6.1 SOLUTIONS ABOUT ORDINARY POINTS

REVIEW MATERIAL

- Power Series (see any Calculus Text)

INTRODUCTION In Section 4.3 we saw that solving a homogeneous linear DE with *constant coefficients* was essentially a problem in algebra. By finding the roots of the auxiliary equation, we could write a general solution of the DE as a linear combination of the elementary functions x^k, $x^k e^{\alpha x}$, $x^k e^{\alpha x}\cos\beta x$, and $x^k e^{\alpha x}\sin\beta x$, where k is a nonnegative integer. But as was pointed out in the introduction to Section 4.7, *most* linear higher-order DEs with *variable coefficients* cannot be solved in terms of elementary functions. A usual course of action for equations of this sort is to assume a solution in the form of infinite series and proceed in a manner similar to the method of undetermined coefficients (Section 4.4). In this section we consider linear second-order DEs with variable coefficients that possess solutions in the form of *power series*.

We begin with a brief review of some of the important facts about power series. For a more comprehensive treatment of the subject you should consult a calculus text.

6.1.1 REVIEW OF POWER SERIES

Recall from calculus that a power series in $x - a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

Such a series is also said to be a **power series centered at *a*.** For example, the power series $\sum_{n=0}^{\infty}(x+1)^n$ is centered at $a = -1$. In this section we are concerned mainly with power series in x, in other words, power series such as $\sum_{n=1}^{\infty} 2^{n-1}x^n = x + 2x^2 + 4x^3 + \cdots$ that are centered at $a = 0$. The following list summarizes some important facts about power series.

- **Convergence** A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is convergent at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges—that is, $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty}\sum_{n=0}^{N} c_n(x-a)^n$ exists. If the limit does not exist at x, then the series is said to be divergent.
- **Interval of Convergence** Every power series has an interval of convergence. The interval of convergence is the set of all real numbers x for which the series converges.
- **Radius of Convergence** Every power series has a radius of convergence R. If $R > 0$, then the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$. If the series converges only at its center a, then $R = 0$. If the series converges for all x, then we write $R = \infty$. Recall that the absolute-value inequality $|x-a| < R$ is equivalent to the simultaneous inequality $a - R < x < a + R$. A power series might or might not converge at the endpoints $a - R$ and $a + R$ of this interval.
- **Absolute Convergence** Within its interval of convergence a power series converges absolutely. In other words, if x is a number in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty}|c_n(x-a)^n|$ converges. See Figure 6.1.1.
- **Ratio Test** Convergence of a power series can often be determined by the ratio test. Suppose that $c_n \neq 0$ for all n and that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L.$$

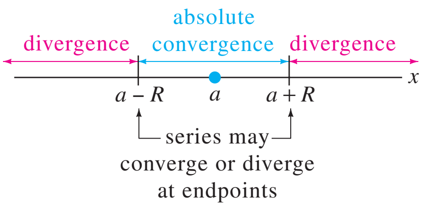

FIGURE 6.1.1 Absolute convergence within the interval of convergence and divergence outside of this interval

If $L < 1$, the series converges absolutely; if $L > 1$, the series diverges; and if $L = 1$, the test is inconclusive. For example, for the power series $\sum_{n=1}^{\infty}(x-3)^n/2^n n$ the ratio test gives

$$\lim_{n\to\infty}\left|\frac{\dfrac{(x-3)^{n+1}}{2^{n+1}(n+1)}}{\dfrac{(x-3)^n}{2^n n}}\right| = |x-3|\lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}|x-3|;$$

the series converges absolutely for $\frac{1}{2}|x-3| < 1$ or $|x-3| < 2$ or $1 < x < 5$. This last inequality defines the *open* interval of convergence. The series diverges for $|x-3| > 2$, that is, for $x > 5$ or $x < 1$. At the left endpoint $x = 1$ of the open interval of convergence, the series of constants $\sum_{n=1}^{\infty}((-1)^n/n)$ is convergent by the alternating series test. At the right endpoint $x = 5$, the series $\sum_{n=1}^{\infty}(1/n)$ is the divergent harmonic series. The interval of convergence of the series is $[1, 5)$, and the radius of convergence is $R = 2$.

- **A Power Series Defines a Function** A power series defines a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. If the radius of convergence is $R > 0$, then f is continuous, differentiable, and integrable on the interval $(a-R, a+R)$. Moreover, $f'(x)$ and $\int f(x)dx$ can be found by term-by-term differentiation and integration. Convergence at an endpoint may be either lost by differentiation or gained through integration. If $y = \sum_{n=0}^{\infty} c_n x^n$ is a power series in x, then the first two derivatives are $y' = \sum_{n=0}^{\infty} nx^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1)x^{n-2}$. Notice that the first term in the first derivative and the first two terms in the second derivative are zero. We omit these zero terms and write

$$y' = \sum_{n=1}^{\infty} c_n nx^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}. \tag{1}$$

These results are important and will be used shortly.
- **Identity Property** If $\sum_{n=0}^{\infty} c_n(x-a)^n = 0$, $R > 0$ for all numbers x in the interval of convergence, then $c_n = 0$ for all n.
- **Analytic at a Point** A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive or infinite radius of convergence. In calculus it is seen that functions such as e^x, $\cos x$, $\sin x$, $\ln(1-x)$, and so on can be represented by Taylor series. Recall, for example, that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{2}$$

for $|x| < \infty$. These Taylor series centered at 0, called Maclaurin series, show that e^x, $\sin x$, and $\cos x$ are analytic at $x = 0$.
- **Arithmetic of Power Series** Power series can be combined through the operations of addition, multiplication, and division. The procedures for power series are similar to those by which two polynomials are added, multiplied, and divided—that is, we add coefficients of like powers of x, use the distributive law and collect like terms, and perform long division. For example, using the series in (2), we have

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) \\ &= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \cdots \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots. \end{aligned}$$

Since the power series for e^x and $\sin x$ converge for $|x| < \infty$, the product series converges on the same interval. Problems involving multiplication or division of power series can be done with minimal fuss by using a CAS.

SHIFTING THE SUMMATION INDEX For the remainder of this section, as well as this chapter, it is important that you become adept at simplifying the sum of two or more power series, each expressed in summation (sigma) notation, to an expression with a single Σ. As the next example illustrates, combining two or more summations as a single summation often requires a reindexing—that is, a shift in the index of summation.

EXAMPLE 1 Adding Two Power Series

Write $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$ as a single power series whose general term involves x^k.

SOLUTION To add the two series, it is necessary that both summation indices start with the same number and that the powers of x in each series be "in phase"; that is, if one series starts with a multiple of, say, x to the first power, then we want the other series to start with the same power. Note that in the given problem the first series starts with x^0, whereas the second series starts with x^1. By writing the first term of the first series outside the summation notation,

series starts with x for $n = 3$ ↓ series starts with x for $n = 0$ ↓

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2 \cdot 1 c_2 x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1},$$

we see that both series on the right-hand side start with the same power of x—namely, x^1. Now to get the same summation index, we are inspired by the exponents of x; we let $k = n - 2$ in the first series and at the same time let $k = n + 1$ in the second series. The right-hand side becomes

same

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k. \tag{3}$$

same

Remember that the summation index is a "dummy" variable; the fact that $k = n - 1$ in one case and $k = n + 1$ in the other should cause no confusion if you keep in mind that it is the *value* of the summation index that is important. In both cases k takes on the same successive values $k = 1, 2, 3, \ldots$ when n takes on the values $n = 2, 3, 4, \ldots$ for $k = n - 1$ and $n = 0, 1, 2, \ldots$ for $k = n + 1$. We are now in a position to add the series in (3) term by term:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} + c_{k-1}]x^k. \tag{4}$$ ■

If you are not convinced of the result in (4), then write out a few terms on both sides of the equality.

6.1.2 POWER SERIES SOLUTIONS

A DEFINITION Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{5}$$

is put into standard form

$$y'' + P(x)y' + Q(x)y = 0 \tag{6}$$

by dividing by the leading coefficient $a_2(x)$. We have the following definition.

DEFINITION 6.1.1 Ordinary and Singular Points

A point x_0 is said to be an **ordinary point** of the differential equation (5) if both $P(x)$ and $Q(x)$ in the standard form (6) are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

Every finite value of x is an ordinary point of the differential equation $y'' + (e^x)y' + (\sin x)y = 0$. In particular, $x = 0$ is an ordinary point because, as we have already seen in (2), both e^x and $\sin x$ are analytic at this point. The negation in the second sentence of Definition 6.1.1 stipulates that if at least one of the functions $P(x)$ and $Q(x)$ in (6) fails to be analytic at x_0, then x_0 is a singular point. Note that $x = 0$ is a singular point of the differential equation $y'' + (e^x)y' + (\ln x)y = 0$ because $Q(x) = \ln x$ is discontinuous at $x = 0$ and so cannot be represented by a power series in x.

POLYNOMIAL COEFFICIENTS We shall be interested primarily in the case when (5) has polynomial coefficients. A polynomial is analytic at any value x, and a rational function is analytic *except* at points where its denominator is zero. Thus if $a_2(x)$, $a_1(x)$, and $a_0(x)$ are *polynomials* with no common factors, then both rational functions $P(x) = a_1(x)/a_2(x)$ and $Q(x) = a_0(x)/a_2(x)$ are analytic except where $a_2(x) = 0$. It follows, then, that:

> $x = x_0$ is an ordinary point of (5) if $a_2(x_0) \neq 0$ whereas $x = x_0$ is a singular point of (5) if $a_2(x_0) = 0$.

For example, the only singular points of the equation $(x^2 - 1)y'' + 2xy' + 6y = 0$ are solutions of $x^2 - 1 = 0$ or $x = \pm 1$. All other finite values* of x are ordinary points. Inspection of the Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$ shows that it has a singular point at $x = 0$. Singular points need not be real numbers. The equation $(x^2 + 1)y'' + xy' - y = 0$ has singular points at the solutions of $x^2 + 1 = 0$—namely, $x = \pm i$. All other values of x, real or complex, are ordinary points.

We state the following theorem about the existence of power series solutions without proof.

THEOREM 6.1.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (5), we can always find two linearly independent solutions in the form of a power series centered at x_0, that is, $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$. A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

*For our purposes, ordinary points and singular points will always be finite points. It is possible for an ODE to have, say, a singular point at infinity.

A solution of the form $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$ is said to be a **solution about the ordinary point x_0.** The distance R in Theorem 6.1.1 is the *minimum value* or the *lower bound* for the radius of convergence of series solutions of the differential equation about x_0.

In the next example we use the fact that in the complex plane the distance between two complex numbers $a + bi$ and $c + di$ is just the distance between the two points (a, b) and (c, d).

EXAMPLE 2 Lower Bound for Radius of Convergence

The complex numbers $1 \pm 2i$ are singular points of the differential equation $(x^2 - 2x + 5)y'' + xy' - y = 0$. Because $x = 0$ is an ordinary point of the equation, Theorem 6.1.1 guarantees that we can find two power series solutions about 0, that is, solutions that look like $y = \sum_{n=0}^{\infty} c_n x^n$. Without actually finding these solutions, we know that *each* series must converge *at least* for $|x| < \sqrt{5}$ because $R = \sqrt{5}$ is the distance in the complex plane from 0 (the point (0, 0)) to either of the numbers $1 + 2i$ (the point (1, 2)) or $1 - 2i$ (the point $(1, -2)$). However, one of these two solutions is valid on an interval much larger than $-\sqrt{5} < x < \sqrt{5}$; in actual fact this solution is valid on $(-\infty, \infty)$ because it can be shown that one of the two power series solutions about 0 reduces to a polynomial. Therefore we also say that $\sqrt{5}$ is the lower bound for the radius of convergence of series solutions of the differential equation about 0.

If we seek solutions of the given DE about a different ordinary point, say, $x = -1$, then each series $y = \sum_{n=0}^{\infty} c_n(x + 1)^n$ converges at least for $|x| < 2\sqrt{2}$ because the distance from -1 to either $1 + 2i$ or $1 - 2i$ is $R = \sqrt{8} = 2\sqrt{2}$. ■

NOTE In the examples that follow, as well as in Exercises 6.1, we shall, for the sake of simplicity, find power series solutions only about the ordinary point $x = 0$. If it is necessary to find a power series solution of a linear DE about an ordinary point $x_0 \neq 0$, we can simply make the change of variable $t = x - x_0$ in the equation (this translates $x = x_0$ to $t = 0$), find solutions of the new equation of the form $y = \sum_{n=0}^{\infty} c_n t^n$, and then resubstitute $t = x - x_0$.

FINDING A POWER SERIES SOLUTION The actual determination of a power series solution of a homogeneous linear second-order DE is quite analogous to what we did in Section 4.4 in finding particular solutions of nonhomogeneous DEs by the method of undetermined coefficients. Indeed, the power series method of solving a linear DE with variable coefficients is often described as "the method of undetermined *series* coefficients." In brief, here is the idea: We substitute $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation, combine series as we did in Example 1, and then equate all coefficients to the right-hand side of the equation to determine the coefficients c_n. But because the right-hand side is zero, the last step requires, by the identity property in the preceding bulleted list, that all coefficients of x must be equated to zero. No, this does *not* mean that all coefficients *are* zero; this would not make sense—after all, Theorem 6.1.1 guarantees that we can find two solutions. Example 3 illustrates how the single assumption that $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ leads to two sets of coefficients, so we have two distinct power series $y_1(x)$ and $y_2(x)$, both expanded about the ordinary point $x = 0$. The general solution of the differential equation is $y = C_1 y_1(x) + C_2 y_2(x)$; indeed, it can be shown that $C_1 = c_0$ and $C_2 = c_1$.

EXAMPLE 3 Power Series Solutions

Solve $y'' + xy = 0$.

SOLUTION Since there are no finite singular points, Theorem 6.1.1 guarantees two power series solutions centered at 0, convergent for $|x| < \infty$. Substituting

$y = \sum_{n=0}^{\infty} c_n x^n$ and the second derivative $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$ (see (1)) into the differential equation gives

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}. \quad (7)$$

In Example 1 we already added the last two series on the right-hand side of the equality in (7) by shifting the summation index. From the result given in (4),

$$y'' + xy = 2c_2 + \sum_{k=1}^{\infty} [(k+1)(k+2)c_{k+2} + c_{k-1}]x^k = 0. \quad (8)$$

At this point we invoke the identity property. Since (8) is identically zero, it is necessary that the coefficient of each power of x be set equal to zero—that is, $2c_2 = 0$ (it is the coefficient of x^0), and

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0, \qquad k = 1, 2, 3, \ldots. \quad (9)$$

Now $2c_2 = 0$ obviously dictates that $c_2 = 0$. But the expression in (9), called a **recurrence relation,** determines the c_k in such a manner that we can choose a certain subset of the set of coefficients to be *nonzero.* Since $(k+1)(k+2) \neq 0$ for all values of k, we can solve (9) for c_{k+2} in terms of c_{k-1}:

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, \qquad k = 1, 2, 3, \ldots. \quad (10)$$

This relation generates consecutive coefficients of the assumed solution one at a time as we let k take on the successive integers indicated in (10):

$$k = 1, \qquad c_3 = -\frac{c_0}{2 \cdot 3}$$

$$k = 2, \qquad c_4 = -\frac{c_1}{3 \cdot 4}$$

$$k = 3, \qquad c_5 = -\frac{c_2}{4 \cdot 5} = 0 \qquad \leftarrow c_2 \text{ is zero}$$

$$k = 4, \qquad c_6 = -\frac{c_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} c_0$$

$$k = 5, \qquad c_7 = -\frac{c_4}{6 \cdot 7} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1$$

$$k = 6, \qquad c_8 = -\frac{c_5}{7 \cdot 8} = 0 \qquad \leftarrow c_5 \text{ is zero}$$

$$k = 7, \qquad c_9 = -\frac{c_6}{8 \cdot 9} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0$$

$$k = 8, \qquad c_{10} = -\frac{c_7}{9 \cdot 10} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1$$

$$k = 9, \qquad c_{11} = -\frac{c_8}{10 \cdot 11} = 0 \qquad \leftarrow c_8 \text{ is zero}$$

and so on. Now substituting the coefficients just obtained into the original assumption

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + \cdots,$$

we get

$$y = c_0 + c_1x + 0 - \frac{c_0}{2\cdot 3}x^3 - \frac{c_1}{3\cdot 4}x^4 + 0 + \frac{c_0}{2\cdot 3\cdot 5\cdot 6}x^6$$
$$+ \frac{c_1}{3\cdot 4\cdot 6\cdot 7}x^7 + 0 - \frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 - \frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + 0 + \cdots.$$

After grouping the terms containing c_0 and the terms containing c_1, we obtain $y = c_0y_1(x) + c_1y_2(x)$, where

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \cdots = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{2\cdot 3\cdots(3k-1)(3k)}x^{3k}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + \cdots = x + \sum_{k=1}^{\infty}\frac{(-1)^k}{3\cdot 4\cdots(3k)(3k+1)}x^{3k+1}.$$

Because the recursive use of (10) leaves c_0 and c_1 completely undetermined, they can be chosen arbitrarily. As was mentioned prior to this example, the linear combination $y = c_0y_1(x) + c_1y_2(x)$ actually represents the general solution of the differential equation. Although we know from Theorem 6.1.1 that each series solution converges for $|x| < \infty$, this fact can also be verified by the ratio test. ■

The differential equation in Example 3 is called **Airy's equation** and is encountered in the study of diffraction of light, diffraction of radio waves around the surface of the Earth, aerodynamics, and the deflection of a uniform thin vertical column that bends under its own weight. Other common forms of Airy's equation are $y'' - xy = 0$ and $y'' + \alpha^2xy = 0$. See Problem 41 in Exercises 6.3 for an application of the last equation.

EXAMPLE 4 Power Series Solution

Solve $(x^2 + 1)y'' + xy' - y = 0$.

SOLUTION As we have already seen on page 223, the given differential equation has singular points at $x = \pm i$, and so a power series solution centered at 0 will converge at least for $|x| < 1$, where 1 is the distance in the complex plane from 0 to either i or $-i$. The assumption $y = \sum_{n=0}^{\infty} c_nx^n$ and its first two derivatives (see (1)) lead to

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + x\sum_{n=1}^{\infty} nc_nx^{n-1} - \sum_{n=0}^{\infty} c_nx^n$$
$$= \sum_{n=2}^{\infty} n(n-1)c_nx^n + \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + \sum_{n=1}^{\infty} nc_nx^n - \sum_{n=0}^{\infty} c_nx^n$$
$$= 2c_2x^0 - c_0x^0 + 6c_3x + c_1x - c_1x + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_nx^n}_{k=n}$$
$$+ \underbrace{\sum_{n=4}^{\infty} n(n-1)c_nx^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty} nc_nx^n}_{k=n} - \underbrace{\sum_{n=2}^{\infty} c_nx^n}_{k=n}$$
$$= 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty}[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k$$
$$= 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0.$$

From this identity we conclude that $2c_2 - c_0 = 0$, $6c_3 = 0$, and

$$(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0.$$

Thus
$$\begin{aligned} c_2 &= \frac{1}{2}c_0 \\ c_3 &= 0 \\ c_{k+2} &= \frac{1-k}{k+2}c_k, \qquad k = 2, 3, 4, \ldots. \end{aligned}$$

Substituting $k = 2, 3, 4, \ldots$ into the last formula gives

$$\begin{aligned}
c_4 &= -\frac{1}{4}c_2 = -\frac{1}{2\cdot 4}c_0 = -\frac{1}{2^2 2!}c_0 \\
c_5 &= -\frac{2}{5}c_3 = 0 && \leftarrow c_3 \text{ is zero} \\
c_6 &= -\frac{3}{6}c_4 = \frac{3}{2\cdot 4\cdot 6}c_0 = \frac{1\cdot 3}{2^3 3!}c_0 \\
c_7 &= -\frac{4}{7}c_5 = 0 && \leftarrow c_5 \text{ is zero} \\
c_8 &= -\frac{5}{8}c_6 = -\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}c_0 = -\frac{1\cdot 3\cdot 5}{2^4 4!}c_0 \\
c_9 &= -\frac{6}{9}c_7 = 0, && \leftarrow c_7 \text{ is zero} \\
c_{10} &= -\frac{7}{10}c_8 = \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}c_0 = \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}c_0,
\end{aligned}$$

and so on. Therefore

$$\begin{aligned}
y &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9 + c_{10}x^{10} + \cdots \\
&= c_0\left[1 + \frac{1}{2}x^2 - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^6 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^8 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^{10} - \cdots\right] + c_1x \\
&= c_0y_1(x) + c_1y_2(x).
\end{aligned}$$

The solutions are the polynomial $y_2(x) = x$ and the power series

$$y_1(x) = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n}, \qquad |x| < 1. \qquad \blacksquare$$

EXAMPLE 5 Three-Term Recurrence Relation

If we seek a power series solution $y = \sum_{n=0}^{\infty} c_nx^n$ for the differential equation

$$y'' - (1+x)y = 0,$$

we obtain $c_2 = \frac{1}{2}c_0$ and the three-term recurrence relation

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, \qquad k = 1, 2, 3, \ldots.$$

It follows from these two results that all coefficients c_n, for $n \geq 3$, are expressed in terms of *both* c_0 and c_1. To simplify life, we can first choose $c_0 \neq 0$, $c_1 = 0$; this yields coefficients for one solution expressed entirely in terms of c_0. Next, if we choose $c_0 = 0$, $c_1 \neq 0$, then coefficients for the other solution are expressed in terms of c_1. Using $c_2 = \frac{1}{2}c_0$ in both cases, the recurrence relation for $k = 1, 2, 3, \ldots$ gives

$c_0 \neq 0,\ c_1 = 0$	$c_0 = 0,\ c_1 \neq 0$
$c_2 = \dfrac{1}{2}c_0$	$c_2 = \dfrac{1}{2}c_0 = 0$
$c_3 = \dfrac{c_1 + c_0}{2 \cdot 3} = \dfrac{c_0}{2 \cdot 3} = \dfrac{c_0}{6}$	$c_3 = \dfrac{c_1 + c_0}{2 \cdot 3} = \dfrac{c_1}{2 \cdot 3} = \dfrac{c_1}{6}$
$c_4 = \dfrac{c_2 + c_1}{3 \cdot 4} = \dfrac{c_0}{2 \cdot 3 \cdot 4} = \dfrac{c_0}{24}$	$c_4 = \dfrac{c_2 + c_1}{3 \cdot 4} = \dfrac{c_1}{3 \cdot 4} = \dfrac{c_1}{12}$
$c_5 = \dfrac{c_3 + c_2}{4 \cdot 5} = \dfrac{c_0}{4 \cdot 5}\left[\dfrac{1}{6} + \dfrac{1}{2}\right] = \dfrac{c_0}{30}$	$c_5 = \dfrac{c_3 + c_2}{4 \cdot 5} = \dfrac{c_1}{4 \cdot 5 \cdot 6} = \dfrac{c_1}{120}$

and so on. Finally, we see that the general solution of the equation is $y = c_0 y_1(x) + c_1 y_2(x)$, where

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$$

and

$$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots.$$

Each series converges for all finite values of x. ■

NONPOLYNOMIAL COEFFICIENTS The next example illustrates how to find a power series solution about the ordinary point $x_0 = 0$ of a differential equation when its coefficients are not polynomials. In this example we see an application of the multiplication of two power series.

EXAMPLE 6 DE with Nonpolynomial Coefficients

Solve $y'' + (\cos x)y = 0$.

SOLUTION We see that $x = 0$ is an ordinary point of the equation because, as we have already seen, $\cos x$ is analytic at that point. Using the Maclaurin series for $\cos x$ given in (2), along with the usual assumption $y = \sum_{n=0}^{\infty} c_n x^n$ and the results in (1), we find

$$\begin{aligned} y'' + (\cos x)y &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\sum_{n=0}^{\infty} c_n x^n \\ &= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) \\ &= 2c_2 + c_0 + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3 + \cdots = 0. \end{aligned}$$

It follows that

$$2c_2 + c_0 = 0, \qquad 6c_3 + c_1 = 0, \qquad 12c_4 + c_2 - \frac{1}{2}c_0 = 0, \qquad 20c_5 + c_3 - \frac{1}{2}c_1 = 0,$$

and so on. This gives $c_2 = -\frac{1}{2}c_0$, $c_3 = -\frac{1}{6}c_1$, $c_4 = \frac{1}{12}c_0$, $c_5 = \frac{1}{30}c_1$, ... By grouping terms, we arrive at the general solution $y = c_0 y_1(x) + c_1 y_2(x)$, where

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots \quad \text{and} \quad y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots.$$

Because the differential equation has no finite singular points, both power series converge for $|x| < \infty$. ■

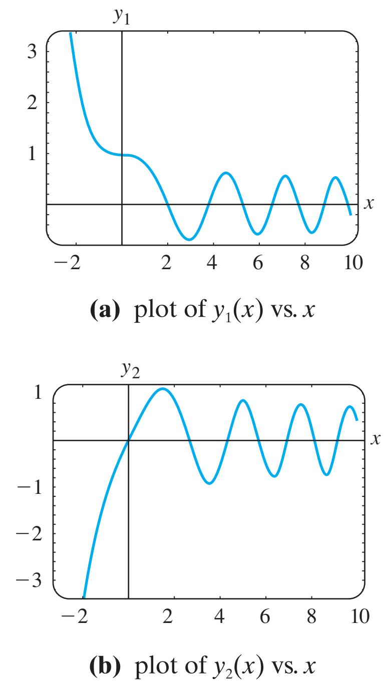

FIGURE 6.1.2 Numerical solution curves for Airy's DE

SOLUTION CURVES The approximate graph of a power series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ can be obtained in several ways. We can always resort to graphing the terms in the sequence of partial sums of the series—in other words, the graphs of the polynomials $S_N(x) = \sum_{n=0}^{N} c_n x^n$. For large values of N, $S_N(x)$ should give us an indication of the behavior of $y(x)$ near the ordinary point $x = 0$. We can also obtain an approximate or numerical solution curve by using a solver as we did in Section 4.9. For example, if you carefully scrutinize the series solutions of Airy's equation in Example 3, you should see that $y_1(x)$ and $y_2(x)$ are, in turn, the solutions of the initial-value problems

$$\begin{aligned} y'' + xy &= 0, \quad y(0) = 1, \quad y'(0) = 0, \\ y'' + xy &= 0, \quad y(0) = 0, \quad y'(0) = 1. \end{aligned} \tag{11}$$

The specified initial conditions "pick out" the solutions $y_1(x)$ and $y_2(x)$ from $y = c_0 y_1(x) + c_1 y_2(x)$, since it should be apparent from our basic series assumption $y = \sum_{n=0}^{\infty} c_n x^n$ that $y(0) = c_0$ and $y'(0) = c_1$. Now if your numerical solver requires a system of equations, the substitution $y' = u$ in $y'' + xy = 0$ gives $y'' = u' = -xy$, and so a system of two first-order equations equivalent to Airy's equation is

$$\begin{aligned} y' &= u \\ u' &= -xy. \end{aligned} \tag{12}$$

Initial conditions for the system in (12) are the two sets of initial conditions in (11) rewritten as $y(0) = 1$, $u(0) = 0$, and $y(0) = 0$, $u(0) = 1$. The graphs of $y_1(x)$ and $y_2(x)$ shown in Figure 6.1.2 were obtained with the aid of a numerical solver. The fact that the numerical solution curves appear to be oscillatory is consistent with the fact that Airy's equation appeared in Section 5.1 (page 186) in the form $mx'' + ktx = 0$ as a model of a spring whose "spring constant" $K(t) = kt$ increases with time.

REMARKS

(*i*) In the problems that follow, do not expect to be able to write a solution in terms of summation notation in each case. Even though we can generate as many terms as desired in a series solution $y = \sum_{n=0}^{\infty} c_n x^n$ either through the use of a recurrence relation or, as in Example 6, by multiplication, it might not be possible to deduce any general term for the coefficients c_n. We might have to settle, as we did in Examples 5 and 6, for just writing out the first few terms of the series.

(*ii*) A point x_0 is an ordinary point of a *nonhomogeneous* linear second-order DE $y'' + P(x)y' + Q(x)y = f(x)$ if $P(x)$, $Q(x)$, and $f(x)$ are analytic at x_0. Moreover, Theorem 6.1.1 extends to such DEs; in other words, we can find power series solutions $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ of nonhomogeneous linear DEs in the same manner as in Examples 3–6. See Problem 36 in Exercises 6.1.

EXERCISES 6.1

Answers to selected odd-numbered problems begin on page ANS-8.

6.1.1 REVIEW OF POWER SERIES

In Problems 1–4 find the radius of convergence and interval of convergence for the given power series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$ **2.** $\displaystyle\sum_{n=0}^{\infty} \frac{(100)^n}{n!}(x+7)^n$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{10^k}(x-5)^k$ **4.** $\displaystyle\sum_{k=0}^{\infty} k!(x-1)^k$

In Problems 5 and 6 the given function is analytic at $x = 0$. Find the first four terms of a power series in x. Perform the multiplication by hand or use a CAS, as instructed.

5. $\sin x \cos x$ **6.** $e^{-x}\cos x$

In Problems 7 and 8 the given function is analytic at $x = 0$. Find the first four terms of a power series in x. Perform the long division by hand or use a CAS, as instructed. Give the open interval of convergence.

7. $\dfrac{1}{\cos x}$ **8.** $\dfrac{1-x}{2+x}$

In Problems 9 and 10 rewrite the given power series so that its general term involves x^k.

9. $\displaystyle\sum_{n=1}^{\infty} nc_n x^{n+2}$ **10.** $\displaystyle\sum_{n=3}^{\infty} (2n-1)c_n x^{n-3}$

In Problems 11 and 12 rewrite the given expression as a single power series whose general term involves x^k.

11. $\displaystyle\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$

12. $\displaystyle\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$

In Problems 13 and 14 verify by direct substitution that the given power series is a particular solution of the indicated differential equation.

13. $\displaystyle y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad (x+1)y'' + y' = 0$

14. $\displaystyle y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}, \quad xy'' + y' + xy = 0$

6.1.2 POWER SERIES SOLUTIONS

In Problems 15 and 16 without actually solving the given differential equation, find a lower bound for the radius of convergence of power series solutions about the ordinary point $x = 0$. About the ordinary point $x = 1$.

15. $(x^2 - 25)y'' + 2xy' + y = 0$

16. $(x^2 - 2x + 10)y'' + xy' - 4y = 0$

In Problems 17–28 find two power series solutions of the given differential equation about the ordinary point $x = 0$.

17. $y'' - xy = 0$ **18.** $y'' + x^2y = 0$

19. $y'' - 2xy' + y = 0$ **20.** $y'' - xy' + 2y = 0$

21. $y'' + x^2y' + xy = 0$ **22.** $y'' + 2xy' + 2y = 0$

23. $(x - 1)y'' + y' = 0$ **24.** $(x + 2)y'' + xy' - y = 0$

25. $y'' - (x + 1)y' - y = 0$

26. $(x^2 + 1)y'' - 6y = 0$

27. $(x^2 + 2)y'' + 3xy' - y = 0$

28. $(x^2 - 1)y'' + xy' - y = 0$

In Problems 29–32 use the power series method to solve the given initial-value problem.

29. $(x - 1)y'' - xy' + y = 0, \quad y(0) = -2, y'(0) = 6$

30. $(x + 1)y'' - (2 - x)y' + y = 0, \quad y(0) = 2, y'(0) = -1$

31. $y'' - 2xy' + 8y = 0, \quad y(0) = 3, y'(0) = 0$

32. $(x^2 + 1)y'' + 2xy' = 0, \quad y(0) = 0, y'(0) = 1$

In Problems 33 and 34 use the procedure in Example 6 to find two power series solutions of the given differential equation about the ordinary point $x = 0$.

33. $y'' + (\sin x)y = 0$ **34.** $y'' + e^x y' - y = 0$

Discussion Problems

35. Without actually solving the differential equation $(\cos x)y'' + y' + 5y = 0$, find a lower bound for the radius of convergence of power series solutions about $x = 0$. About $x = 1$.

36. How can the method described in this section be used to find a power series solution of the *nonhomogeneous* equation $y'' - xy = 1$ about the ordinary point $x = 0$? Of $y'' - 4xy' - 4y = e^x$? Carry out your ideas by solving both DEs.

37. Is $x = 0$ an ordinary or a singular point of the differential equation $xy'' + (\sin x)y = 0$? Defend your answer with sound mathematics.

38. For purposes of this problem, ignore the graphs given in Figure 6.1.2. If Airy's DE is written as $y'' = -xy$, what can we say about the shape of a solution curve if $x > 0$ and $y > 0$? If $x > 0$ and $y < 0$?

Computer Lab Assignments

39. **(a)** Find two power series solutions for $y'' + xy' + y = 0$ and express the solutions $y_1(x)$ and $y_2(x)$ in terms of summation notation.

(b) Use a CAS to graph the partial sums $S_N(x)$ for $y_1(x)$. Use $N = 2, 3, 5, 6, 8, 10$. Repeat using the partial sums $S_N(x)$ for $y_2(x)$.

(c) Compare the graphs obtained in part (b) with the curve obtained by using a numerical solver. Use the initial-conditions $y_1(0) = 1$, $y_1'(0) = 0$, and $y_2(0) = 0$, $y_2'(0) = 1$.

(d) Reexamine the solution $y_1(x)$ in part (a). Express this series as an elementary function. Then use (5) of Section 4.2 to find a second solution of the equation. Verify that this second solution is the same as the power series solution $y_2(x)$.

40. **(a)** Find one more nonzero term for each of the solutions $y_1(x)$ and $y_2(x)$ in Example 6.

(b) Find a series solution $y(x)$ of the initial-value problem $y'' + (\cos x)y = 0$, $y(0) = 1$, $y'(0) = 1$.

(c) Use a CAS to graph the partial sums $S_N(x)$ for the solution $y(x)$ in part (b). Use $N = 2, 3, 4, 5, 6, 7$.

(d) Compare the graphs obtained in part (c) with the curve obtained using a numerical solver for the initial-value problem in part (b).

6.2 SOLUTIONS ABOUT SINGULAR POINTS

REVIEW MATERIAL

- Section 4.2 (especially (5) of that section)

INTRODUCTION The two differential equations

$$y'' + xy = 0 \qquad \text{and} \qquad xy'' + y = 0$$

are similar only in that they are both examples of simple linear second-order DEs with variable coefficients. That is all they have in common. Since $x = 0$ is an *ordinary point* of $y'' + xy = 0$, we saw in Section 6.1 that there was no problem in finding two distinct power series solutions centered at that point. In contrast, because $x = 0$ is a *singular point* of $xy'' + y = 0$, finding two infinite series—notice that we did not say *power series*—solutions of the equation about that point becomes a more difficult task.

The solution method that is discussed in this section does not always yield two infinite series solutions. When only one solution is found, we can use the formula given in (5) of Section 4.2 to find a second solution.

A DEFINITION A singular point x_0 of a linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is further classified as either regular or irregular. The classification again depends on the functions P and Q in the standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

DEFINITION 6.2.1 Regular and Irregular Singular Points

A singular point x_0 is said to be a **regular singular point** of the differential equation (1) if the functions $p(x) = (x - x_0)\,P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the equation.

The second sentence in Definition 6.2.1 indicates that if one or both of the functions $p(x) = (x - x_0)\,P(x)$ and $q(x) = (x - x_0)^2Q(x)$ fail to be analytic at x_0, then x_0 is an irregular singular point.

6.3 SPECIAL FUNCTIONS

REVIEW MATERIAL

- Sections 6.1 and 6.2

INTRODUCTION The two differential equations

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1}$$

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \tag{2}$$

occur in advanced studies in applied mathematics, physics, and engineering. They are called **Bessel's equation of order ν** and **Legendre's equation of order n**, respectively. When we solve (1) we shall assume that $\nu \geq 0$, whereas in (2) we shall consider only the case when n is a nonnegative integer.

6.3.1 BESSEL'S EQUATION

THE SOLUTION Because $x = 0$ is a regular singular point of Bessel's equation, we know that there exists at least one solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Substituting the last expression into (1) gives

$$\begin{aligned} x^2y'' + xy' + (x^2 - \nu^2)y &= \sum_{n=0}^{\infty} c_n(n + r)(n + r - 1)x^{n+r} + \sum_{n=0}^{\infty} c_n(n + r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= c_0(r^2 - r + r - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n + r)(n + r - 1) + (n + r) - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= c_0(r^2 - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n + r)^2 - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}. \end{aligned} \tag{3}$$

From (3) we see that the indicial equation is $r^2 - \nu^2 = 0$, so the indicial roots are $r_1 = \nu$ and $r_2 = -\nu$. When $r_1 = \nu$, (3) becomes

$$\begin{aligned} x^\nu \sum_{n=1}^{\infty} c_n n(n + 2\nu)x^n + x^\nu \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= x^\nu \left[(1 + 2\nu)c_1 x + \underbrace{\sum_{n=2}^{\infty} c_n n(n + 2\nu)x^n}_{k = n - 2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{k = n}\right] \\ &= x^\nu \left[(1 + 2\nu)c_1 x + \sum_{k=0}^{\infty} [(k + 2)(k + 2 + 2\nu)c_{k+2} + c_k]x^{k+2}\right] = 0. \end{aligned}$$

Therefore by the usual argument we can write $(1 + 2\nu)c_1 = 0$ and

$$(k + 2)(k + 2 + 2\nu)c_{k+2} + c_k = 0$$

or

$$c_{k+2} = \frac{-c_k}{(k + 2)(k + 2 + 2\nu)}, \qquad k = 0, 1, 2, \ldots. \tag{4}$$

The choice $c_1 = 0$ in (4) implies that $c_3 = c_5 = c_7 = \cdots = 0$, so for $k = 0, 2, 4, \ldots$ we find, after letting $k + 2 = 2n$, $n = 1, 2, 3, \ldots$, that

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n + \nu)}. \tag{5}$$

Thus
$$c_2 = -\frac{c_0}{2^2 \cdot 1 \cdot (1+\nu)}$$
$$c_4 = -\frac{c_2}{2^2 \cdot 2(2+\nu)} = \frac{c_0}{2^4 \cdot 1 \cdot 2(1+\nu)(2+\nu)}$$
$$c_6 = -\frac{c_4}{2^2 \cdot 3(3+\nu)} = -\frac{c_0}{2^6 \cdot 1 \cdot 2 \cdot 3(1+\nu)(2+\nu)(3+\nu)}$$
$$\vdots$$
$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)}, \qquad n = 1, 2, 3, \ldots. \tag{6}$$

It is standard practice to choose c_0 to be a specific value, namely,

$$c_0 = \frac{1}{2^\nu \Gamma(1+\nu)},$$

where $\Gamma(1+\nu)$ is the gamma function. See Appendix I. Since this latter function possesses the convenient property $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, we can reduce the indicated product in the denominator of (6) to one term. For example,

$$\Gamma(1+\nu+1) = (1+\nu)\Gamma(1+\nu)$$
$$\Gamma(1+\nu+2) = (2+\nu)\Gamma(2+\nu) = (2+\nu)(1+\nu)\Gamma(1+\nu).$$

Hence we can write (6) as

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!(1+\nu)(2+\nu)\cdots(n+\nu)\Gamma(1+\nu)} = \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)}$$

for $n = 0, 1, 2, \ldots$.

BESSEL FUNCTIONS OF THE FIRST KIND Using the coefficients c_{2n} just obtained and $r = \nu$, a series solution of (1) is $y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$. This solution is usually denoted by $J_\nu(x)$:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{7}$$

If $\nu \geq 0$, the series converges at least on the interval $[0, \infty)$. Also, for the second exponent $r_2 = -\nu$ we obtain, in exactly the same manner,

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{8}$$

The functions $J_\nu(x)$ and $J_{-\nu}(x)$ are called **Bessel functions of the first kind** of order ν and $-\nu$, respectively. Depending on the value of ν, (8) may contain negative powers of x and hence converges on $(0, \infty)$.*

Now some care must be taken in writing the general solution of (1). When $\nu = 0$, it is apparent that (7) and (8) are the same. If $\nu > 0$ and $r_1 - r_2 = \nu - (-\nu) = 2\nu$ is not a positive integer, it follows from Case I of Section 6.2 that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions of (1) on $(0, \infty)$, and so the general solution on the interval is $y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$. But we also know from Case II of Section 6.2 that when $r_1 - r_2 = 2\nu$ is a positive integer, a second series solution of (1) *may* exist. In this second case we distinguish two possibilities. When $\nu = m =$ positive integer, $J_{-m}(x)$ defined by (8) and $J_m(x)$ are not linearly independent solutions. It can be shown that J_{-m} is a constant multiple of J_m (see Property (*i*) on page 245). In addition, $r_1 - r_2 = 2\nu$ can be a positive integer when ν is half an odd positive integer. It can be shown in this

*When we replace x by $|x|$, the series given in (7) and (8) converge for $0 < |x| < \infty$.

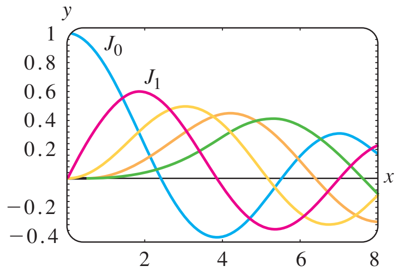

FIGURE 6.3.1 Bessel functions of the first kind for $n = 0, 1, 2, 3, 4$

latter event that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. In other words, the general solution of (1) on $(0, \infty)$ is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \qquad \nu \neq \text{integer}. \tag{9}$$

The graphs of $y = J_0(x)$ and $y = J_1(x)$ are given in Figure 6.3.1.

EXAMPLE 1 Bessel's Equation of Order $\frac{1}{2}$

By identifying $\nu^2 = \frac{1}{4}$ and $\nu = \frac{1}{2}$, we can see from (9) that the general solution of the equation $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ on $(0, \infty)$ is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$. ■

BESSEL FUNCTIONS OF THE SECOND KIND If $\nu \neq$ integer, the function defined by the linear combination

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \tag{10}$$

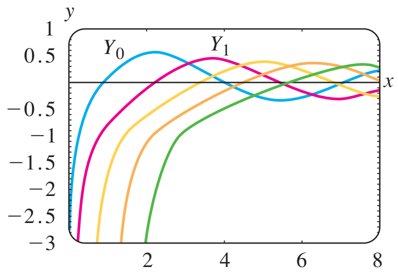

FIGURE 6.3.2 Bessel functions of the second kind for $n = 0, 1, 2, 3, 4$

and the function $J_\nu(x)$ are linearly independent solutions of (1). Thus another form of the general solution of (1) is $y = c_1 J_\nu(x) + c_2 Y_\nu(x)$, provided that $\nu \neq$ integer. As $\nu \to m$, m an integer, (10) has the indeterminate form $0/0$. However, it can be shown by L'Hôpital's Rule that $\lim_{\nu \to m} Y_\nu(x)$ exists. Moreover, the function

$$Y_m(x) = \lim_{\nu \to m} Y_\nu(x)$$

and $J_m(x)$ are linearly independent solutions of $x^2y'' + xy' + (x^2 - m^2)y = 0$. Hence for *any* value of ν the general solution of (1) on $(0, \infty)$ can be written as

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x). \tag{11}$$

$Y_\nu(x)$ is called the **Bessel function of the second kind** of order ν. Figure 6.3.2 shows the graphs of $Y_0(x)$ and $Y_1(x)$.

EXAMPLE 2 Bessel's Equation of Order 3

By identifying $\nu^2 = 9$ and $\nu = 3$, we see from (11) that the general solution of the equation $x^2y'' + xy' + (x^2 - 9)y = 0$ on $(0, \infty)$ is $y = c_1 J_3(x) + c_2 Y_3(x)$. ■

DEs SOLVABLE IN TERMS OF BESSEL FUNCTIONS Sometimes it is possible to transform a differential equation into equation (1) by means of a change of variable. We can then express the solution of the original equation in terms of Bessel functions. For example, if we let $t = \alpha x$, $\alpha > 0$, in

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0, \tag{12}$$

then by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \alpha\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \alpha^2\frac{d^2y}{dt^2}.$$

Accordingly, (12) becomes

$$\left(\frac{t}{\alpha}\right)^2\alpha^2\frac{d^2y}{dt^2} + \left(\frac{t}{\alpha}\right)\alpha\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \quad \text{or} \quad t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0.$$

The last equation is Bessel's equation of order ν with solution $y = c_1 J_\nu(t) + c_2 Y_\nu(t)$. By resubstituting $t = \alpha x$ in the last expression, we find that the general solution of (12) is

$$y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x). \tag{13}$$

Equation (12), called the **parametric Bessel equation of order ν,** and its general solution (13) are very important in the study of certain boundary-value problems involving partial differential equations that are expressed in cylindrical coordinates.

Another equation that bears a resemblance to (1) is the **modified Bessel equation of order ν,**

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0. \tag{14}$$

This DE can be solved in the manner just illustrated for (12). This time if we let $t = ix$, where $i^2 = -1$, then (14) becomes

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0.$$

Because solutions of the last DE are $J_\nu(t)$ and $Y_\nu(t)$, *complex-valued* solutions of (14) are $J_\nu(ix)$ and $Y_\nu(ix)$. A real-valued solution, called the **modified Bessel function of the first kind** of order ν, is defined in terms of $J_\nu(ix)$:

$$I_\nu(x) = i^{-\nu}J_\nu(ix). \tag{15}$$

See Problem 21 in Exercises 6.3. Analogous to (10), the **modified Bessel function of the second kind** of order $\nu \neq$ integer is defined to be

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}, \tag{16}$$

and for integer $\nu = n$,

$$K_n(x) = \lim_{\nu\to n} K_\nu(x).$$

Because I_ν and K_ν are linearly independent on the interval $(0, \infty)$ for any value of ν, the general solution of (14) is

$$y = c_1I_\nu(x) + c_2K_\nu(x). \tag{17}$$

Yet another equation, important because many DEs fit into its form by appropriate choices of the parameters, is

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right)y = 0, \qquad p \ge 0. \tag{18}$$

Although we shall not supply the details, the general solution of (18),

$$y = x^a\left[c_1J_p(bx^c) + c_2Y_p(bx^c)\right], \tag{19}$$

can be found by means of a change in both the independent and the dependent variables: $z = bx^c$, $y(x) = \left(\frac{z}{b}\right)^{a/c} w(z)$. If p is not an integer, then Y_p in (19) can be replaced by J_{-p}.

EXAMPLE 3 Using (18)

Find the general solution of $xy'' + 3y' + 9y = 0$ on $(0, \infty)$.

SOLUTION By writing the given DE as

$$y'' + \frac{3}{x}y' + \frac{9}{x}y = 0,$$

we can make the following identifications with (18):

$$1 - 2a = 3, \qquad b^2c^2 = 9, \qquad 2c - 2 = -1, \qquad \text{and} \qquad a^2 - p^2c^2 = 0.$$

The first and third equations imply that $a = -1$ and $c = \frac{1}{2}$. With these values the second and fourth equations are satisfied by taking $b = 6$ and $p = 2$. From (19) we find that the general solution of the given DE on the interval $(0, \infty)$ is $y = x^{-1}[c_1 J_2(6x^{1/2}) + c_2 Y_2(6x^{1/2})]$. ■

EXAMPLE 4 The Aging Spring Revisited

Recall that in Section 5.1 we saw that one mathematical model for the free undamped motion of a mass on an aging spring is given by $mx'' + ke^{-\alpha t}x = 0$, $\alpha > 0$. We are now in a position to find the general solution of the equation. It is left as a problem to show that the change of variables $s = \dfrac{2}{\alpha}\sqrt{\dfrac{k}{m}}\,e^{-\alpha t/2}$ transforms the differential equation of the aging spring into

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0.$$

The last equation is recognized as (1) with $\nu = 0$ and where the symbols x and s play the roles of y and x, respectively. The general solution of the new equation is $x = c_1J_0(s) + c_2Y_0(s)$. If we resubstitute s, then the general solution of $mx'' + ke^{-\alpha t}x = 0$ is seen to be

$$x(t) = c_1J_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + c_2Y_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right).$$

See Problems 33 and 39 in Exercises 6.3. ■

The other model that was discussed in Section 5.1 of a spring whose characteristics change with time was $mx'' + ktx = 0$. By dividing through by m, we see that the equation $x'' + \dfrac{k}{m}tx = 0$ is Airy's equation $y'' + \alpha^2xy = 0$. See Example 3 in Section 6.1. The general solution of Airy's differential equation can also be written in terms of Bessel functions. See Problems 34, 35, and 40 in Exercises 6.3.

PROPERTIES We list below a few of the more useful properties of Bessel functions of order m, $m = 0, 1, 2, \ldots$:

$$(i)\;\; J_{-m}(x) = (-1)^mJ_m(x), \qquad (ii)\;\; J_m(-x) = (-1)^mJ_m(x),$$

$$(iii)\;\; J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0, \end{cases} \qquad (iv)\;\; \lim_{x\to 0^+} Y_m(x) = -\infty.$$

Note that Property (*ii*) indicates that $J_m(x)$ is an even function if m is an even integer and an odd function if m is an odd integer. The graphs of $Y_0(x)$ and $Y_1(x)$ in Figure 6.3.2 illustrate Property (*iv*), namely, $Y_m(x)$ is unbounded at the origin. This last fact is not obvious from (10). The solutions of the Bessel equation of order 0 can be obtained by using the solutions $y_1(x)$ in (21) and $y_2(x)$ in (22) of Section 6.2. It can be shown that (21) of Section 6.2 is $y_1(x) = J_0(x)$, whereas (22) of that section is

$$y_2(x) = J_0(x)\ln x - \sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}.$$

The Bessel function of the second kind of order 0, $Y_0(x)$, is then defined to be the linear combination $Y_0(x) = \dfrac{2}{\pi}(\gamma - \ln 2)y_1(x) + \dfrac{2}{\pi}y_2(x)$ for $x > 0$. That is,

$$Y_0(x) = \frac{2}{\pi}J_0(x)\left[\gamma + \ln\frac{x}{2}\right] - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k},$$

where $\gamma = 0.57721566\ldots$ is **Euler's constant.** Because of the presence of the logarithmic term, it is apparent that $Y_0(x)$ is discontinuous at $x = 0$.

NUMERICAL VALUES The first five nonnegative zeros of $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ are given in Table 6.1. Some additional function values of these four functions are given in Table 6.2.

TABLE 6.1 Zeros of J_0, J_1, Y_0, and Y_1

$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
2.4048	0.0000	0.8936	2.1971
5.5201	3.8317	3.9577	5.4297
8.6537	7.0156	7.0861	8.5960
11.7915	10.1735	10.2223	11.7492
14.9309	13.3237	13.3611	14.8974

TABLE 6.2 Numerical Values of J_0, J_1, Y_0, and Y_1

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0	1.0000	0.0000	—	—
1	0.7652	0.4401	0.0883	−0.7812
2	0.2239	0.5767	0.5104	−0.1070
3	−0.2601	0.3391	0.3769	0.3247
4	−0.3971	−0.0660	−0.0169	0.3979
5	−0.1776	−0.3276	−0.3085	0.1479
6	0.1506	−0.2767	−0.2882	−0.1750
7	0.3001	−0.0047	−0.0259	−0.3027
8	0.1717	0.2346	0.2235	−0.1581
9	−0.0903	0.2453	0.2499	0.1043
10	−0.2459	0.0435	0.0557	0.2490
11	−0.1712	−0.1768	−0.1688	0.1637
12	0.0477	−0.2234	−0.2252	−0.0571
13	0.2069	−0.0703	−0.0782	−0.2101
14	0.1711	0.1334	0.1272	−0.1666
15	−0.0142	0.2051	0.2055	0.0211

DIFFERENTIAL RECURRENCE RELATION Recurrence formulas that relate Bessel functions of different orders are important in theory and in applications. In the next example we derive a **differential recurrence relation.**

EXAMPLE 5 Derivation Using the Series Definition

Derive the formula $xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$.

SOLUTION It follows from (7) that

$$\begin{aligned}
xJ'_\nu(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+\nu)}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} \\
&= \nu\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} + 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} \\
&= \nu J_\nu(x) + x\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu-1}}_{k=n-1} \\
&= \nu J_\nu(x) - x\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(2+\nu+k)}\left(\frac{x}{2}\right)^{2k+\nu+1} = \nu J_\nu(x) - xJ_{\nu+1}(x).
\end{aligned}$$

■

The result in Example 5 can be written in an alternative form. Dividing $xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x)$ by x gives

$$J'_\nu(x) - \frac{\nu}{x}J_\nu(x) = -J_{\nu+1}(x).$$

This last expression is recognized as a linear first-order differential equation in $J_\nu(x)$. Multiplying both sides of the equality by the integrating factor $x^{-\nu}$ then yields

$$\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x). \tag{20}$$

It can be shown in a similar manner that

$$\frac{d}{dx}[x^{\nu}J_\nu(x)] = x^{\nu}J_{\nu-1}(x). \tag{21}$$

See Problem 27 in Exercises 6.3. The differential recurrence relations (20) and (21) are also valid for the Bessel function of the second kind $Y_\nu(x)$. Observe that when $\nu = 0$, it follows from (20) that

$$J_0'(x) = -J_1(x) \qquad \text{and} \qquad Y_0'(x) = -Y_1(x). \tag{22}$$

An application of these results is given in Problem 39 of Exercises 6.3.

SPHERICAL BESSEL FUNCTIONS When the order ν is half an odd integer, that is, $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, the Bessel functions of the first kind $J_\nu(x)$ can be expressed in terms of the elementary functions $\sin x$, $\cos x$, and powers of x. Such Bessel functions are called **spherical Bessel functions.** Let's consider the case when $\nu = \frac{1}{2}$. From (7),

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(1 + \frac{1}{2} + n\right)} \left(\frac{x}{2}\right)^{2n+1/2}.$$

In view of the property $\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$ and the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ the values of $\Gamma\left(1 + \frac{1}{2} + n\right)$ for $n = 0$, $n = 1$, $n = 2$, and $n = 3$ are, respectively,

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(1 + \tfrac{1}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(1 + \tfrac{3}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \frac{3}{2^2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \Gamma\left(1 + \tfrac{5}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \frac{5 \cdot 3}{2^3}\sqrt{\pi} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^3 4 \cdot 2}\sqrt{\pi} = \frac{5!}{2^5 2!}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{9}{2}\right) = \Gamma\left(1 + \tfrac{7}{2}\right) = \tfrac{7}{2}\Gamma\left(\tfrac{7}{2}\right) = \frac{7 \cdot 5}{2^6 \cdot 2!}\sqrt{\pi} = \frac{7 \cdot 6 \cdot 5!}{2^6 \cdot 6 \cdot 2!}\sqrt{\pi} = \frac{7!}{2^7 3!}\sqrt{\pi}.$$

In general, $$\Gamma\left(1 + \frac{1}{2} + n\right) = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}.$$

Hence $$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\dfrac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Since the infinite series in the last line is the Maclaurin series for $\sin x$, we have shown that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \tag{23}$$

It is left as an exercise to show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{24}$$

See Problems 31 and 32 in Exercises 6.3.

6.3.2 LEGENDRE'S EQUATION

THE SOLUTION Since $x = 0$ is an ordinary point of Legendre's equation (2), we substitute the series $y = \sum_{k=0}^{\infty} c_k x^k$, shift summation indices, and combine series to get

$$(1 - x^2)y'' - 2xy' + n(n+1)y = [n(n+1)c_0 + 2c_2] + [(n-1)(n+2)c_1 + 6c_3]x$$
$$+ \sum_{j=2}^{\infty} [(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j]x^j = 0$$

which implies that

$$n(n+1)c_0 + 2c_2 = 0$$
$$(n-1)(n+2)c_1 + 6c_3 = 0$$
$$(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j = 0$$

or

$$c_2 = -\frac{n(n+1)}{2!}c_0$$
$$c_3 = -\frac{(n-1)(n+2)}{3!}c_1$$
$$c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j, \qquad j = 2, 3, 4, \ldots. \tag{25}$$

If we let j take on the values 2, 3, 4, . . . , the recurrence relation (25) yields

$$c_4 = -\frac{(n-2)(n+3)}{4 \cdot 3}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0$$
$$c_5 = -\frac{(n-3)(n+4)}{5 \cdot 4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1$$
$$c_6 = -\frac{(n-4)(n+5)}{6 \cdot 5}c_4 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0$$
$$c_7 = -\frac{(n-5)(n+6)}{7 \cdot 6}c_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1$$

and so on. Thus for at least $|x| < 1$ we obtain two linearly independent power series solutions:

$$\begin{aligned} y_1(x) &= c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 \right.\\ &\qquad \left. - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right] \\ y_2(x) &= c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 \right.\\ &\qquad \left. - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]. \end{aligned} \tag{26}$$

Notice that if n is an even integer, the first series terminates, whereas $y_2(x)$ is an infinite series. For example, if $n = 4$, then

$$y_1(x) = c_0\left[1 - \frac{4 \cdot 5}{2!}x^2 + \frac{2 \cdot 4 \cdot 5 \cdot 7}{4!}x^4\right] = c_0\left[1 - 10x^2 + \frac{35}{3}x^4\right].$$

Similarly, when n is an odd integer, the series for $y_2(x)$ terminates with x^n; that is, *when n is a nonnegative integer, we obtain an nth-degree polynomial solution* of Legendre's equation.

Because we know that a constant multiple of a solution of Legendre's equation is also a solution, it is traditional to choose specific values for c_0 or c_1, depending on whether n is an even or odd positive integer, respectively. For $n = 0$ we choose $c_0 = 1$, and for $n = 2, 4, 6, \ldots$

$$c_0 = (-1)^{n/2}\frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n},$$

whereas for $n = 1$ we choose $c_1 = 1$, and for $n = 3, 5, 7, \ldots$

$$c_1 = (-1)^{(n-1)/2}\frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}.$$

For example, when $n = 4$, we have

$$y_1(x) = (-1)^{4/2}\frac{1 \cdot 3}{2 \cdot 4}\left[1 - 10x^2 + \frac{35}{3}x^4\right] = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

LEGENDRE POLYNOMIALS These specific nth-degree polynomial solutions are called **Legendre polynomials** and are denoted by $P_n(x)$. From the series for $y_1(x)$ and $y_2(x)$ and from the above choices of c_0 and c_1 we find that the first several Legendre polynomials are

$$\begin{aligned} &P_0(x) = 1, && P_1(x) = x, \\ &P_2(x) = \frac{1}{2}(3x^2 - 1), && P_3(x) = \frac{1}{2}(5x^3 - 3x), \\ &P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), && P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned} \tag{27}$$

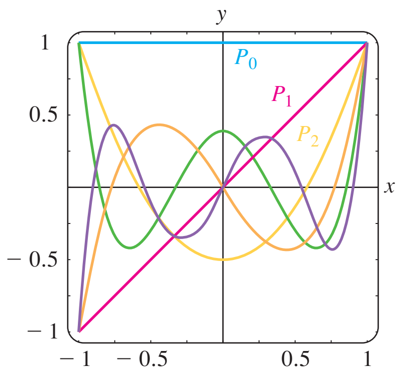

FIGURE 6.3.3 Legendre polynomials for $n = 0, 1, 2, 3, 4, 5$

Remember, $P_0(x), P_1(x), P_2(x), P_3(x), \ldots$ are, in turn, particular solutions of the differential equations

$$\begin{aligned} n = 0{:}&\quad (1 - x^2)y'' - 2xy' = 0, \\ n = 1{:}&\quad (1 - x^2)y'' - 2xy' + 2y = 0, \\ n = 2{:}&\quad (1 - x^2)y'' - 2xy' + 6y = 0, \\ n = 3{:}&\quad (1 - x^2)y'' - 2xy' + 12y = 0, \\ \vdots&\qquad\qquad \vdots \end{aligned} \tag{28}$$

The graphs, on the interval $[-1, 1]$, of the six Legendre polynomials in (27) are given in Figure 6.3.3.

PROPERTIES You are encouraged to verify the following properties using the Legendre polynomials in (27).

$$(i)\ \ P_n(-x) = (-1)^n P_n(x)$$

$$(ii)\ \ P_n(1) = 1 \qquad\qquad (iii)\ \ P_n(-1) = (-1)^n$$

$$(iv)\ \ P_n(0) = 0, \quad n \text{ odd} \qquad\qquad (v)\ \ P_n'(0) = 0, \quad n \text{ even}$$

Property (i) indicates, as is apparent in Figure 6.3.3, that $P_n(x)$ is an even or odd function according to whether n is even or odd.

RECURRENCE RELATION Recurrence relations that relate Legendre polynomials of different degrees are also important in some aspects of their applications. We state, without proof, the three-term recurrence relation

$$(k + 1)P_{k+1}(x) - (2k + 1)xP_k(x) + kP_{k-1}(x) = 0, \tag{29}$$

which is valid for $k = 1, 2, 3, \ldots$. In (27) we listed the first six Legendre polynomials. If, say, we wish to find $P_6(x)$, we can use (29) with $k = 5$. This relation expresses $P_6(x)$ in terms of the known $P_4(x)$ and $P_5(x)$. See Problem 45 in Exercises 6.3.

Another formula, although not a recurrence relation, can generate the Legendre polynomials by differentiation. **Rodrigues' formula** for these polynomials is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \qquad n = 0, 1, 2, \ldots. \tag{30}$$

See Problem 48 in Exercises 6.3.

REMARKS

(*i*) Although we have assumed that the parameter n in Legendre's differential equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$, represented a nonnegative integer, in a more general setting n can represent any real number. Any solution of Legendre's equation is called a **Legendre function.** If n is *not* a nonnegative integer, then both Legendre functions $y_1(x)$ and $y_2(x)$ given in (26) are infinite series convergent on the open interval $(-1, 1)$ and divergent (unbounded) at $x = \pm 1$. If n is a nonnegative integer, then as we have just seen one of the Legendre functions in (26) is a polynomial and the other is an infinite series convergent for $-1 < x < 1$. You should be aware of the fact that Legendre's equation possesses solutions that are bounded on the *closed* interval $[-1, 1]$ only in the case when $n = 0, 1, 2, \ldots$. More to the point, the only Legendre functions that are bounded on the closed interval $[-1, 1]$ are the Legendre polynomials $P_n(x)$ or constant multiples of these polynomials. See Problem 47 in Exercises 6.3 and Problem 24 in Chapter 6 in Review.

(*ii*) In the *Remarks* at the end of Section 2.3 we mentioned the branch of mathematics called **special functions.** Perhaps a better appellation for this field of applied mathematics might be *named functions,* since many of the functions studied bear proper names: Bessel functions, Legendre functions, Airy functions, Chebyshev polynomials, Gauss's hypergeometric function, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, Mathieu functions, Weber functions, and so on. Historically, special functions were the by-product of necessity; someone needed a solution of a very specialized differential equation that arose from an attempt to solve a physical problem.

EXERCISES 6.3

Answers to selected odd-numbered problems begin on page ANS-10.

6.3.1 BESSEL'S EQUATION

In Problems 1–6 use (1) to find the general solution of the given differential equation on $(0, \infty)$.

1. $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

2. $x^2y'' + xy' + (x^2 - 1)y = 0$

3. $4x^2y'' + 4xy' + (4x^2 - 25)y = 0$

4. $16x^2y'' + 16xy' + (16x^2 - 1)y = 0$

5. $xy'' + y' + xy = 0$

6. $\dfrac{d}{dx}[xy'] + \left(x - \dfrac{4}{x}\right)y = 0$

In Problems 7–10 use (12) to find the general solution of the given differential equation on $(0, \infty)$.

7. $x^2y'' + xy' + (9x^2 - 4)y = 0$

8. $x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

9. $x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$

10. $x^2y'' + xy' + (2x^2 - 64)y = 0$

In Problems 11 and 12 use the indicated change of variable to find the general solution of the given differential equation on $(0, \infty)$.

11. $x^2y'' + 2xy' + \alpha^2x^2y = 0; \quad y = x^{-1/2}v(x)$

12. $x^2y'' + \left(\alpha^2x^2 - \nu^2 + \frac{1}{4}\right)y = 0; \quad y = \sqrt{x}\,v(x)$

In Problems 13–20 use (18) to find the general solution of the given differential equation on $(0, \infty)$.

13. $xy'' + 2y' + 4y = 0$ **14.** $xy'' + 3y' + xy = 0$

15. $xy'' - y' + xy = 0$ **16.** $xy'' - 5y' + xy = 0$

17. $x^2y'' + (x^2 - 2)y = 0$

18. $4x^2y'' + (16x^2 + 1)y = 0$

19. $xy'' + 3y' + x^3y = 0$

20. $9x^2y'' + 9xy' + (x^6 - 36)y = 0$

21. Use the series in (7) to verify that $I_\nu(x) = i^{-\nu} J_\nu(ix)$ is a real function.

22. Assume that b in equation (18) can be pure imaginary, that is, $b = \beta i, \beta > 0, i^2 = -1$. Use this assumption to express the general solution of the given differential equation in terms the modified Bessel functions I_n and K_n.

(a) $y'' - x^2y = 0$ **(b)** $xy'' + y' - 7x^3y = 0$

In Problems 23–26 first use (18) to express the general solution of the given differential equation in terms of Bessel functions. Then use (23) and (24) to express the general solution in terms of elementary functions.

23. $y'' + y = 0$

24. $x^2y'' + 4xy' + (x^2 + 2)y = 0$

25. $16x^2y'' + 32xy' + (x^4 - 12)y = 0$

26. $4x^2y'' - 4xy' + (16x^2 + 3)y = 0$

27. **(a)** Proceed as in Example 5 to show that

$$xJ'_\nu(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x).$$

[*Hint*: Write $2n + \nu = 2(n + \nu) - \nu$.]

(b) Use the result in part (a) to derive (21).

28. Use the formula obtained in Example 5 along with part (a) of Problem 27 to derive the recurrence relation

$$2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x).$$

In Problems 29 and 30 use (20) or (21) to obtain the given result.

29. $\int_0^x rJ_0(r)\,dr = xJ_1(x)$ **30.** $J'_0(x) = J_{-1}(x) = -J_1(x)$

31. Proceed as on page 247 to derive the elementary form of $J_{-1/2}(x)$ given in (24).

32. **(a)** Use the recurrence relation in Problem 28 along with (23) and (24) to express $J_{3/2}(x)$, $J_{-3/2}(x)$, and $J_{5/2}(x)$ in terms of $\sin x$, $\cos x$, and powers of x.

(b) Use a graphing utility to graph $J_{1/2}(x)$, $J_{-1/2}(x)$, $J_{3/2}(x)$, $J_{-3/2}(x)$, and $J_{5/2}(x)$.

33. Use the change of variables $s = \dfrac{2}{\alpha}\sqrt{\dfrac{k}{m}}\,e^{-\alpha t/2}$ to show that the differential equation of the aging spring $mx'' + ke^{-\alpha t}x = 0, \alpha > 0$, becomes

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0.$$

34. Show that $y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2xy = 0$, $x > 0$, whenever w is a solution of Bessel's equation of order $\frac{1}{3}$, that is, $t^2w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0$, $t > 0$. [*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$.]

35. **(a)** Use the result of Problem 34 to express the general solution of Airy's differential equation for $x > 0$ in terms of Bessel functions.

(b) Verify the results in part (a) using (18).

36. Use the Table 6.1 to find the first three positive eigenvalues and corresponding eigenfunctions of the boundary-value problem

$$xy'' + y' + \lambda xy = 0,$$

$$y(x),\ y'(x) \text{ bounded as } x \to 0^+, \quad y(2) = 0.$$

[*Hint*: By identifying $\lambda = \alpha^2$, the DE is the parametric Bessel equation of order zero.]

37. **(a)** Use (18) to show that the general solution of the differential equation $xy'' + \lambda y = 0$ on the interval $(0, \infty)$ is

$$y = c_1\sqrt{x}J_1\left(2\sqrt{\lambda x}\right) + c_2\sqrt{x}Y_1\left(2\sqrt{\lambda x}\right).$$

(b) Verify by direct substitution that $y = \sqrt{x}J_1(2\sqrt{x})$ is a particular solution of the DE in the case $\lambda = 1$.

Computer Lab Assignments

38. Use a CAS to graph the modified Bessel functions $I_0(x)$, $I_1(x)$, $I_2(x)$ and $K_0(x)$, $K_1(x)$, $K_2(x)$. Compare these graphs with those shown in Figures 6.3.1 and 6.3.2. What major difference is apparent between Bessel functions and the modified Bessel functions?

39. **(a)** Use the general solution given in Example 4 to solve the IVP

$$4x'' + e^{-0.1t}x = 0, \quad x(0) = 1, \quad x'(0) = -\tfrac{1}{2}.$$

Also use $J'_0(x) = -J_1(x)$ and $Y'_0(x) = -Y_1(x)$ along with Table 6.1 or a CAS to evaluate coefficients.

(b) Use a CAS to graph the solution obtained in part (a) for $0 \le t \le \infty$.

40. **(a)** Use the general solution obtained in Problem 35 to solve the IVP

$$4x'' + tx = 0, \quad x(0.1) = 1, \quad x'(0.1) = -\tfrac{1}{2}.$$

Use a CAS to evaluate coefficients.

(b) Use a CAS to graph the solution obtained in part (a) for $0 \le t \le 200$.

41. **Column Bending Under Its Own Weight** A uniform thin column of length L, positioned vertically with one end embedded in the ground, will deflect, or bend away, from the vertical under the influence of its own weight when its length or height exceeds a certain critical value. It can be shown that the angular deflection $\theta(x)$ of the column from the vertical at a point $P(x)$ is a solution of the boundary-value problem:

$$EI\frac{d^2\theta}{dx^2} + \delta g(L - x)\theta = 0, \quad \theta(0) = 0, \quad \theta'(L) = 0,$$

where E is Young's modulus, I is the cross-sectional moment of inertia, δ is the constant linear density, and x is the distance along the column measured from its base. See Figure 6.3.4. The column will bend only for those values of L for which the boundary-value problem has a nontrivial solution.

(a) Restate the boundary-value problem by making the change of variables $t = L - x$. Then use the results of a problem earlier in this exercise set to express the general solution of the differential equation in terms of Bessel functions.

(b) Use the general solution found in part (a) to find a solution of the BVP and an equation which defines the critical length L, that is, the smallest value of L for which the column will start to bend.

(c) With the aid of a CAS, find the critical length L of a solid steel rod of radius $r = 0.05$ in., $\delta g = 0.28\,A$ lb/in., $E = 2.6 \times 10^7$ lb/in.^{2}, $A = \pi r^2$, and $I = \frac{1}{4}\pi r^4$.

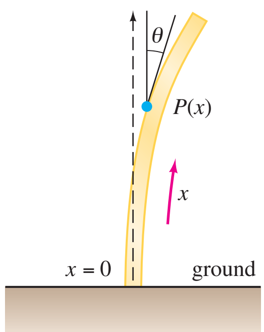

FIGURE 6.3.4 Beam in Problem 41

42. **Buckling of a Thin Vertical Column** In Example 3 of Section 5.2 we saw that when a constant vertical compressive force, or load, P was applied to a thin column of uniform cross section and hinged at both ends, the deflection $y(x)$ is a solution of the BVP:

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

(a) If the bending stiffness factor EI is proportional to x, then $EI(x) = kx$, where k is a constant of proportionality. If $EI(L) = kL = M$ is the maximum stiffness factor, then $k = M/L$ and so $EI(x) = Mx/L$. Use the information in Problem 37 to find a solution of

$$M\frac{x}{L}\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0$$

if it is known that $\sqrt{x}\,Y_1(2\sqrt{\lambda x})$ is *not* zero at $x = 0$.

(b) Use Table 6.1 to find the Euler load P_1 for the column.

(c) Use a CAS to graph the first buckling mode $y_1(x)$ corresponding to the Euler load P_1. For simplicity assume that $c_1 = 1$ and $L = 1$.

43. **Pendulum of Varying Length** For the simple pendulum described on page 209 of Section 5.3, suppose that the rod holding the mass m at one end is replaced by a flexible wire or string and that the wire is strung over a pulley at the point of support O in Figure 5.3.3. In this manner, while it is in motion in a vertical plane, the mass m can be raised or lowered. In other words, the length $l(t)$ of the pendulum varies with time. Under the same assumptions leading to equation (6) in Section 5.3, it can be shown* that the differential equation for the displacement angle θ is now

$$l\theta'' + 2l'\theta' + g\sin\theta = 0.$$

(a) If l increases at constant rate v and if $l(0) = l_0$, show that a linearization of the foregoing DE is

$$(l_0 + vt)\theta'' + 2v\theta' + g\theta = 0. \tag{31}$$

(b) Make the change of variables $x = (l_0 + vt)/v$ and show that (31) becomes

$$\frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} + \frac{g}{vx}\theta = 0.$$

(c) Use part (b) and (18) to express the general solution of equation (31) in terms of Bessel functions.

(d) Use the general solution obtained in part (c) to solve the initial-value problem consisting of equation (31) and the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$. [*Hints*: To simplify calculations, use a further change of variable $u = \dfrac{2}{v}\sqrt{g(l_0 + vt)} = 2\sqrt{\dfrac{g}{v}}\,x^{1/2}$.

*See *Mathematical Methods in Physical Sciences,* Mary Boas, John Wiley & Sons, Inc., 1966. Also see the article by Borelli, Coleman, and Hobson in *Mathematics Magazine,* vol. 58, no. 2, March 1985.

Also, recall that (20) holds for both $J_1(u)$ and $Y_1(u)$. Finally, the identity

$$J_1(u)Y_2(u) - J_2(u)Y_1(u) = -\frac{2}{\pi u}$$ will be helpful.]

(e) Use a CAS to graph the solution $\theta(t)$ of the IVP in part (d) when $l_0 = 1$ ft, $\theta_0 = \frac{1}{10}$ radian, and $v = \frac{1}{60}$ ft/s. Experiment with the graph using different time intervals such as [0, 10], [0, 30], and so on.

(f) What do the graphs indicate about the displacement angle $\theta(t)$ as the length l of the wire increases with time?

6.3.2 LEGENDRE'S EQUATION

44. **(a)** Use the explicit solutions $y_1(x)$ and $y_2(x)$ of Legendre's equation given in (26) and the appropriate choice of c_0 and c_1 to find the Legendre polynomials $P_6(x)$ and $P_7(x)$.

(b) Write the differential equations for which $P_6(x)$ and $P_7(x)$ are particular solutions.

45. Use the recurrence relation (29) and $P_0(x) = 1$, $P_1(x) = x$, to generate the next six Legendre polynomials.

46. Show that the differential equation

$$\sin\theta \frac{d^2y}{d\theta^2} + \cos\theta \frac{dy}{d\theta} + n(n+1)(\sin\theta)y = 0$$

can be transformed into Legendre's equation by means of the substitution $x = \cos\theta$.

47. Find the first three positive values of λ for which the problem

$$(1 - x^2)y'' - 2xy' + \lambda y = 0,$$

$$y(0) = 0, \quad y(x), y'(x) \text{ bounded on } [-1,1]$$

has nontrivial solutions.

Computer Lab Assignments

48. For purposes of this problem ignore the list of Legendre polynomials given on page 249 and the graphs given in Figure 6.3.3. Use Rodrigues' formula (30) to generate the Legendre polynomials $P_1(x), P_2(x), \ldots, P_7(x)$. Use a CAS to carry out the differentiations and simplifications.

49. Use a CAS to graph $P_1(x), P_2(x), \ldots, P_7(x)$ on the interval $[-1, 1]$.

50. Use a root-finding application to find the zeros of $P_1(x), P_2(x), \ldots, P_7(x)$. If the Legendre polynomials are built-in functions of your CAS, find zeros of Legendre polynomials of higher degree. Form a conjecture about the location of the zeros of any Legendre polynomial $P_n(x)$, and then investigate to see whether it is true.

CHAPTER 6 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-10.

In Problems 1 and 2 answer true or false without referring back to the text.

1. The general solution of $x^2y'' + xy' + (x^2 - 1)y = 0$ is $y = c_1J_1(x) + c_2J_{-1}(x)$. ________

2. Because $x = 0$ is an irregular singular point of $x^3y'' - xy' + y = 0$, the DE possesses no solution that is analytic at $x = 0$. ________

3. Both power series solutions of $y'' + \ln(x + 1)y' + y = 0$ centered at the ordinary point $x = 0$ are guaranteed to converge for all x in which *one* of the following intervals?

(a) $(-\infty, \infty)$ **(b)** $(-1, \infty)$
(c) $[-\frac{1}{2}, \frac{1}{2}]$ **(d)** $[-1, 1]$

4. $x = 0$ is an ordinary point of a certain linear differential equation. After the assumed solution $y = \sum_{n=0}^{\infty} c_nx^n$ is substituted into the DE, the following algebraic system is obtained by equating the coefficients of x^0, x^1, x^2, and x^3 to zero:

$$\begin{aligned} 2c_2 + 2c_1 + c_0 &= 0 \\ 6c_3 + 4c_2 + c_1 &= 0 \\ 12c_4 + 6c_3 + c_2 - \tfrac{1}{3}c_1 &= 0 \\ 20c_5 + 8c_4 + c_3 - \tfrac{2}{3}c_2 &= 0. \end{aligned}$$

Bearing in mind that c_0 and c_1 are arbitrary, write down the first five terms of two power series solutions of the differential equation.

5. Suppose the power series $\sum_{k=0}^{\infty} c_k(x - 4)^k$ is known to converge at -2 and diverge at 13. Discuss whether the series converges at -7, 0, 7, 10, and 11. Possible answers are *does, does not, might.*

6. Use the Maclaurin series for $\sin x$ and $\cos x$ along with long division to find the first three nonzero terms of a power series in x for the function $f(x) = \dfrac{\sin x}{\cos x}$.

In Problems 7 and 8 construct a linear second-order differential equation that has the given properties.

7. A regular singular point at $x = 1$ and an irregular singular point at $x = 0$

8. Regular singular points at $x = 1$ and at $x = -3$

In Problems 9–14 use an appropriate infinite series method about $x = 0$ to find two solutions of the given differential equation.

9. $2xy'' + y' + y = 0$

10. $y'' - xy' - y = 0$

11. $(x - 1)y'' + 3y = 0$

12. $y'' - x^2y' + xy = 0$

13. $xy'' - (x + 2)y' + 2y = 0$

14. $(\cos x)y'' + y = 0$

In Problems 15 and 16 solve the given initial-value problem.

15. $y'' + xy' + 2y = 0, \quad y(0) = 3, y'(0) = -2$

16. $(x + 2)y'' + 3y = 0, \quad y(0) = 0, y'(0) = 1$

17. Without actually solving the differential equation $(1 - 2\sin x)y'' + xy = 0$, find a lower bound for the radius of convergence of power series solutions about the ordinary point $x = 0$.

18. Even though $x = 0$ is an ordinary point of the differential equation, explain why it is not a good idea to try to find a solution of the IVP

$$y'' + xy' + y = 0, \quad y(1) = -6, \quad y'(1) = 3$$

of the form $y = \sum_{n=0}^{\infty} c_n x^n$. Using power series, find a better way to solve the problem.

In Problems 19 and 20 investigate whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation. [*Hint*: Recall the Maclaurin series for $\cos x$ and e^x.]

19. $xy'' + (1 - \cos x)y' + x^2y = 0$

20. $(e^x - 1 - x)y'' + xy = 0$

21. Note that $x = 0$ is an ordinary point of the differential equation $y'' + x^2y' + 2xy = 5 - 2x + 10x^3$. Use the assumption $y = \sum_{n=0}^{\infty} c_n x^n$ to find the general solution $y = y_c + y_p$ that consists of three power series centered at $x = 0$.

22. The first-order differential equation $dy/dx = x^2 + y^2$ cannot be solved in terms of elementary functions. However, a solution can be expressed in terms of Bessel functions.

(a) Show that the substitution $y = -\dfrac{1}{u}\dfrac{du}{dx}$ leads to the equation $u'' + x^2u = 0$.

(b) Use (18) in Section 6.3 to find the general solution of $u'' + x^2u = 0$.

(c) Use (20) and (21) in Section 6.3 in the forms

$$J_\nu'(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$$

and $$J_\nu'(x) = -\frac{\nu}{x}J_\nu(x) + J_{\nu-1}(x)$$

as an aid to show that a one-parameter family of solutions of $dy/dx = x^2 + y^2$ is given by

$$y = x\,\frac{J_{3/4}\left(\frac{1}{2}x^2\right) - cJ_{-3/4}\left(\frac{1}{2}x^2\right)}{cJ_{1/4}\left(\frac{1}{2}x^2\right) + J_{-1/4}\left(\frac{1}{2}x^2\right)}.$$

23. (a) Use (23) and (24) of Section 6.3 to show that

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x.$$

(b) Use (15) of Section 6.3 to show that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x \quad \text{and} \quad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x.$$

(c) Use part (b) to show that

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}.$$

24. (a) From (27) and (28) of Section 6.3 we know that when $n = 0$, Legendre's differential equation $(1 - x^2)y'' - 2xy' = 0$ has the polynomial solution $y = P_0(x) = 1$. Use (5) of Section 4.2 to show that a second Legendre function satisfying the DE for $-1 < x < 1$ is

$$y = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right).$$

(b) We also know from (27) and (28) of Section 6.3 that when $n = 1$, Legendre's differential equation $(1 - x^2)y'' - 2xy' + 2y = 0$ possesses the polynomial solution $y = P_1(x) = x$. Use (5) of Section 4.2 to show that a second Legendre function satisfying the DE for $-1 < x < 1$ is

$$y = \frac{x}{2}\ln\left(\frac{1 + x}{1 - x}\right) - 1.$$

(c) Use a graphing utility to graph the logarithmic Legendre functions given in parts (a) and (b).

25. (a) Use binomial series to formally show that

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

(b) Use the result obtained in part (a) to show that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. See Properties (*ii*) and (*iii*) on page 249.