# Lecture 12

# The Inverse of a Matrix

In this lecture and the next, we will consider only square matrices and we will investigate the matrix analogue of the reciprocal or multiplicative inverse of a nonzero real number.

**Inverse of a square Matrix**

If ***A*** is an $n \times n$ matrix, A matrix C of order $n \times n$ is called multiplicative inverse of A if

$AC = CA = I$ where $I$ is the $n \times n$ identity matrix.

**Invertible Matrix**

If the inverse of a square matrix exists, it is called an invertible matrix.

In this case, we say that ***A*** is **invertible** and we call ***C*** an **inverse** of ***A***.

**Note:** If ***B*** is another inverse of ***A***, then we would have

$$B = BI = B(AC) = (BA)C = IC = C.$$

Thus when ***A*** is **invertible**, its inverse is unique.

The inverse of ***A*** is denoted by $\boldsymbol{A^{-1}}$, so that

$$AA^{-1} = I \qquad and \qquad A^{-1}A = I$$

**Note:** A matrix that is not invertible is sometimes called a **singular** matrix, and an invertible matrix is called a **non-singular** matrix.

**Warning**

By No means $A^{-1}.A = A.A^{-1} = I \quad The\, identity\, matrix \;\; A^{-1} = \dfrac{1}{A}$, as in case of real number, we have $3^{-1} = \dfrac{1}{3}$.

$A^{-1}$ is, in fact the $n \times n$ matrix corresponding to the $n \times n$ matrix A, which satisfies the property

$$A^{-1}.A = A.A^{-1} = I \quad The\, identity\, matrix$$

**Example 1** If $A=\begin{bmatrix}2 & 5\\-3 & -7\end{bmatrix}$ *and* $C=\begin{bmatrix}-7 & -5\\3 & 2\end{bmatrix}$, then

$$AC=\begin{bmatrix}2 & 5\\-3 & -7\end{bmatrix}\begin{bmatrix}-7 & -5\\3 & 2\end{bmatrix}=\begin{bmatrix}-14+15 & -10+10\\21-21 & 15-14\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \quad and$$

$$CA=\begin{bmatrix}-7 & -5\\3 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\-3 & -7\end{bmatrix}=\begin{bmatrix}-14+15 & -35+35\\6-6 & 15-14\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Thus **C =** $\boldsymbol{A^{-1}}$.

**Theorem** Let $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$.

If $ad-bc\neq 0$, then ***A*** is invertible or non singular and $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

If $ad-bc=0$, then ***A*** is not invertible or singular.

The quantity $ad-bc$ is called the **determinant of *A***, and we write

$$\det A=ad-bc$$

This implies that a $2\times 2$ matrix ***A*** is invertible if and only if $\det A\neq 0$.

**Example 2** Find the inverse of $A=\begin{bmatrix}3 & 4\\5 & 6\end{bmatrix}$.

**Solution** We have det ***A*** = 3(6) – 4(5) = $-2\neq 0$.

Hence ***A*** is invertible $A^{-1}=\dfrac{1}{-2}\begin{bmatrix}6 & -4\\-5 & 3\end{bmatrix}=\begin{bmatrix}6/(-2) & -4/(-2)\\-5/(-2) & 3/(-2)\end{bmatrix}=\begin{bmatrix}-3 & 2\\5/2 & -3/2\end{bmatrix}$

The next theorem provides three useful facts about invertible matrices.

**Theorem**

a. If ***A*** is an invertible matrix, then $\boldsymbol{A^{-1}}$ is invertible and $(A^{-1})^{-1}=A$

b. If ***A*** and ***B*** are $n\times n$ invertible matrices, then so is ***AB***, and the inverse of ***AB*** is the product of the inverses of ***A*** and ***B*** in the reverse order. That is $(AB)^{-1}=B^{-1}A^{-1}$

c. If ***A*** is an invertible matrix, then so is $\boldsymbol{A^T}$, and the inverse of $\boldsymbol{A^T}$ is the transpose of $\boldsymbol{A^{-1}}$. That is $(A^T)^{-1}=(A^{-1})^T$

**Proof**

(a) We must find a matrix ***C*** such that $A^{-1}C = I \quad and \quad CA^{-1} = I$

However, we already know that these equations are satisfied with ***A*** in place of ***C***. Hence ***A***$^{-1}$ is invertible and ***A*** is its inverse.

(b) We use the associative law for multiplication:

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$ .

Hence ***AB*** is invertible, and its inverse is $B^{-1}A^{-1}$ i.e $(AB)^{-1} = B^{-1}A^{-1}$

**Generalization**

Similarly we can prove the same results for more than two matrices i.e

$$\left((A_1)(A_2)(A_3)...(A_n)\right)^{-1} = A_n^{-1}A_{n-1}^{-1}... \; A_3^{-1}A_2^{-1}A_1^{-1}$$

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

**__Example 3__ (Inverse of a Transpose).** Consider a general $2\times2$ invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad and \qquad A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since A is invertible, its determinant *(ad – bc)* is nonzero. But the determinant of ***A***$^t$ is also *(ad – bc )*, so ***A***$^t$ is also invertible. It follows that

$$(A^t)^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{c}{ad-bc} \\ -\dfrac{b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} \text{---------------(1)}$$

Now $$A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Therefore, $(A^{-1})^t = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{c}{ad-bc} \\ -\dfrac{b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$ -----------------(2)

From (1) and (2), we have

$$(A^t)^{-1} = (A^{-1})^t .$$

**<u>Example 4</u> (The Inverse of a Product).** Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix},$$

Here

$$(AB)^{-1} = \frac{1}{|AB|} Adj(AB) = \frac{1}{-2}\begin{bmatrix} 8 & -6 \\ -9 & 7 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj(A) = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix},$$

$$B^{-1} = \frac{1}{|B|} Adj(B) = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix},$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$

**Theorem:** If A is invertible and n is a non-negative integer, then:
**(a)** $\boldsymbol{A^n}$ is invertible and $\boldsymbol{(A^n)^{-1} = A^{-n} = (A^{-1})^n}$
**(b)** $k\boldsymbol{A}$ is invertible for any nonzero scalar $k$, and $(k\boldsymbol{A})^{-1} = k^{-1}\boldsymbol{A}^{-1}$.

**Example 5 (Related to the above theorem)**

(a) Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad then\ A^{-1} = \frac{1}{|A|} Adj(A) = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Now $$A^{-3} = \left(A^{-1}\right)^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also, $$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$(A^3)^{-1} = \frac{1}{(11)(41)-(30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

(b)

Take $A=\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}$ and k=3

kA=3A=$\begin{bmatrix}3 & 6\\9 & 3\end{bmatrix}$, $(kA)^{-1}=(3A)^{-1}=\frac{1}{9-54}\begin{bmatrix}3 & -6\\-9 & 3\end{bmatrix}=\begin{bmatrix}-1/15 & 2/15\\1/5 & -1/15\end{bmatrix}$------(1)

$A^{-1}=-\frac{1}{5}\begin{bmatrix}1 & -2\\-3 & 1\end{bmatrix}$

A= *So* $k^{-1}A^{-1}=3^{-1}A^{-1}=\frac{1}{3}.\frac{-1}{5}\begin{bmatrix}1 & -2\\-3 & 1\end{bmatrix}=\begin{bmatrix}-1/15 & 2/15\\1/5 & -1/15\end{bmatrix}$------------------------(2)

From (1) and (2), we have
$(3A)^{-1}=3^{-1}A^{-1}$

There is an important connection between invertible matrices and row operations that leads to a method for computing inverses. As we shall see, an invertible matrix ***A*** is row equivalent to an identity matrix, and we can find $\boldsymbol{A^{-1}}$ by watching the row reduction of ***A*** to ***I***.

**Elementary Matrices**

As we have studied that there are three types of elementary row operations that can be performed on a matrix:
There are three types of elementary operations

- Interchanging of any two rows
- Multiplication to a row by a nonzero constant
- Adding a multiple of one row to another

**Elementary matrix**

An elementary matrix is a matrix that results from applying a single elementary row operation to an identity matrix.

Some examples are given below:

$$\begin{bmatrix}1 & 0\\0 & -3\end{bmatrix},\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{bmatrix},\begin{bmatrix}1 & 0 & 3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

| | | | |
|---|---|---|---|
| Multiply the second row of $I_2$ by -3. | Interchange the second and fourth rows of $I_4$. | Add 3 times the third row of $I_3$ to the first row. | Multiply the first row of $I_3$ by 1. |

***From Def it is clear that elementary matrices are always square.***

Elementary matrices are important because they can be used to execute elementary row operations by matrix multiplication.

**Theorem** If ***A*** is an $n \times n$ identity matrix, and if the elementary matrix ***E*** results by performing a certain row operation on the identity matrix, then the product *EA* is the matrix that results when the same row operation is performed on ***A***.

In short, this theorem states that an elementary row operation can be performed on a matrix ***A*** using a left multiplication by and appropriate elementary matrix.

**Example 6** **(Performing Row Operations by Matrix Multiplication)**. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

Find an elementary matrix ***E*** such that ***EA*** is the matrix that results by adding 4 times the first row of ***A*** to the third row.

**Solution**: The matrix ***E*** must be $3 \times 3$ to conform to the product ***EA***. Thus, we obtain ***E*** by adding 4 times the first row of $I_3$ to the third row. This gives us $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

As a check, the product ***EA*** is $EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 5 & 4 & 12 & 12 \end{bmatrix}$

So left multiplication by ***E*** does, in fact, add 4 times the first row of ***A*** to the third row.

If an elementary row operation is applied to an identity matrix ***I*** to produce an elementary matrix ***E***, then there is a second row operation that, when applied to ***E***, produces ***I*** back again.

For example, if ***E*** is obtained by multiplying the *i-th* row of ***I*** by a nonzero scalar c, then ***I*** can be recovered by multiplying the *i-th* row of ***E*** by 1/c. The following table explains how to recover the identity matrix from an elementary matrix for each of the three elementary row operations. The operations on the right side of this table are called the **inverse operations** of the corresponding operations on the left side.

| **Row operation on I that produces E** | **Row operation on E that reproduces I** |
|---|---|
| Multiply row *i* by $c \neq 0$ | Multiply row *i* by 1/c |
| Interchange rows *i* and *j* | Interchange rows *i* and *j* |
| Add c times row *i* to row *j* | Add –c times row *i* to row *j* |

**Example 7 (Recovering Identity Matrices from Elementary Matrices)**. Here are three examples that use inverses of row operations to recover the identity matrix from

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply the second row by 7.  Multiply the second row by 1/7.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Interchange the first and second rows.  Interchange the first and second rows.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add 5 times the second row to the first.  Add -5 times the second row to the first.

The next theorem is the basic result on the invertibility of elementary matrices.

**Theorem** An elementary matrix is invertible and the inverse is also an elementary matrix.

**Example 8** Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute ***E₁A, E₂A, E₃A*** and describe how these products can be obtained by elementary row operations on ***A***.

**Solution** We have

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, \quad E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}, \; E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

Addition of (–4) times row 1 of $A$ to row 3 produces $E_1A$. (This is a row replacement operation.) An interchange of rows 1 and 2 of $A$ produces $E_2A$ and multiplication of row 3 of $A$ by 5 produces $E_3A$.

**Left-multiplication** (that is, multiplication on the left) by $E_1$ in Example 8 has the same effect on any $3\times n$ matrix. It adds – 4 times row 1 to row 3. In particular, since $E_1 I = E_1$, we see that $E_1$ itself is produced by the same row operation on the identity. Thus Example 8 illustrates the following general fact about elementary matrices.

**Note:** Since row operations are reversible, elementary matrices are invertible, for if $E$ is produced by a row operation on $I$, then there is another row operation of the same type that changes $E$ back into $I$. Hence there is an elementary matrix $F$ such that $FE = I$. Since $E$ and $F$ correspond to reverse operations, $EF = I$.

Each elementary matrix $E$ is invertible. The inverse of $E$ is the elementary matrix of the same type that transforms $E$ back into $I$.

**Example** Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

**Solution**: To transform $E_1$ into $I$, add + 4 times row 1 to row 3.

The elementary matrix which does that is $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}$

**Theorem** An $n\times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to $I_n$, and in this case, any sequence of elementary row operations that reduces $A$ to $I_n$ also transforms $I_n$ into $A^{-1}$.

**An Algorithm for Finding $A^{-1}$** If we place $A$ and $I$ side-by-side to form an augmented matrix $[A \quad I]$, then row operations on this matrix produce identical operations on $A$ and $I$. Then either there are row operations that transform $A$ to $I_n$, and $I_n$ to $A^{-1}$, or else $A$ is not invertible.

| **Algorithm for Finding $A^{-1}$** |
|---|
| Row reduce the augmented matrix $[A \quad I]$. If $A$ is row equivalent to $I$, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, $A$ does not have an inverse. |

**Example 9** Find the inverse of the matrix $A=\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 3\\ 4 & -3 & 8\end{bmatrix}$, if it exists.

**Solution** $[A \quad I]=\begin{bmatrix}0 & 1 & 2 & 1 & 0 & 0\\ 1 & 0 & 3 & 0 & 1 & 0\\ 4 & -3 & 8 & 0 & 0 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 4 & -3 & 8 & 0 & 0 & 1\end{bmatrix}R_{12}$

$$\begin{array}{cc} -4R_1+R_3 & 3R_2+R_3 \end{array}$$

$$\sim\begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & -3 & -4 & 0 & -4 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 2 & 3 & -4 & 1\end{bmatrix}$$

$$\begin{array}{cc} -2R_3+R_2 & -3R_3+R_1 \end{array}$$

$$\sim\begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 1 & 3/2 & -2 & 1/2\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0 & -9/2 & 7 & -3/2\\ 0 & 1 & 0 & -2 & 4 & -1\\ 0 & 0 & 1 & 3/2 & -2 & 1/2\end{bmatrix}$$

Since ***A*** ~ **I**, we conclude that ***A*** is invertible, and $A^{-1}=\begin{bmatrix}-9/2 & 7 & -3/2\\ -2 & 4 & -1\\ 3/2 & -2 & 1/2\end{bmatrix}$

It is a good idea to check the final answer:

$$AA^{-1}=\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 3\\ 4 & -3 & 8\end{bmatrix}\begin{bmatrix}-9/2 & 7 & -3/2\\ -2 & 4 & -1\\ 3/2 & -2 & 1/2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

It is not necessary to check that $A^{-1}A = I$ since ***A*** is invertible.

**Example 10** Find the inverse of the matrix $A=\begin{bmatrix}1 & 2 & -3 & 1\\ -1 & 3 & -3 & -2\\ 2 & 0 & 1 & 5\\ 3 & 1 & -2 & 5\end{bmatrix}$, if it exists.

Consider $detA = \begin{vmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & -3 & -2 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -6 & -1 \\ 0 & -4 & 7 & 3 \\ 0 & -5 & 7 & 2 \end{vmatrix}$

operating $R_2 + R_1, R_3 - 2R_1, R_4 - 3R_1$

Expand from first column $= \begin{vmatrix} 5 & -6 & -1 \\ -4 & 7 & 3 \\ -5 & 7 & 2 \end{vmatrix} = \begin{vmatrix} 5 & -6 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 5(1-2)+6(1-0)-1(1-0) = 0$

As the given matrix is singular, so it is not invertible.

**<u>Example 11</u>** Find the inverse of the given matrix if possible $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

**Solution** $\det A = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = -1$

As the given matrix is non-singular therefore, inverse is possible.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$Multiply R_3\ by -1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

$R_1 - R_3, R_2 + R_3$

Hence the inverse of matrix A is $A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

**<u>Example 12</u>** Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 5 & 2 & 3 \end{bmatrix}$

**Solution** $\det A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 5 & 2 & 3 \end{vmatrix} = 6$

As the given matrix is non-singular, therefore, inverse of the matrix is possible.
We reduce it to reduce echelon form.

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & -8 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - 5R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1/2 \\ 0 & -8 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

multiply 2nd row by -1/2

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

$R_3 + 8R_2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 0 \\ -1 & 4/3 & -1/3 \end{bmatrix}$$

Multiply 3rd row by -1/3

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & -\frac{8}{3} & \frac{2}{3} \\ 3/2 & -7/6 & 1/6 \\ -1 & 4/3 & -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1/3 & 1/3 \\ 3/2 & -7/6 & 1/6 \\ -1 & 4/3 & -1/3 \end{bmatrix}$$

$R_2 - (1/2)R_3, R_1 - 2R_3$

Hence the inverse of the original matrix $A^{-1} = \begin{bmatrix} 0 & -1/3 & 1/3 \\ 3/2 & -7/6 & 1/6 \\ -1 & 4/3 & -1/3 \end{bmatrix}$

## Exercises

In exercises 1 to 4, find the inverses of the matrices, if they exist. Use elementary row operations.

1. $\begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & 8 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

7. Let $A = \begin{bmatrix} -1 & -5 & -7 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$. Find the third column of $A^{-1}$ without computing the other columns.

8. Let $A = \begin{bmatrix} -25 & -9 & -27 \\ 546 & 180 & 537 \\ 154 & 50 & 149 \end{bmatrix}$. Find the second and third columns of $A^{-1}$ without computing the first column.

9. Find an elementary matrix E that satisfies the equation.
(a) EA = B (b) EB = A (c) EA = C (d) EC = A
where $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$.

10. Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

   (a) Find elementary matrices $E_1$ and $E_2$ such that $E_2E_1A=I$.
   (b) Write $A^{-1}$ as a product of two elementary matrices.

(c) Write A as a product of two elementary matrices.
In exercises 11 and 12, express A and $A^{-1}$ as products of elementary matrices.

11. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

13. Factor the matrix $A = \begin{bmatrix} 0 & 1 & 7 & 8 \\ 1 & 3 & 3 & 8 \\ -2 & -5 & 1 & -8 \end{bmatrix}$ as A = EFGR, where E, F, and G are elementary matrices and R is in row echelon form.

# Lecture 13

# Characterizations of Invertible Matrices

This chapter involves a few techniques of solving the system of n linear equations in **n** unknowns and transformation associated with a matrix.

**Solving Linear Systems by Matrix Inversion**

**Theorem**

Let $\boldsymbol{A}$ be an $n \times n$ invertible matrix. For any $\boldsymbol{b} \in \boldsymbol{R}^n$, the equation $\boldsymbol{Ax} = \boldsymbol{b}$ …..(1) has the unique solution i.e. $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

**Proof**

Since A is invertible and $\boldsymbol{b} \in \boldsymbol{R}^n$ be any vector. Then, we must have a matrix $A^{-1}b$ which is a solution of eq. (1) .i.e. $\boldsymbol{Ax = A\ (A^{-1}b) = Ib = b}$.

**Uniqueness**

For uniqueness, we assume that there is another solution $\boldsymbol{u}$. Indeed, it is a solution of eq.(1) so it must be $\boldsymbol{u = A^{-1}b}$, it means x = $\boldsymbol{A^{-1}b}$ = $\boldsymbol{u}$. This shows that $\boldsymbol{u = x}$.

**Theorem**

Let A and B be the square matrices such that AB = I. Then, A and B are invertible with B = $A^{-1}$ and A = $B^{-1}$

**Example 1**

Solve the system of linear equations

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 5x_1 + x_2 + 3x_3 &= 3 \\ x_1 + \qquad 4x_3 &= 6 \end{aligned}$$

**Solution**

Consider the linear system

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 5x_1 + x_2 + 3x_3 &= 3 \\ x_1 + \qquad 4x_3 &= 6 \end{aligned}$$

The Matrix form of system is $\boldsymbol{Ax = b}$, where

$$A=\begin{bmatrix}2&1&1\\5&1&3\\1&0&4\end{bmatrix},\qquad x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\qquad b=\begin{bmatrix}1\\3\\6\end{bmatrix}$$

Here, det (***A***) = 2(4) – 1 (20 – 3) + 1(0 – 1) = 8 – 17 – 1 = -10 $\neq$ 0

So, ***A*** is invertible. Now, we apply the inversion algorithm:

$$\left[\begin{array}{ccc|ccc}2&1&1&1&0&0\\5&1&3&0&1&0\\1&0&4&0&0&1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}1&0&4&0&0&1\\5&1&3&0&1&0\\2&1&1&1&0&0\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}1&0&4&0&0&1\\0&1&-17&0&1&-5\\0&1&-7&1&0&-2\end{array}\right],\quad -5R_1+R_2,\quad -2R_1+R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&4&0&0&1\\0&1&-17&0&1&-5\\0&0&10&1&-1&3\end{array}\right],\quad -R_2+R_3,$$

$$\left[\begin{array}{cccccc}1&0&4&0&0&1\\0&1&-17&0&1&-5\\0&0&1&\frac{1}{10}&\frac{-1}{10}&\frac{3}{10}\end{array}\right],\quad \frac{R_3}{10}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{-2}{5}&\frac{2}{5}&\frac{-1}{5}\\0&1&0&\frac{17}{10}&\frac{-7}{10}&\frac{1}{10}\\0&0&1&\frac{1}{10}&\frac{-1}{10}&\frac{3}{10}\end{array}\right],\quad -4R_3+R_1\quad ,-17R_3+R_2$$

Hence, $A^{-1} = \begin{bmatrix} \frac{-2}{5} & \frac{2}{5} & \frac{-1}{5} \\ \frac{17}{10} & \frac{-7}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{-1}{10} & \frac{3}{10} \end{bmatrix}$

Thus, the solution of the linear system is $x = A^{-1}b = A^{-1} = \begin{bmatrix} \frac{-2}{5} & \frac{2}{5} & \frac{-1}{5} \\ \frac{17}{10} & \frac{-7}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{-1}{10} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{-2}{5} \\ \frac{1}{5} \\ \frac{8}{5} \end{bmatrix}$

**Example # 2** Solve the system $\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ x_1 + \quad\quad 8x_3 = 17 \end{cases}$ by inversion method.

**Solution** Consider the linear system $\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ x_1 + \quad\quad 8x_3 = 17 \end{cases}$

This system can be written in matrix form as ***Ax = b***, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

Here, [det (***A***)] = 40 – 2 (16 – 3) + 3(0 – 5) = 40 – 26 – 15 = -1 $\neq$ 0
Therefore, ***A*** is invertible.

Now, we apply the inversion algorithm:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] -2R_1 + R_2, \quad -1R_1 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] 2R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] -1R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] 3R_3 + R_2, \quad -3R_3 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] -2R_2 + R_1$$

Hence, $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Thus, the solution of the linear system is $x = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Thus $x_1 = 1, x_2 = -1, x_3 = 2$.

**Note:** It is only applicable when the number of equations = number of unknown and fails if given matrix is not invertible.

**Example 3**

Solve the system of linear equation

$$x_1 + 6x_2 + 4x_3 = 2$$
$$2x_1 + 4x_2 - x_3 = 3$$
$$-x_1 + 2x_2 + 5x_3 = 3$$

**Solution**

The matrix coefficient

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

det (***A***) = 1(20 + 2) – 6(10 – 1) + 4 (4+4)

$= 22 - 54 + 32$

$= 0$

Thus, A is not invertible. Hence, the inversion method fails.

**Solution for the system of a $n \times n$ Homogeneous linear equations with an Invertible Coefficient Matrix:-**

Let see if the system is considered as homogeneous then what does the above theorem say?

**Theorem:-**

Let ***Ax = 0*** be a homogeneous linear system of **n** equations and **n** unknowns. Then, the coefficient matrix ***A*** is invertible iff this system has only a trivial solution.

**<u>Example 4</u>**

State whether the following system of linear equation has a solution or not?

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \\ x_1 \qquad - 4x_3 &= 0 \end{aligned}$$

**<u>Solution</u>**

We see

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -3 & 1 \\ 1 & 0 & -4 \end{bmatrix}$$ is an invertible matrix (det (A) $\neq 0$ )

Thus, this homogeneous linear system has only the trivial solution.

**<u>Example 5</u>**

Solve $\begin{cases} x_1 + x_2 + x_3 = 8 \\ \quad 2x_2 + 3x_3 = 24 \\ 5x_1 + 5x_2 + x_3 = 8 \end{cases}$

**<u>Solution</u>** This system can be written in matrix form as ***Ax = b***, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix}$$

Here [det (***A***)] = 1(2 – 15) – 1(0 – 15) + 1(0 – 10) = -13 + 15 – 10 = -8 $\neq 0$
Therefore, ***A*** is invertible.

Now, we apply the inversion algorithm:

$$\begin{array}{c} A \qquad\qquad I_3 \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \quad -5R_1 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \quad \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \quad -1R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] \quad -\frac{1}{4}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] \quad -\frac{3}{2}R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] \quad \frac{1}{2}R_3 + R_1$$

Hence, $$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

Thus, the solution of the linear system is

$$x = A^{-1}b = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

Thus $x_1 = 0, x_2 = 0, x_3 = 8$.

**Theorem (Invertible Matrix Theorem)** Let ***A*** be a square $n \times n$ matrix. Then the following statements are equivalent. (Means if any one holds then all are true).

(a) ***A*** is an invertible matrix.

(b) ***A*** is row equivalent to the $n \times n$ identity matrix.

(c) ***A*** has **n** pivot positions.

(d) The equation ***Ax = 0*** has only the trivial solution.

(e) The columns of ***A*** form a linearly independent set.

(f) The linear transformation $x \to Ax$ is one-to-one.

(g) The equation ***Ax = b*** has at least one solution for each ***b*** in $\boldsymbol{R^n}$.

(h) The columns of ***A*** span $\boldsymbol{R^n}$.

(i) The linear transformation $x \to Ax$ maps $\boldsymbol{R^n}$ onto $\boldsymbol{R^n}$.

(j) There is a $n \times n$ matrix ***C*** such that ***CA = I***.

(k) There is a $n \times n$ matrix ***D*** such that ***AD = I.***

(l) $\boldsymbol{A^T}$ is an invertible matrix.

**Example 6**

Show that the matrix $A = \begin{bmatrix} 1 & 0 & -4 \\ 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ is invertible by using Invertible Matrix Theorem

**Solution**

By row equivalent,

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}, -R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 9 \\ 0 & 1 & 2 \end{bmatrix}, -R_2 + R_3$$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 9 \\ 0 & 0 & -7 \end{bmatrix}$$

It shows that A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem (c).

**Example 7**

Use the Invertible Matrix Theorem to decide if ***A*** is invertible, where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

**Solution**

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$ Here, ***A*** has three pivot positions and hence is invertible by the Invertible Matrix Theorem (c).

**Example 7** Find $A^t$ and show that $A^t$ is an invertible matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

**Solution**

$$A^t = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Now, by row equivalent of A,

$$A=\begin{bmatrix}1&2&1&0\\2&4&1&1\\1&0&1&1\\0&1&1&1\end{bmatrix}\overset{R_2-2R_1,\,R_3-R_1}{\sim}\begin{bmatrix}1&2&1&0\\0&0&-1&1\\0&-2&0&1\\0&1&1&1\end{bmatrix}\overset{R_{23}}{\sim}\begin{bmatrix}1&2&1&0\\0&-2&0&1\\0&0&-1&1\\0&1&1&1\end{bmatrix}$$

$$\overset{-\frac{1}{2}R_2}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&-\frac{1}{2}\\0&0&-1&1\\0&1&1&1\end{bmatrix}\overset{R_4-R_2}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&-\frac{1}{2}\\0&0&-1&1\\0&0&1&\frac{3}{2}\end{bmatrix}\overset{(-1)R_3}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&\frac{-1}{2}\\0&0&1&-1\\0&0&1&\frac{3}{2}\end{bmatrix}\overset{-R_3+R_4}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&\frac{-1}{2}\\0&0&1&-1\\0&0&0&\frac{5}{2}\end{bmatrix}$$

Here ***A*** has 4 pivot positions so by Invertible Matrix Theorem (c) A is invertible. Thus, by (l) $A^t$ is invertible.

**Example 8**

Use the Invertible Matrix Theorem to decide if ***A*** is invertible, where

$$A=\begin{bmatrix}1&2&1&0\\2&4&1&2\\1&0&1&2\\0&1&1&2\end{bmatrix}$$

**Solution**

$$A=\begin{bmatrix}1&2&1&0\\2&4&1&2\\1&0&1&2\\0&1&1&2\end{bmatrix}\overset{R_2-2R_1,\,R_3-R_1}{\sim}\begin{bmatrix}1&2&1&0\\0&0&-1&2\\0&-2&0&2\\0&1&1&2\end{bmatrix}\overset{R_{23}}{\sim}\begin{bmatrix}1&2&1&0\\0&-2&0&2\\0&0&-1&2\\0&1&1&2\end{bmatrix}$$

$$\overset{-\frac{1}{2}R_2}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&-1\\0&0&-1&2\\0&1&1&2\end{bmatrix}\overset{R_4-R_2}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&-1\\0&0&-1&2\\0&0&1&3\end{bmatrix}\overset{(-1)R_3}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&-1\\0&0&1&-2\\0&0&1&3\end{bmatrix}\overset{-R_3+R_4}{\sim}\begin{bmatrix}1&2&1&0\\0&1&0&-1\\0&0&1&-2\\0&0&0&5\end{bmatrix}$$

Here, ***A*** has 4 pivot positions and hence is invertible by Invertible Matrix Theorem (c).

**Solving Multiple Linear Systems with a Common Coefficient Matrix**

This technique is used in solving a sequence of linear systems

$$Ax_1 = b_1,\ Ax_2 = b_2,\ \cdots\ ,\ Ax_k = b_k \qquad (1)$$

where coefficient matrix ***A*** *remains* same and off course if it is invertible, then we have a sequence of solutions. i.e.

$x_1 = A^{-1}b_1,\ x_2 = A^{-1}b_2,\ \cdots\ ,\ x_k = A^{-1}b_k$.

**Find a Matrix from Linear Transformation**

We can find a matrix corresponding to every transformation. In this section we will learn how to find a matrix attached with a linear transformation.

**Example 9**

Let L be the linear transformation from $R^2$ to $P_2$ (Polynomials of order 2) defined by

$T(x, y) = x\,y\,t + (x + y)t^2$

Find the matrix representing T with respect to the standard bases.

**Solution**

Let A = {(1,0),(0,1)} be the basis of $R^2$, then

$T(1,0) = t^2 = (0,0,1)$ (This triple represents the coefficients of polynomial $t^2$)

i.e. $t^2 = 0.1 + 0.t + 1.t^2$

Similarly, $T(0, 1) = t^2 = (0,0,1)$. Hence, the matrix is given by

$$A=\begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix}$$

Now, we will proceed with a more complicated example.

**Example 10**

Let T be the linear transformation from $R^2$ to $R^2$ such that T(x, y) = (x , y + 2x). Find a matrix A for T.

**Solution**

This matrix is found by finding T (1, 0) = (1, 2) and T (0, 1) = (0, 1)The matrix is $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

**Important Note**

It should be clear that the Invertible Matrix Theorem applies only to square matrices. For example, if the columns of a $4 \times 3$ matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude any thing about the existence or nonexistence of solutions to equations of the form $\boldsymbol{Ax = b}$.

**Invertible Linear Transformations**

Recall that matrix multiplication corresponds to composition of linear transformations. When a matrix $\boldsymbol{A}$ is invertible, the equation $\boldsymbol{A^{-1}Ax = x}$ can be viewed as a statement about linear transformations. See Figure 2.

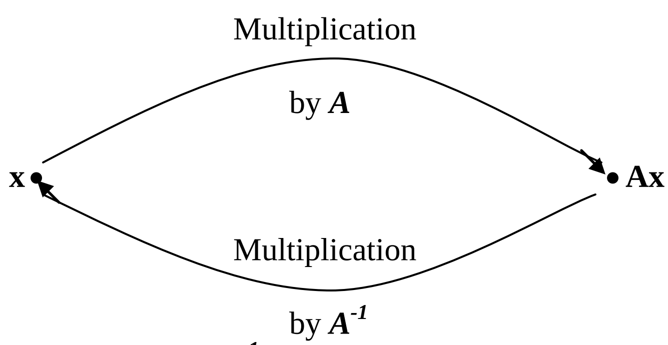


**Figure 2** $\boldsymbol{A^{-1}}$ transforms $\boldsymbol{Ax}$ back to $\boldsymbol{x}$

**Definition**

A linear transformation $T : R^n \to R^n$ be a linear Transformation and A be a standard matrix for T. Then, T is invertible if and only if A is an invertible matrix in that case linear transformation S given by S(x) = $A^{-1}$x is a unique function satisfying (1) and (2)

$$S(T(x)) = x \ \forall \ \ x \in R^n \qquad (1)$$

$$T(S(x)) = x \ \forall \ \ x \in R^n \qquad (2)$$

**Important Note**

If the inverse of a linear transformation exits then it is unique.

**Proposition**

Let $T : R^n \rightarrow R^m$ be linear transformation, given as $T(x) = Ax \quad , \forall\ x \in R^n$, where A is a $m \times n$ matrix. The mapping T is invertible if the system $y = Ax$ has a unique solution.

**Case 1:**

If $m < n$, then the system $Ax = y$ has either no solution or infinitely many solutions, for any y in $R^{m.}$ Therefore, $y = Ax$ is non-invertible.

**Case 2:**

If $m = n$, then the system $Ax = y$ has a unique solution if and only if Rank (A) = n.

**Case 3:**

If $m > n$, then the transformation $y = Ax$ is non-invertible because we can find a vector y in $R^m$ such that $Ax = y$ is inconsistent.

## Exercises

1.Solve the system of linear equations by inverse matrix method.

$$x_1 + x_2 + 4x_3 = 2$$
$$2x_2 + 3x_3 = 4$$
$$5x_1 + x_2 - x_3 = 3$$

2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, b_1 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}, b_2 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, b_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, and\ b_4 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$

(a) Find $A^{-1}$ and use it to solve the equations Ax = $b_1$, Ax = $b_2$, Ax = $b_3$, Ax = $b_4$.

(b) Solve the four equations in part (a) by row reducing the augmented matrix $\begin{bmatrix} A & b_1 & b_2 & b_3 & b_4 \end{bmatrix}$.

3. (a) Solve the two systems of linear equations

$$x_1 + 2x_2 + x_3 = -1$$
$$x_1 + 3x_2 + 2x_3 = 3$$
$$x_2 + 2x_3 = 4$$

&

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 + 3x_2 + 2x_3 = 0$$
$$x_2 + 2x_3 = 4$$

by row reduction.

(b) Write the systems in (a) as Ax = $b_1$ and Ax = $b_2$, and then solve each of them by the method of inversion.

Determine which of the matrices in exercises 4 to 10 are invertible?

4. $\begin{bmatrix} -4 & 16 \\ 3 & -9 \end{bmatrix}$ 5. $\begin{bmatrix} 5 & 0 & 3 \\ 7 & 0 & 2 \\ 9 & 0 & 1 \end{bmatrix}$ 6. $\begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}$ 7. $\begin{bmatrix} 5 & -9 & 3 \\ 0 & 3 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & -1 \\ -2 & -6 & 3 & 2 \\ 3 & 5 & 8 & -3 \end{bmatrix}$ 9. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{bmatrix}$ 10. $\begin{bmatrix} 7 & -6 & -4 & 1 \\ -5 & 1 & 0 & -2 \\ 10 & 11 & 7 & -3 \\ 19 & 9 & 7 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 5 & 4 & 3 & 6 & 3 \\ 7 & 6 & 5 & 9 & 5 \\ 8 & 6 & 4 & 10 & 4 \\ 9 & -8 & 9 & -5 & 8 \\ 10 & 8 & 7 & -9 & 7 \end{bmatrix}$

12. Suppose that ***A*** and ***B*** are $n \times n$ matrices and the equation ***ABx = 0*** has a nontrivial solution. What can you say about the matrix ***AB***?

13**.** What can we say about a one-to-one linear transformation ***T*** from $\boldsymbol{R^n}$ into $\boldsymbol{R^n}$?

14. Let $T : R^2 \to R^2$ be a linear transformation given as $T(x) = 5x$, then find a matrix A of linear transformation T.

In exercises 15 and 16, T is a linear transformation from $R^2$ into $R^2$. Show that T is invertible.

15. $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$

16. $T(x_1, x_2) = (6x_1 - 8x_2, -5x_1 + 7x_2)$

17. Let $T : R^n \to R^n$ be a linear transformation and let ***A*** be the standard matrix for ***T***. Then ***T*** is invertible if and only if ***A*** is an invertible matrix.