \in V$ with $u \neq v$ implies $Tu \neq Tv$. Equivalently if $Tu = Tv$ then $u = v$.

T is said to be one-to-one or bijective if

i) T is one-one

ii)T is onto

Theorem Let $T : R^n \to R^m$ be a linear transformation and let $\boldsymbol{A}$ be the standard matrix for $\boldsymbol{T}$. Then

(a) $\boldsymbol{T}$ maps $\boldsymbol{R^n}$ onto $\boldsymbol{R^m}$ if and only if the columns of $\boldsymbol{A}$ span $\boldsymbol{R^m}$;

(b) $\boldsymbol{T}$ is one-to-one if and only if the columns of **A** are linearly independent.

Proof:

(a) The columns of $\boldsymbol{A}$ span $\boldsymbol{R^m}$ if and only if for each $\boldsymbol{b}$ the equation $\boldsymbol{Ax = b}$ is consistent – in other words, if and only if for every $\boldsymbol{b}$, the equation $\boldsymbol{T(x) = b}$ has at least one solution. This is true if and only if $\boldsymbol{T}$ maps $\boldsymbol{R^n}$ onto $\boldsymbol{R^m}$.

(b) The equations $\boldsymbol{T(x) = 0}$ and $\boldsymbol{Ax = 0}$ are the same except for notation. So $\boldsymbol{T}$ is one-to-one if and only if $\boldsymbol{Ax = 0}$ has only the trivial solution. This happens if and only if the columns of $\boldsymbol{A}$ are linearly independent.

We can also write column vectors in rows, using parentheses and commas. Also, when we apply a linear transformation $\boldsymbol{T}$ to a vector – say, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2)$ we write $T(x_1, x_2)$ instead of the more formal $T\left((x_1, x_2)\right)$.

Example 7 Let $\boldsymbol{T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)}$.

Show that $\boldsymbol{T}$ is a one-to-one linear transformation.
Does $\boldsymbol{T}$ map $\boldsymbol{R^2}$ onto $\boldsymbol{R^3}$?

Solution When ***x*** and ***T (x)*** are written as column vectors, it is easy to see that ***T*** is described by the equation

$$\begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} \qquad (4)$$

so ***T*** is indeed a linear transformation, with its standard matrix ***A*** shown in (4). The columns of ***A*** are linearly independent because they are not multiples. Hence ***T*** is one-to-one. To decide if ***T*** is onto $\boldsymbol{R^3}$, we examine the span of the columns of ***A***. Since ***A*** is 3×2, the columns of ***A*** span $\boldsymbol{R^3}$ if and only if ***A*** has 3 pivot positions. This is impossible, since ***A*** has only 2 columns. So the columns of ***A*** do not span $\boldsymbol{R^3}$ and the associated linear transformation is not onto $\boldsymbol{R^3}$.

Exercises

1. Let $T: R^2 \to R^2$ be transformation that first performs a horizontal shear that maps e_2 into $e_2 - .5e_1$ (but leaves e_1 unchanged) and then reflects the result in the $\boldsymbol{x_2}$ – axis. Assuming that ***T*** is linear, find its standard matrix.

Assume that T is a linear transformation. Find the standard matrix of T.

2. $T: R^2 \to R^3, T(1,0) = (4,-1,2) \; and \; T(0,1) = (-5,3,-6)$

3. $T: R^3 \to R^2, T(e_1) = (1,4), T(e_2) = (-2,9), and \; T(e_3) = (3,-8),$ where e_1, e_2, and e_3 are the columns of the identity matrix.

4. $T: R^2 \to R^2$ rotates points clockwise through π radians.

5. $T: R^2 \to R^2$ is a "vertical shear" transformation that maps e_1 into $e_1 + 2e_2$ but leaves the vector e_2 unchanged.

6. $T: R^2 \to R^2$ is a "horizontal shear" transformation that maps e_2 into $e_2 - 3e_1$ but leaves the vector e_1 unchanged.

7. $T: R^3 \to R^3$ projects each point (x_1, x_2, x_3) vertically onto the x_1x_2-plane (where x_3=0).

8. $T: R^2 \to R^2$ first performs a vertical shear mapping e_1 into $e_1 - 3e_2$(leaving e_2 unchanged) and then reflects the result in the x_2-axis.

9. $T: R^2 \to R^2$ first rotates points counterclockwise through $\pi/4$ radians and then reflects the result in the x_2-axis.

Show that T is a linear transformation by finding a matrix that implements the mapping. Note that x_1, x_2, … are not vectors but are entries in vectors.

10. $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 + x_3, x_3 + x_4, 0)$

11. $T(x_1, x_2, x_3) = (3x_2 - x_3, x_1 + 4x_2 + x_3)$

12. $T(x_1, x_2, x_3, x_4) = 3x_1 - 4x_2 + 8x_4$

13. Let $T: R^2 \to R^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 7x_2)$. Find ***x*** such that T (x) =(-2, -5).

13. Let $T : R^2 \to R^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 + 2x_2, -x_1 - 3x_2, -3x_1 - 2x_2)$. Find ***x*** such that T (x) =(-4, 7, 0).

In exercises 14 and 15, let T be the linear transformation whose standard matrix is given.

14. Decide if T is one-to-one mapping. Justify your answer.

$$\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$$

15. Decide if T maps R^5 onto R^5. Justify your answer.

$$\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$$