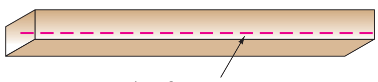
5.2 LINEAR MODELS: BOUNDARY-VALUE PROBLEMS

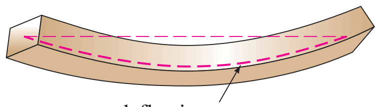
REVIEW MATERIAL

- Problems 37–40 in Exercises 4.3
- Problems 37–40 in Exercises 4.4

INTRODUCTION The preceding section was devoted to systems in which a second-order mathematical model was accompanied by initial conditions—that is, side conditions that are specified on the unknown function and its first derivative at a single point. But often the mathematical description of a physical system demands that we solve a homogeneous linear differential equation subject to boundary conditions—that is, conditions specified on the unknown function, or on one of its derivatives, or even on a linear combination of the unknown function and one of its derivatives at two (or more) different points.



(a)



(b)

FIGURE 5.2.1 Deflection of a homogeneous beam

DEFLECTION OF A BEAM Many structures are constructed by using girders or beams, and these beams deflect or distort under their own weight or under the influence of some external force. As we shall now see, this deflection $y(x)$ is governed by a relatively simple linear fourth-order differential equation.

To begin, let us assume that a beam of length L is homogeneous and has uniform cross sections along its length. In the absence of any load on the beam (including its weight), a curve joining the centroids of all its cross sections is a straight line called the **axis of symmetry.** See Figure 5.2.1(a). If a load is applied to the beam in a vertical plane containing the axis of symmetry, the beam, as shown in Figure 5.2.1(b), undergoes a distortion, and the curve connecting the centroids of all cross sections is called the **deflection curve** or **elastic curve.** The deflection curve approximates the shape of the beam. Now suppose that the x-axis coincides with the axis of symmetry and that the deflection $y(x)$, measured from this axis, is positive if downward. In the theory of elasticity it is shown that the bending moment $M(x)$ at a point x along the beam is related to the load per unit length $w(x)$ by the equation

$$\frac{d^2M}{dx^2} = w(x). \tag{1}$$

In addition, the bending moment $M(x)$ is proportional to the curvature κ of the elastic curve

$$M(x) = EI\kappa, \tag{2}$$

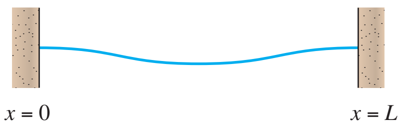
where E and I are constants; E is Young's modulus of elasticity of the material of the beam, and I is the moment of inertia of a cross section of the beam (about an axis known as the neutral axis). The product EI is called the **flexural rigidity** of the beam.

Now, from calculus, curvature is given by $\kappa = y''/[1 + (y')^2]^{3/2}$. When the deflection $y(x)$ is small, the slope $y' \approx 0$, and so $[1 + (y')^2]^{3/2} \approx 1$. If we let $\kappa \approx y''$, equation (2) becomes $M = EI\,y''$. The second derivative of this last expression is

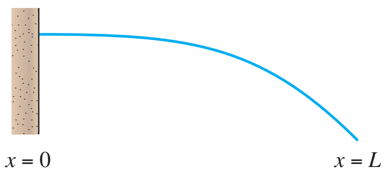
$$\frac{d^2M}{dx^2} = EI\,\frac{d^2}{dx^2}\,y'' = EI\,\frac{d^4y}{dx^4}. \tag{3}$$

Using the given result in (1) to replace d^2M/dx^2 in (3), we see that the deflection $y(x)$ satisfies the fourth-order differential equation

$$EI\,\frac{d^4y}{dx^4} = w(x). \tag{4}$$



(a) embedded at both ends



(b) cantilever beam: embedded at the left end, free at the right end

(c) simply supported at both ends

FIGURE 5.2.2 Beams with various end conditions

TABLE 5.1

Ends of the Beam	Boundary Conditions
embedded	$y = 0, \quad y' = 0$
free	$y'' = 0, \quad y''' = 0$
simply supported or hinged	$y = 0, \quad y'' = 0$

Boundary conditions associated with equation (4) depend on how the ends of the beam are supported. A cantilever beam is **embedded** or **clamped** at one end and **free** at the other. A diving board, an outstretched arm, an airplane wing, and a balcony are common examples of such beams, but even trees, flagpoles, skyscrapers, and the George Washington Monument can act as cantilever beams because they are embedded at one end and are subject to the bending force of the wind. For a cantilever beam the deflection $y(x)$ must satisfy the following two conditions at the embedded end $x = 0$:

- $y(0) = 0$ because there is no deflection, and
- $y'(0) = 0$ because the deflection curve is tangent to the x-axis (in other words, the slope of the deflection curve is zero at this point).

At $x = L$ the free-end conditions are

- $y''(L) = 0$ because the bending moment is zero, and
- $y'''(L) = 0$ because the shear force is zero.

The function $F(x) = dM/dx = EI\, d^3y/dx^3$ is called the shear force. If an end of a beam is **simply supported** or **hinged** (also called **pin supported** and **fulcrum supported**) then we must have $y = 0$ and $y'' = 0$ at that end. Table 5.1 summarizes the boundary conditions that are associated with (4). See Figure 5.2.2.

EXAMPLE 1 An Embedded Beam

A beam of length L is embedded at both ends. Find the deflection of the beam if a constant load w_0 is uniformly distributed along its length—that is, $w(x) = w_0$, $0 < x < L$.

SOLUTION From (4) we see that the deflection $y(x)$ satisfies

$$EI\frac{d^4y}{dx^4} = w_0.$$

Because the beam is embedded at both its left end ($x = 0$) and its right end ($x = L$), there is no vertical deflection and the line of deflection is horizontal at these points. Thus the boundary conditions are

$$y(0) = 0, \qquad y'(0) = 0, \qquad y(L) = 0, \qquad y'(L) = 0.$$

We can solve the nonhomogeneous differential equation in the usual manner (find y_c by observing that $m = 0$ is root of multiplicity four of the auxiliary equation $m^4 = 0$ and then find a particular solution y_p by undetermined coefficients), or we can simply integrate the equation $d^4y/dx^4 = w_0/EI$ four times in succession. Either way, we find the general solution of the equation $y = y_c + y_p$ to be

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4.$$

Now the conditions $y(0) = 0$ and $y'(0) = 0$ give, in turn, $c_1 = 0$ and $c_2 = 0$, whereas the remaining conditions $y(L) = 0$ and $y'(L) = 0$ applied to $y(x) = c_3x^2 + c_4x^3 + \dfrac{w_0}{24EI}x^4$ yield the simultaneous equations

$$c_3L^2 + c_4L^3 + \frac{w_0}{24EI}L^4 = 0$$

$$2c_3L + 3c_4L^2 + \frac{w_0}{6EI}L^3 = 0.$$

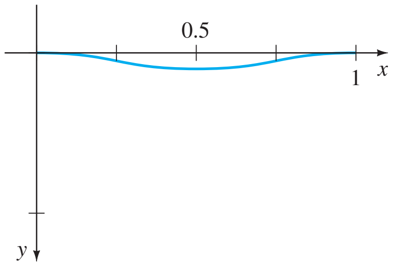


FIGURE 5.2.3 Deflection curve for Example 1

Solving this system gives $c_3 = w_0L^2/24EI$ and $c_4 = -w_0L/12EI$. Thus the deflection is

$$y(x) = \frac{w_0L^2}{24EI}x^2 - \frac{w_0L}{12EI}x^3 + \frac{w_0}{24EI}x^4$$

or $y(x) = \dfrac{w_0}{24EI}x^2(x-L)^2$. By choosing $w_0 = 24EI$, and $L = 1$, we obtain the deflection curve in Figure 5.2.3. ■

EIGENVALUES AND EIGENFUNCTIONS Many applied problems demand that we solve a two-point boundary-value problem (BVP) involving a linear differential equation that contains a parameter λ. We seek the values of λ for which the boundary-value problem has *nontrivial,* that is, *nonzero,* solutions.

EXAMPLE 2 Nontrivial Solutions of a BVP

Solve the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0.$$

SOLUTION We shall consider three cases: $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

CASE I: For $\lambda = 0$ the solution of $y'' = 0$ is $y = c_1x + c_2$. The conditions $y(0) = 0$ and $y(L) = 0$ applied to this solution imply, in turn, $c_2 = 0$ and $c_1 = 0$. Hence for $\lambda = 0$ the only solution of the boundary-value problem is the trivial solution $y = 0$.

CASE II: For $\lambda < 0$ it is convenient to write $\lambda = -\alpha^2$, where α denotes a positive number. With this notation the roots of the auxiliary equation $m^2 - \alpha^2 = 0$ are $m_1 = \alpha$ and $m_2 = -\alpha$. Since the interval on which we are working is finite, we choose to write the general solution of $y'' - \alpha^2y = 0$ as $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. Now $y(0)$ is

■ Note that we use hyperbolic functions here. Reread "Two Equations Worth Knowing" on page 135.

$$y(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1 \cdot 1 + c_2 \cdot 0 = c_1,$$

and so $y(0) = 0$ implies that $c_1 = 0$. Thus $y = c_2 \sinh \alpha x$. The second condition, $y(L) = 0$, demands that $c_2 \sinh \alpha L = 0$. For $\alpha \neq 0$, $\sinh \alpha L \neq 0$; consequently, we are forced to choose $c_2 = 0$. Again the only solution of the BVP is the trivial solution $y = 0$.

CASE III: For $\lambda > 0$ we write $\lambda = \alpha^2$, where α is a positive number. Because the auxiliary equation $m^2 + \alpha^2 = 0$ has complex roots $m_1 = i\alpha$ and $m_2 = -i\alpha$, the general solution of $y'' + \alpha^2y = 0$ is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. As before, $y(0) = 0$ yields $c_1 = 0$, and so $y = c_2 \sin \alpha x$. Now the last condition $y(L) = 0$, or

$$c_2 \sin \alpha L = 0,$$

is satisfied by choosing $c_2 = 0$. But this means that $y = 0$. If we require $c_2 \neq 0$, then $\sin \alpha L = 0$ is satisfied whenever αL is an integer multiple of π.

$$\alpha L = n\pi \quad \text{or} \quad \alpha = \frac{n\pi}{L} \quad \text{or} \quad \lambda_n = \alpha_n^2 = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Therefore for any real nonzero c_2, $y = c_2 \sin(n\pi x/L)$ is a solution of the problem for each n. Because the differential equation is homogeneous, any constant multiple of a solution is also a solution, so we may, if desired, simply take $c_2 = 1$. In other words, for each number in the sequence

$$\lambda_1 = \frac{\pi^2}{L^2}, \quad \lambda_2 = \frac{4\pi^2}{L^2}, \quad \lambda_3 = \frac{9\pi^2}{L^2}, \ \cdots,$$

the *corresponding* function in the sequence

$$y_1 = \sin\frac{\pi}{L}x, \quad y_2 = \sin\frac{2\pi}{L}x, \quad y_3 = \sin\frac{3\pi}{L}x, \cdots,$$

is a nontrivial solution of the original problem. ■

The numbers $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$ for which the boundary-value problem in Example 2 possesses nontrivial solutions are known as **eigenvalues.** The nontrivial solutions that depend on these values of λ_n, $y_n = c_2 \sin(n\pi x/L)$ or simply $y_n = \sin(n\pi x/L)$, are called **eigenfunctions.**

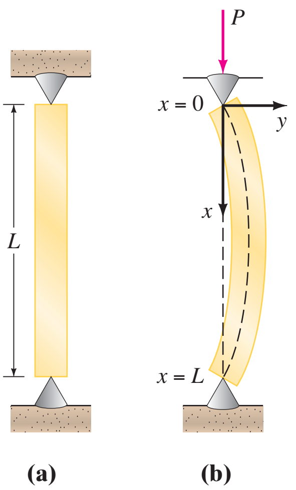


FIGURE 5.2.4 Elastic column buckling under a compressive force

BUCKLING OF A THIN VERTICAL COLUMN In the eighteenth century Leonhard Euler was one of the first mathematicians to study an eigenvalue problem in analyzing how a thin elastic column buckles under a compressive axial force.

Consider a long, slender vertical column of uniform cross section and length L. Let $y(x)$ denote the deflection of the column when a constant vertical compressive force, or load, P is applied to its top, as shown in Figure 5.2.4. By comparing bending moments at any point along the column, we obtain

$$EI\frac{d^2y}{dx^2} = -Py \quad \text{or} \quad EI\frac{d^2y}{dx^2} + Py = 0, \tag{5}$$

where E is Young's modulus of elasticity and I is the moment of inertia of a cross section about a vertical line through its centroid.

EXAMPLE 3 The Euler Load

Find the deflection of a thin vertical homogeneous column of length L subjected to a constant axial load P if the column is hinged at both ends.

SOLUTION The boundary-value problem to be solved is

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

First note that $y = 0$ is a perfectly good solution of this problem. This solution has a simple intuitive interpretation: If the load P is not great enough, there is no deflection. The question then is this: For what values of P will the column bend? In mathematical terms: For what values of P does the given boundary-value problem possess nontrivial solutions?

By writing $\lambda = P/EI$, we see that

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

is identical to the problem in Example 2. From Case III of that discussion we see that the deflections are $y_n(x) = c_2 \sin(n\pi x/L)$ corresponding to the eigenvalues $\lambda_n = P_n/EI = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$. Physically, this means that the column will buckle or deflect only when the compressive force is one of the values $P_n = n^2\pi^2 EI/L^2$, $n = 1, 2, 3, \ldots$. These different forces are called **critical loads.** The deflection corresponding to the smallest critical load $P_1 = \pi^2 EI/L^2$, called the **Euler load,** is $y_1(x) = c_2 \sin(\pi x/L)$ and is known as the **first buckling mode.** ■

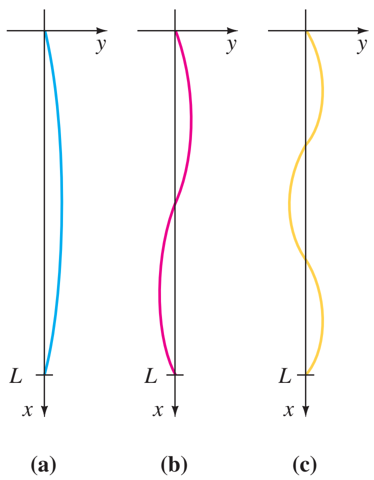


FIGURE 5.2.5 Deflection curves corresponding to compressive forces P_1, P_2, P_3

The deflection curves in Example 3 corresponding to $n = 1$, $n = 2$, and $n = 3$ are shown in Figure 5.2.5. Note that if the original column has some sort of physical restraint put on it at $x = L/2$, then the smallest critical load will be $P_2 = 4\pi^2 EI/L^2$, and the deflection curve will be as shown in Figure 5.2.5(b). If restraints are put on the column at $x = L/3$ and at $x = 2L/3$, then the column will not buckle until the critical load $P_3 = 9\pi^2 EI/L^2$ is applied, and the deflection curve will be as shown in Figure 5.2.5(c). See Problem 23 in Exercises 5.2.

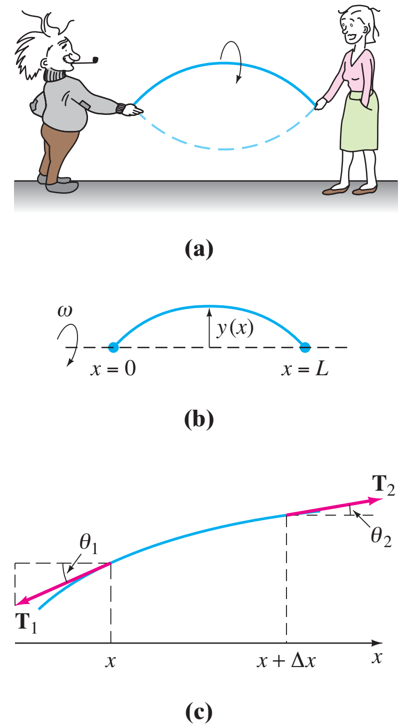


FIGURE 5.2.6 Rotating string and forces acting on it

ROTATING STRING The simple linear second-order differential equation

$$y'' + \lambda y = 0 \tag{6}$$

occurs again and again as a mathematical model. In Section 5.1 we saw (6) in the forms $d^2x/dt^2 + (k/m)x = 0$ and $d^2q/dt^2 + (1/LC)q = 0$ as models for, respectively, the simple harmonic motion of a spring/mass system and the simple harmonic response of a series circuit. It is apparent when the model for the deflection of a thin column in (5) is written as $d^2y/dx^2 + (P/EI)y = 0$ that it is the same as (6). We encounter the basic equation (6) one more time in this section: as a model that defines the deflection curve or the shape $y(x)$ assumed by a rotating string. The physical situation is analogous to when two people hold a jump rope and twirl it in a synchronous manner. See Figures 5.2.6(a) and 5.2.6(b).

Suppose a string of length L with constant linear density ρ (mass per unit length) is stretched along the x-axis and fixed at $x = 0$ and $x = L$. Suppose the string is then rotated about that axis at a constant angular speed ω. Consider a portion of the string on the interval $[x, x + \Delta x]$, where Δx is small. If the magnitude T of the tension $\mathbf{T}$, acting tangential to the string, is constant along the string, then the desired differential equation can be obtained by equating two different formulations of the net force acting on the string on the interval $[x, x + \Delta x]$. First, we see from Figure 5.2.6(c) that the net vertical force is

$$F = T\sin\theta_2 - T\sin\theta_1. \tag{7}$$

When angles θ_1 and θ_2 (measured in radians) are small, we have $\sin\theta_2 \approx \tan\theta_2$ and $\sin\theta_1 \approx \tan\theta_1$. Moreover, since $\tan\theta_2$ and $\tan\theta_1$ are, in turn, slopes of the lines containing the vectors $\mathbf{T}_2$ and $\mathbf{T}_1$, we can also write

$$\tan\theta_2 = y'(x + \Delta x) \qquad \text{and} \qquad \tan\theta_1 = y'(x).$$

Thus (7) becomes

$$F \approx T[y'(x + \Delta x) - y'(x)]. \tag{8}$$

Second, we can obtain a different form of this same net force using Newton's second law, $F = ma$. Here the mass of the string on the interval is $m = \rho\,\Delta x$; the centripetal acceleration of a body rotating with angular speed ω in a circle of radius r is $a = r\omega^2$. With Δx small we take $r = y$. Thus the net vertical force is also approximated by

$$F \approx -(\rho\,\Delta x)y\omega^2, \tag{9}$$

where the minus sign comes from the fact that the acceleration points in the direction opposite to the positive y-direction. Now by equating (8) and (9), we have

$$T[y'(x + \Delta x) - y'(x)] = -(\rho\Delta x)y\omega^2 \qquad \text{or} \qquad T\overbrace{\frac{y'(x + \Delta x) - y'(x)}{\Delta x}}^{\text{difference quotient}} + \rho\omega^2 y = 0. \tag{10}$$

For Δx close to zero the difference quotient in (10) is approximately the second derivative d^2y/dx^2. Finally, we arrive at the model

$$T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0. \tag{11}$$

Since the string is anchored at its ends $x = 0$ and $x = L$, we expect that the solution $y(x)$ of equation (11) should also satisfy the boundary conditions $y(0) = 0$ and $y(L) = 0$.

REMARKS

(*i*) Eigenvalues are not always easily found, as they were in Example 2; you might have to approximate roots of equations such as $\tan x = -x$ or $\cos x \cosh x = 1$. See Problems 34–38 in Exercises 5.2.

(*ii*) Boundary conditions applied to a general solution of a linear differential equation can lead to a homogeneous algebraic system of linear equations in which the unknowns are the coefficients c_i in the general solution. A homogeneous algebraic system of linear equations is always consistent because it possesses at least a trivial solution. But a homogeneous system of n linear equations in n unknowns has a nontrivial solution if and only if the determinant of the coefficients equals zero. You might need to use this last fact in Problems 19 and 20 in Exercises 5.2.

EXERCISES 5.2

Answers to selected odd-numbered problems begin on page ANS-8.

Deflection of a Beam

In Problems 1–5 solve equation (4) subject to the appropriate boundary conditions. The beam is of length L, and w_0 is a constant.

1. (a) The beam is embedded at its left end and free at its right end, and $w(x) = w_0$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 24EI$ and $L = 1$.

2. (a) The beam is simply supported at both ends, and $w(x) = w_0$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 24EI$ and $L = 1$.

3. (a) The beam is embedded at its left end and simply supported at its right end, and $w(x) = w_0$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 48EI$ and $L = 1$.

4. (a) The beam is embedded at its left end and simply supported at its right end, and $w(x) = w_0 \sin(\pi x/L)$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 2\pi^3 EI$ and $L = 1$.

(c) Use a root-finding application of a CAS (or a graphic calculator) to approximate the point in the graph in part (b) at which the maximum deflection occurs. What is the maximum deflection?

5. (a) The beam is simply supported at both ends, and $w(x) = w_0 x$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 36EI$ and $L = 1$.

(c) Use a root-finding application of a CAS (or a graphic calculator) to approximate the point in the graph in part (b) at which the maximum deflection occurs. What is the maximum deflection?

6. (a) Find the maximum deflection of the cantilever beam in Problem 1.

(b) How does the maximum deflection of a beam that is half as long compare with the value in part (a)?

(c) Find the maximum deflection of the simply supported beam in Problem 2.

(d) How does the maximum deflection of the simply supported beam in part (c) compare with the value of maximum deflection of the embedded beam in Example 1?

7. A cantilever beam of length L is embedded at its right end, and a horizontal tensile force of P pounds is applied to its free left end. When the origin is taken at its free end, as shown in Figure 5.2.7, the deflection $y(x)$ of the beam can be shown to satisfy the differential equation

$$EIy'' = Py - w(x)\frac{x}{2}.$$

Find the deflection of the cantilever beam if $w(x) = w_0 x$, $0 < x < L$, and $y(0) = 0$, $y'(L) = 0$.

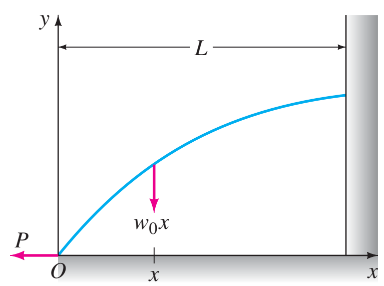


FIGURE 5.2.7 Deflection of cantilever beam in Problem 7

8. When a compressive instead of a tensile force is applied at the free end of the beam in Problem 7, the differential equation of the deflection is

$$EIy'' = -Py - w(x)\frac{x}{2}.$$

Solve this equation if $w(x) = w_0 x$, $0 < x < L$, and $y(0) = 0$, $y'(L) = 0$.

Eigenvalues and Eigenfunctions

In Problems 9–18 find the eigenvalues and eigenfunctions for the given boundary-value problem.

9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$

10. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi/4) = 0$

11. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$

12. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi/2) = 0$

13. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$

14. $y'' + \lambda y = 0, \quad y(-\pi) = 0, \quad y(\pi) = 0$

15. $y'' + 2y' + (\lambda + 1)y = 0, \quad y(0) = 0, \quad y(5) = 0$

16. $y'' + (\lambda + 1)y = 0, \quad y'(0) = 0, \quad y'(1) = 0$

17. $x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(e^{\pi}) = 0$

18. $x^2y'' + xy' + \lambda y = 0, \quad y'(e^{-1}) = 0, \quad y(1) = 0$

In Problems 19 and 20 find the eigenvalues and eigenfunctions for the given boundary-value problem. Consider only the case $\lambda = \alpha^4$, $\alpha > 0$.

19. $y^{(4)} - \lambda y = 0, \quad y(0) = 0, \quad y''(0) = 0, \quad y(1) = 0, \quad y''(1) = 0$

20. $y^{(4)} - \lambda y = 0, \quad y'(0) = 0, \quad y'''(0) = 0, \quad y(\pi) = 0, \quad y''(\pi) = 0$

Buckling of a Thin Column

21. Consider Figure 5.2.5. Where should physical restraints be placed on the column if we want the critical load to be P_4? Sketch the deflection curve corresponding to this load.

22. The critical loads of thin columns depend on the end conditions of the column. The value of the Euler load P_1 in Example 3 was derived under the assumption that the column was hinged at both ends. Suppose that a thin vertical homogeneous column is embedded at its base ($x = 0$) and free at its top ($x = L$) and that a constant axial load P is applied to its free end. This load either causes a small deflection δ as shown in Figure 5.2.8 or does not cause such a deflection. In either case the differential equation for the deflection $y(x)$ is

$$EI\frac{d^2y}{dx^2} + Py = P\delta.$$

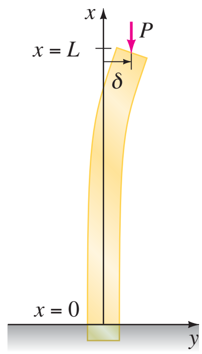


FIGURE 5.2.8 Deflection of vertical column in Problem 22

(a) What is the predicted deflection when $\delta = 0$?

(b) When $\delta \neq 0$, show that the Euler load for this column is one-fourth of the Euler load for the hinged column in Example 3.

23. As was mentioned in Problem 22, the differential equation (5) that governs the deflection $y(x)$ of a thin elastic column subject to a constant compressive axial force P is valid only when the ends of the column are hinged. In general, the differential equation governing the deflection of the column is given by

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + P\frac{d^2y}{dx^2} = 0.$$

Assume that the column is uniform (EI is a constant) and that the ends of the column are hinged. Show that the solution of this fourth-order differential equation subject to the boundary conditions $y(0) = 0$, $y''(0) = 0$, $y(L) = 0$, $y''(L) = 0$ is equivalent to the analysis in Example 3.

24. Suppose that a uniform thin elastic column is hinged at the end $x = 0$ and embedded at the end $x = L$.

(a) Use the fourth-order differential equation given in Problem 23 to find the eigenvalues λ_n, the critical loads P_n, the Euler load P_1, and the deflections $y_n(x)$.

(b) Use a graphing utility to graph the first buckling mode.

Rotating String

25. Consider the boundary-value problem introduced in the construction of the mathematical model for the shape of a rotating string:

$$T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0, \quad y(0) = 0, \quad y(L) = 0.$$

For constant T and ρ, define the critical speeds of angular rotation ω_n as the values of ω for which the boundary-value problem has nontrivial solutions. Find the critical speeds ω_n and the corresponding deflections $y_n(x)$.

26. When the magnitude of tension T is not constant, then a model for the deflection curve or shape $y(x)$ assumed by a rotating string is given by

$$\frac{d}{dx}\left[T(x)\frac{dy}{dx}\right] + \rho\omega^2 y = 0.$$

Suppose that $1 < x < e$ and that $T(x) = x^2$.

(a) If $y(1) = 0$, $y(e) = 0$, and $\rho\omega^2 > 0.25$, show that the critical speeds of angular rotation are $\omega_n = \frac{1}{2}\sqrt{(4n^2\pi^2 + 1)/\rho}$ and the corresponding deflections are

$$y_n(x) = c_2 x^{-1/2}\sin(n\pi \ln x), \quad n = 1, 2, 3, \ldots.$$

(b) Use a graphing utility to graph the deflection curves on the interval $[1, e]$ for $n = 1, 2, 3$. Choose $c_2 = 1$.

Miscellaneous Boundary-Value Problems

27. Temperature in a Sphere Consider two concentric spheres of radius $r = a$ and $r = b$, $a < b$. See Figure 5.2.9. The temperature $u(r)$ in the region between the spheres is determined from the boundary-value problem

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0, \quad u(a) = u_0, \quad u(b) = u_1,$$

where u_0 and u_1 are constants. Solve for $u(r)$.

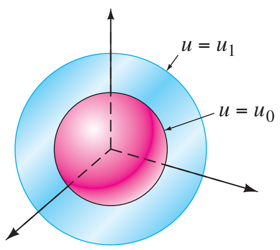


FIGURE 5.2.9 Concentric spheres in Problem 27

28. Temperature in a Ring The temperature $u(r)$ in the circular ring shown in Figure 5.2.10 is determined from the boundary-value problem

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0, \quad u(a) = u_0, \quad u(b) = u_1,$$

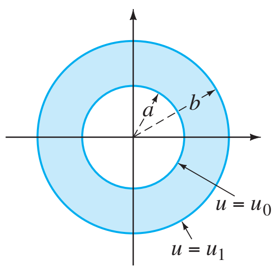


FIGURE 5.2.10 Circular ring in Problem 28

where u_0 and u_1 are constants. Show that

$$u(r) = \frac{u_0\ln(r/b) - u_1\ln(r/a)}{\ln(a/b)}.$$

Discussion Problems

29. Simple Harmonic Motion The model $mx'' + kx = 0$ for simple harmonic motion, discussed in Section 5.1, can be related to Example 2 of this section.

Consider a free undamped spring/mass system for which the spring constant is, say, $k = 10$ lb/ft. Determine those masses m_n that can be attached to the spring so that when each mass is released at the equilibrium position at $t = 0$ with a nonzero velocity v_0, it will then pass through the equilibrium position at $t = 1$ second. How many times will each mass m_n pass through the equilibrium position in the time interval $0 < t < 1$?

30. Damped Motion Assume that the model for the spring/mass system in Problem 29 is replaced by $mx'' + 2x' + kx = 0$. In other words, the system is free but is subjected to damping numerically equal to 2 times the instantaneous velocity. With the same initial conditions and spring constant as in Problem 29, investigate whether a mass m can be found that will pass through the equilibrium position at $t = 1$ second.

In Problems 31 and 32 determine whether it is possible to find values y_0 and y_1 (Problem 31) and values of $L > 0$ (Problem 32) so that the given boundary-value problem has **(a)** precisely one nontrivial solution, **(b)** more than one solution, **(c)** no solution, **(d)** the trivial solution.

31. $y'' + 16y = 0, \quad y(0) = y_0, y(\pi/2) = y_1$

32. $y'' + 16y = 0, \quad y(0) = 1, y(L) = 1$

33. Consider the boundary-value problem

$$y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

(a) The type of boundary conditions specified are called **periodic boundary conditions.** Give a geometric interpretation of these conditions.

(b) Find the eigenvalues and eigenfunctions of the problem.

(c) Use a graphing utility to graph some of the eigenfunctions. Verify your geometric interpretation of the boundary conditions given in part (a).

34. Show that the eigenvalues and eigenfunctions of the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$$

are $\lambda_n = \alpha_n^2$ and $y_n = \sin \alpha_n x$, respectively, where α_n, $n = 1, 2, 3, \ldots$ are the consecutive positive roots of the equation $\tan \alpha = -\alpha$.

Computer Lab Assignments

35. Use a CAS to plot graphs to convince yourself that the equation $\tan \alpha = -\alpha$ in Problem 34 has an infinite number of roots. Explain why the negative roots of the equation can be ignored. Explain why $\lambda = 0$ is not an eigenvalue even though $\alpha = 0$ is an obvious solution of the equation $\tan \alpha = -\alpha$.

36. Use a root-finding application of a CAS to approximate the first four eigenvalues λ_1, λ_2, λ_3, and λ_4 for the BVP in Problem 34.

In Problems 37 and 38 find the eigenvalues and eigenfunctions of the given boundary-value problem. Use a CAS to approximate the first four eigenvalues λ_1, λ_2, λ_3, and λ_4.

37. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) - \frac{1}{2}y'(1) = 0$

38. $y^{(4)} - \lambda y = 0,\ y(0) = 0,\ y'(0) = 0,\ y(1) = 0,\ y'(1) = 0$
[*Hint*: Consider only $\lambda = \alpha^4$, $\alpha > 0$.]

5.3 NONLINEAR MODELS

REVIEW MATERIAL

- Section 4.9

INTRODUCTION In this section we examine some nonlinear higher-order mathematical models. We are able to solve some of these models using the substitution method (leading to reduction of the order of the DE) introduced on page 174. In some cases in which the model cannot be solved, we show how a nonlinear DE can be replaced by a linear DE through a process called *linearization.*

NONLINEAR SPRINGS The mathematical model in (1) of Section 5.1 has the form

$$m\frac{d^2x}{dt^2} + F(x) = 0, \tag{1}$$

where $F(x) = kx$. Because x denotes the displacement of the mass from its equilibrium position, $F(x) = kx$ is Hooke's law—that is, the force exerted by the spring that tends to restore the mass to the equilibrium position. A spring acting under a linear restoring force $F(x) = kx$ is naturally referred to as a **linear spring.** But springs are seldom perfectly linear. Depending on how it is constructed and the material that is used, a spring can range from "mushy," or soft, to "stiff," or hard, so its restorative force may vary from something below to something above that given by the linear law. In the case of free motion, if we assume that a nonaging spring has some nonlinear characteristics, then it might be reasonable to assume that the restorative force of a spring—that is, $F(x)$ in (1)—is proportional to, say, the cube of the displacement x of the mass beyond its equilibrium position or that $F(x)$ is a linear combination of powers of the displacement such as that given by the nonlinear function $F(x) = kx + k_1x^3$. A spring whose mathematical model incorporates a nonlinear restorative force, such as

$$m\frac{d^2x}{dt^2} + kx^3 = 0 \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx + k_1x^3 = 0, \tag{2}$$

is called a **nonlinear spring.** In addition, we examined mathematical models in which damping imparted to the motion was proportional to the instantaneous velocity dx/dt and the restoring force of a spring was given by the linear function $F(x) = kx$. But these were simply assumptions; in more realistic situations damping could be proportional to some power of the instantaneous velocity dx/dt. The nonlinear differential equation

$$m\frac{d^2x}{dt^2} + \beta\left|\frac{dx}{dt}\right|\frac{dx}{dt} + kx = 0 \tag{3}$$

is one model of a free spring/mass system in which the damping force is proportional to the square of the velocity. One can then envision other kinds of models: linear damping and nonlinear restoring force, nonlinear damping and nonlinear restoring force, and so on. The point is that nonlinear characteristics of a physical system lead to a mathematical model that is nonlinear.

Notice in (2) that both $F(x) = kx^3$ and $F(x) = kx + k_1x^3$ are odd functions of x. To see why a polynomial function containing only odd powers of x provides a reasonable model for the restoring force, let us express F as a power series centered at the equilibrium position $x = 0$:

$$F(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots.$$

When the displacements x are small, the values of x^n are negligible for n sufficiently large. If we truncate the power series with, say, the fourth term, then $F(x) = c_0 + c_1x + c_2x^2 + c_3x^3$. For the force at $x > 0$,

$$F(x) = c_0 + c_1x + c_2x^2 + c_3x^3,$$

and for the force at $-x < 0$,

$$F(-x) = c_0 - c_1x + c_2x^2 - c_3x^3$$

to have the same magnitude but act in the opposite direction, we must have $F(-x) = -F(x)$. Because this means that F is an odd function, we must have $c_0 = 0$ and $c_2 = 0$, and so $F(x) = c_1x + c_3x^3$. Had we used only the first two terms in the series, the same argument yields the linear function $F(x) = c_1x$. A restoring force with mixed powers, such as $F(x) = c_1x + c_2x^2$, and the corresponding vibrations are said to be unsymmetrical. In the next discussion we shall write $c_1 = k$ and $c_3 = k_1$.

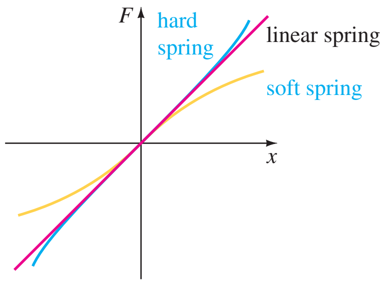


FIGURE 5.3.1 Hard and soft springs

HARD AND SOFT SPRINGS Let us take a closer look at the equation in (1) in the case in which the restoring force is given by $F(x) = kx + k_1x^3$, $k > 0$. The spring is said to be **hard** if $k_1 > 0$ and **soft** if $k_1 < 0$. Graphs of three types of restoring forces are illustrated in Figure 5.3.1. The next example illustrates these two special cases of the differential equation $m\, d^2x/dt^2 + kx + k_1x^3 = 0$, $m > 0$, $k > 0$.

EXAMPLE 1 Comparison of Hard and Soft Springs

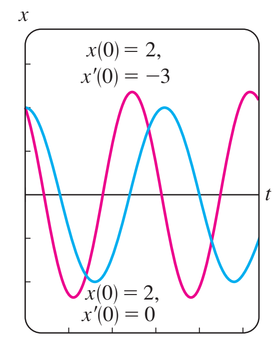
The differential equations

$$\frac{d^2x}{dt^2} + x + x^3 = 0 \tag{4}$$

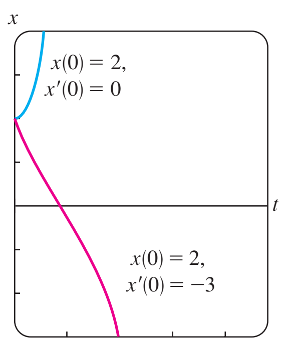
and

$$\frac{d^2x}{dt^2} + x - x^3 = 0 \tag{5}$$

are special cases of the second equation in (2) and are models of a hard spring and a soft spring, respectively. Figure 5.3.2(a) shows two solutions of (4) and Figure 5.3.2(b) shows two solutions of (5) obtained from a numerical solver. The curves shown in red are solutions that satisfy the initial conditions $x(0) = 2$, $x'(0) = -3$; the two curves in blue are solutions that satisfy $x(0) = 2$, $x'(0) = 0$. These solution curves certainly suggest that the motion of a mass on the hard spring is oscillatory, whereas motion of a mass on the soft spring appears to be nonoscillatory. But we must be careful about drawing conclusions based on a couple of numerical solution curves. A more complete picture of the nature of the solutions of both of these equations can be obtained from the qualitative analysis discussed in Chapter 10. ■



(a) hard spring



(b) soft spring

FIGURE 5.3.2 Numerical solution curves

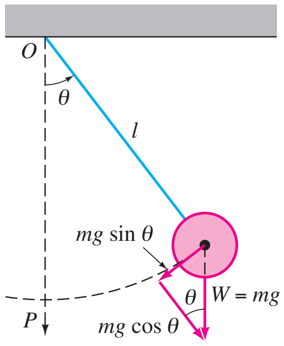


FIGURE 5.3.3 Simple pendulum

NONLINEAR PENDULUM Any object that swings back and forth is called a **physical pendulum.** The **simple pendulum** is a special case of the physical pendulum and consists of a rod of length l to which a mass m is attached at one end. In describing the motion of a simple pendulum in a vertical plane, we make the simplifying assumptions that the mass of the rod is negligible and that no external damping or driving forces act on the system. The displacement angle θ of the pendulum, measured from the vertical as shown in Figure 5.3.3, is considered positive when measured to the right of OP and negative to the left of OP. Now recall the arc s of a circle of radius l is related to the central angle θ by the formula $s = l\theta$. Hence angular acceleration is

$$a = \frac{d^2s}{dt^2} = l\,\frac{d^2\theta}{dt^2}.$$

From Newton's second law we then have

$$F = ma = ml\,\frac{d^2\theta}{dt^2}.$$

From Figure 5.3.3 we see that the magnitude of the tangential component of the force due to the weight W is $mg \sin\theta$. In direction this force is $-mg\sin\theta$ because it points to the left for $\theta > 0$ and to the right for $\theta < 0$. We equate the two different versions of the tangential force to obtain $ml\, d^2\theta/dt^2 = -mg\sin\theta$, or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{6}$$

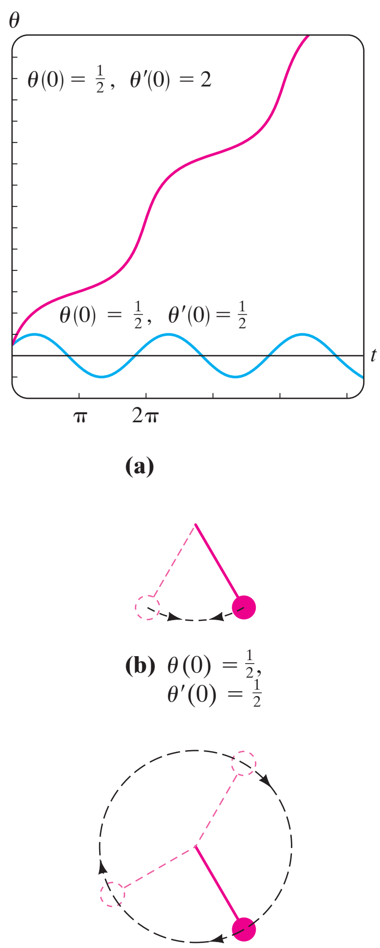
LINEARIZATION Because of the presence of $\sin\theta$, the model in (6) is nonlinear. In an attempt to understand the behavior of the solutions of nonlinear higher-order differential equations, one sometimes tries to simplify the problem by replacing nonlinear terms by certain approximations. For example, the Maclaurin series for $\sin\theta$ is given by

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

so if we use the approximation $\sin\theta \approx \theta - \theta^3/6$, equation (6) becomes $d^2\theta/dt^2 + (g/l)\theta - (g/6l)\theta^3 = 0$. Observe that this last equation is the same as the second nonlinear equation in (2) with $m = 1$, $k = g/l$, and $k_1 = -g/6l$. However, if we assume that the displacements θ are small enough to justify using the replacement $\sin\theta \approx \theta$, then (6) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{7}$$

See Problem 22 in Exercises 5.3. If we set $\omega^2 = g/l$, we recognize (7) as the differential equation (2) of Section 5.1 that is a model for the free undamped vibrations of a linear spring/mass system. In other words, (7) is again the basic linear equation $y'' + \lambda y = 0$ discussed on page 201 of Section 5.2. As a consequence we say that equation (7) is a **linearization** of equation (6). Because the general solution of (7) is $\theta(t) = c_1\cos\omega t + c_2\sin\omega t$, this linearization suggests that for initial conditions amenable to small oscillations the motion of the pendulum described by (6) will be periodic.



(a)

(b) $\theta(0) = \frac{1}{2}$, $\theta'(0) = \frac{1}{2}$

(c) $\theta(0) = \frac{1}{2}$, $\theta'(0) = 2$

FIGURE 5.3.4 Oscillating pendulum in (b); whirling pendulum in (c)

EXAMPLE 2 Two Initial-Value Problems

The graphs in Figure 5.3.4(a) were obtained with the aid of a numerical solver and represent solution curves of (6) when $\omega^2 = 1$. The blue curve depicts the solution of (6) that satisfies the initial conditions $\theta(0) = \frac{1}{2}$, $\theta'(0) = \frac{1}{2}$, whereas the red curve is the solution of (6) that satisfies $\theta(0) = \frac{1}{2}$, $\theta'(0) = 2$. The blue curve represents a periodic solution—the pendulum oscillating back and forth as shown in Figure 5.3.4(b) with an apparent amplitude $A \le 1$. The red curve shows that θ increases without bound as time increases—the pendulum, starting from the same initial displacement, is given an initial velocity of magnitude great enough to send it over the top; in other words, the pendulum is whirling about its pivot as shown in Figure 5.3.4(c). In the absence of damping, the motion in each case is continued indefinitely. ■

TELEPHONE WIRES The first-order differential equation $dy/dx = W/T_1$ is equation (17) of Section 1.3. This differential equation, established with the aid of Figure 1.3.7 on page 25, serves as a mathematical model for the shape of a flexible cable suspended between two vertical supports when the cable is carrying a vertical load. In Section 2.2 we solved this simple DE under the assumption that the vertical load carried by the cables of a suspension bridge was the weight of a horizontal roadbed distributed evenly along the x-axis. With $W = \rho x$, ρ the weight per unit length of the roadbed, the shape of each cable between the vertical supports turned out to be parabolic. We are now in a position to determine the shape of a uniform flexible cable hanging only under its own weight, such as a wire strung between two telephone posts. The vertical load is now the wire itself, and so if ρ is the linear density of the wire (measured, say, in pounds per feet) and s is the length of the segment P_1P_2 in Figure 1.3.7 then $W = \rho s$. Hence

$$\frac{dy}{dx} = \frac{\rho s}{T_1}. \tag{8}$$

Since the arc length between points P_1 and P_2 is given by

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{9}$$

it follows from the fundamental theorem of calculus that the derivative of (9) is

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{10}$$

Differentiating (8) with respect to x and using (10) lead to the second-order equation

$$\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\frac{ds}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{\rho}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{11}$$

In the example that follows we solve (11) and show that the curve assumed by the suspended cable is a **catenary.** Before proceeding, observe that the nonlinear second-order differential equation (11) is one of those equations having the form $F(x, y', y'') = 0$ discussed in Section 4.9. Recall that we have a chance of solving an equation of this type by reducing the order of the equation by means of the substitution $u = y'$.

EXAMPLE 3 An Initial-Value Problem

From the position of the y-axis in Figure 1.3.7 it is apparent that initial conditions associated with the second differential equation in (11) are $y(0) = a$ and $y'(0) = 0$. If we substitute $u = y'$, then the equation in (11) becomes $\dfrac{du}{dx} = \dfrac{\rho}{T_1}\sqrt{1 + u^2}$. Separating variables, we find that

$$\int \frac{du}{\sqrt{1 + u^2}} = \frac{\rho}{T_1}\int dx \quad \text{gives} \quad \sinh^{-1}u = \frac{\rho}{T_1}x + c_1.$$

Now, $y'(0) = 0$ is equivalent to $u(0) = 0$. Since $\sinh^{-1} 0 = 0$, $c_1 = 0$, so $u = \sinh(\rho x/T_1)$. Finally, by integrating both sides of

$$\frac{dy}{dx} = \sinh\frac{\rho}{T_1}x, \qquad \text{we get} \qquad y = \frac{T_1}{\rho}\cosh\frac{\rho}{T_1}x + c_2.$$

Using $y(0) = a$, $\cosh 0 = 1$, the last equation implies that $c_2 = a - T_1/\rho$. Thus we see that the shape of the hanging wire is given by $y = (T_1/\rho)\cosh(\rho x/T_1) + a - T_1/\rho$. ■

In Example 3, had we been clever enough at the start to choose $a = T_1/\rho$, then the solution of the problem would have been simply the hyperbolic cosine $y = (T_1/\rho)\cosh(\rho x/T_1)$.

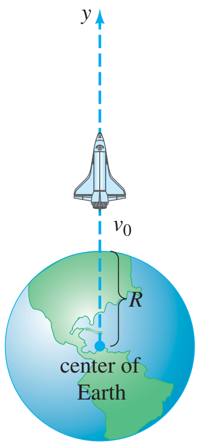


FIGURE 5.3.5 Distance to rocket is large compared to R.

ROCKET MOTION In Section 1.3 we saw that the differential equation of a free-falling body of mass m near the surface of the Earth is given by

$$m\frac{d^2s}{dt^2} = -mg, \qquad \text{or simply} \qquad \frac{d^2s}{dt^2} = -g,$$

where s represents the distance from the surface of the Earth to the object and the positive direction is considered to be upward. In other words, the underlying assumption here is that the distance s to the object is small when compared with the radius R of the Earth; put yet another way, the distance y from the center of the Earth to the object is approximately the same as R. If, on the other hand, the distance y to the object, such as a rocket or a space probe, is large when compared to R, then we combine Newton's second law of motion and his universal law of gravitation to derive a differential equation in the variable y.

Suppose a rocket is launched vertically upward from the ground as shown in Figure 5.3.5. If the positive direction is upward and air resistance is ignored, then the differential equation of motion after fuel burnout is

$$m\frac{d^2y}{dt^2} = -k\frac{Mm}{y^2} \qquad \text{or} \qquad \frac{d^2y}{dt^2} = -k\frac{M}{y^2}, \tag{12}$$

where k is a constant of proportionality, y is the distance from the center of the Earth to the rocket, M is the mass of the Earth, and m is the mass of the rocket. To determine the constant k, we use the fact that when $y = R$, $kMm/R^2 = mg$ or $k = gR^2/M$. Thus the last equation in (12) becomes

$$\frac{d^2y}{dt^2} = -g\frac{R^2}{y^2}. \tag{13}$$

See Problem 14 in Exercises 5.3.

VARIABLE MASS Notice in the preceding discussion that we described the motion of the rocket after it has burned all its fuel, when presumably its mass m is constant. Of course, during its powered ascent the total mass of the rocket varies as its fuel is being expended. The second law of motion, as originally advanced by Newton, states that when a body of mass m moves through a force field with velocity v, the time rate of change of the momentum mv of the body is equal to applied or net force F acting on the body:

$$F = \frac{d}{dt}(mv). \tag{14}$$

If m is constant, then (14) yields the more familiar form $F = m\,dv/dt = ma$, where a is acceleration. We use the form of Newton's second law given in (14) in the next example, in which the mass m of the body is variable.

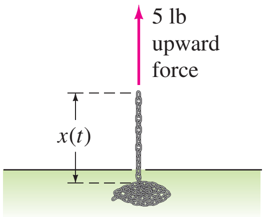


FIGURE 5.3.6 Chain pulled upward by a constant force

EXAMPLE 4 Chain Pulled Upward by a Constant Force

A uniform 10-foot-long chain is coiled loosely on the ground. One end of the chain is pulled vertically upward by means of constant force of 5 pounds. The chain weighs 1 pound per foot. Determine the height of the end above ground level at time t. See Figure 5.3.6.

SOLUTION Let us suppose that $x = x(t)$ denotes the height of the end of the chain in the air at time t, $v = dx/dt$, and the positive direction is upward. For the portion of the chain that is in the air at time t we have the following variable quantities:

$$\begin{aligned} &\textit{weight}: && W = (x \text{ ft}) \cdot (1 \text{ lb/ft}) = x, \\ &\textit{mass}: && m = W/g = x/32, \\ &\textit{net force}: && F = 5 - W = 5 - x. \end{aligned}$$

Thus from (14) we have

$$\frac{d}{dt}\left(\frac{x}{32}v\right) = 5 - x \qquad \text{or} \qquad \overbrace{x\frac{dv}{dt} + v\frac{dx}{dt}}^{\text{Product Rule}} = 160 - 32x. \tag{15}$$

Because $v = dx/dt$, the last equation becomes

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x = 160. \tag{16}$$

The nonlinear second-order differential equation (16) has the form $F(x, x', x'') = 0$, which is the second of the two forms considered in Section 4.9 that can possibly be solved by reduction of order. To solve (16), we revert back to (15) and use $v = x'$ along with the Chain Rule. From $\dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\dfrac{dv}{dx}$ the second equation in (15) can be rewritten as

$$xv\frac{dv}{dx} + v^2 = 160 - 32x. \tag{17}$$

On inspection (17) might appear intractable, since it cannot be characterized as any of the first-order equations that were solved in Chapter 2. However, by rewriting (17) in differential form $M(x,v)dx + N(x,v)dv = 0$, we observe that although the equation

$$(v^2 + 32x - 160)dx + xv\,dv = 0 \tag{18}$$

is not exact, it can be transformed into an exact equation by multiplying it by an integrating factor. From $(M_v - N_x)/N = 1/x$ we see from (13) of Section 2.4 that an integrating factor is $e^{\int dx/x} = e^{\ln x} = x$. When (18) is multiplied by $\mu(x) = x$, the resulting equation is exact (verify). By identifying $\partial f/\partial x = xv^2 + 32x^2 - 160x$, $\partial f/\partial v = x^2v$ and then proceeding as in Section 2.4, we obtain

$$\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = c_1. \tag{19}$$

Since we have assumed that all of the chain is on the floor initially, we have $x(0) = 0$. This last condition applied to (19) yields $c_1 = 0$. By solving the algebraic equation $\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = 0$ for $v = dx/dt > 0$, we get another first-order differential equation,

$$\frac{dx}{dt} = \sqrt{160 - \frac{64}{3}x}.$$

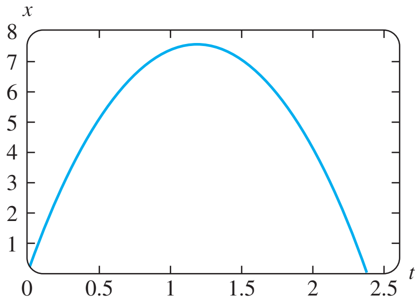


FIGURE 5.3.7 Graph of (21) for $x(t) \geq 0$

The last equation can be solved by separation of variables. You should verify that

$$-\frac{3}{32}\left(160 - \frac{64}{3}x\right)^{1/2} = t + c_2. \tag{20}$$

This time the initial condition $x(0) = 0$ implies that $c_2 = -3\sqrt{10}/8$. Finally, by squaring both sides of (20) and solving for x, we arrive at the desired result,

$$x(t) = \frac{15}{2} - \frac{15}{2}\left(1 - \frac{4\sqrt{10}}{15}t\right)^2. \tag{21}$$

The graph of (21) given in Figure 5.3.7 should not, on physical grounds, be taken at face value. See Problem 15 in Exercises 5.3. ■

EXERCISES 5.3

Answers to selected odd-numbered problems begin on page ANS-8.

To the Instructor In addition to Problems 24 and 25, all or portions of Problems 1–6, 8–13, 15, 20, and 21 could serve as Computer Lab Assignments.

Nonlinear Springs

In Problems 1–4, the given differential equation is model of an undamped spring/mass system in which the restoring force $F(x)$ in (1) is nonlinear. For each equation use a numerical solver to plot the solution curves that satisfy the given initial conditions. If the solutions appear to be periodic use the solution curve to estimate the period T of oscillations.

1. $\dfrac{d^2x}{dt^2} + x^3 = 0,$

$x(0) = 1, x'(0) = 1; \quad x(0) = \frac{1}{2}, x'(0) = -1$

2. $\dfrac{d^2x}{dt^2} + 4x - 16x^3 = 0,$

$x(0) = 1, x'(0) = 1; \quad x(0) = -2, x'(0) = 2$

3. $\dfrac{d^2x}{dt^2} + 2x - x^2 = 0,$

$x(0) = 1, x'(0) = 1; \quad x(0) = \frac{3}{2}, x'(0) = -1$

4. $\dfrac{d^2x}{dt^2} + xe^{0.01x} = 0,$

$x(0) = 1, x'(0) = 1; \quad x(0) = 3, x'(0) = -1$

5. In Problem 3, suppose the mass is released from the initial position $x(0) = 1$ with an initial velocity $x'(0) = x_1$. Use a numerical solver to estimate the smallest value of $|x_1|$ at which the motion of the mass is nonperiodic.

6. In Problem 3, suppose the mass is released from an initial position $x(0) = x_0$ with the initial velocity $x'(0) = 1$. Use a numerical solver to estimate an interval $a \leq x_0 \leq b$ for which the motion is oscillatory.

7. Find a linearization of the differential equation in Problem 4.

8. Consider the model of an undamped nonlinear spring/mass system given by $x'' + 8x - 6x^3 + x^5 = 0$. Use a numerical solver to discuss the nature of the oscillations of the system corresponding to the initial conditions:

$x(0) = 1, x'(0) = 1; \quad x(0) = -2, x'(0) = \frac{1}{2};$

$x(0) = \sqrt{2}, x'(0) = 1; \quad x(0) = 2, x'(0) = \frac{1}{2};$

$x(0) = 2, x'(0) = 0; \quad x(0) = -\sqrt{2}, x'(0) = -1.$

In Problems 9 and 10 the given differential equation is a model of a damped nonlinear spring/mass system. Predict the behavior of each system as $t \to \infty$. For each equation use a numerical solver to obtain the solution curves satisfying the given initial conditions.

9. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + x^3 = 0,$

$x(0) = -3, x'(0) = 4; \quad x(0) = 0, x'(0) = -8$

10. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x - x^3 = 0,$

$x(0) = 0, x'(0) = \frac{3}{2}; \quad x(0) = -1, x'(0) = 1$

11. The model $mx'' + kx + k_1x^3 = F_0\cos\omega t$ of an undamped periodically driven spring/mass system is called **Duffing's differential equation.** Consider the initial-value problem $x'' + x + k_1x^3 = 5\cos t$, $x(0) = 1$, $x'(0) = 0$. Use a numerical solver to investigate the behavior of the system for values of $k_1 > 0$ ranging from $k_1 = 0.01$ to $k_1 = 100$. State your conclusions.

12. **(a)** Find values of $k_1 < 0$ for which the system in Problem 11 is oscillatory.

(b) Consider the initial-value problem

$$x'' + x + k_1x^3 = \cos\tfrac{3}{2}t, \quad x(0) = 0, \quad x'(0) = 0.$$

Find values for $k_1 < 0$ for which the system is oscillatory.

Nonlinear Pendulum

13. Consider the model of the free damped nonlinear pendulum given by

$$\frac{d^2\theta}{dt^2} + 2\lambda\frac{d\theta}{dt} + \omega^2 \sin\theta = 0.$$

Use a numerical solver to investigate whether the motion in the two cases $\lambda^2 - \omega^2 > 0$ and $\lambda^2 - \omega^2 < 0$ corresponds, respectively, to the overdamped and underdamped cases discussed in Section 5.1 for spring/mass systems. Choose appropriate initial conditions and values of λ and ω.

Rocket Motion

14. **(a)** Use the substitution $v = dy/dt$ to solve (13) for v in terms of y. Assuming that the velocity of the rocket at burnout is $v = v_0$ and $y \approx R$ at that instant, show that the approximate value of the constant c of integration is $c = -gR + \frac{1}{2}v_0^2$.

(b) Use the solution for v in part (a) to show that the escape velocity of the rocket is given by $v_0 = \sqrt{2gR}$. [*Hint*: Take $y \to \infty$ and assume $v > 0$ for all time t.]

(c) The result in part (b) holds for any body in the solar system. Use the values $g = 32$ ft/s^2 and $R = 4000$ mi to show that the escape velocity from the Earth is (approximately) $v_0 = 25{,}000$ mi/h.

(d) Find the escape velocity from the Moon if the acceleration of gravity is $0.165g$ and $R = 1080$ mi.

Variable Mass

15. **(a)** In Example 4, how much of the chain would you intuitively expect the constant 5-pound force to be able to lift?

(b) What is the initial velocity of the chain?

(c) Why is the time interval corresponding to $x(t) \geq 0$ given in Figure 5.3.7 not the interval I of definition of the solution (21)? Determine the interval I. How much chain is actually lifted? Explain any difference between this answer and your prediction in part (a).

(d) Why would you expect $x(t)$ to be a periodic solution?

16. A uniform chain of length L, measured in feet, is held vertically so that the lower end just touches the floor. The chain weighs 2 lb/ft. The upper end that is held is released from rest at $t = 0$ and the chain falls straight down. If $x(t)$ denotes the length of the chain on the floor at time t, air resistance is ignored, and the positive direction is taken to be downward, then

$$(L - x)\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 = Lg.$$

(a) Solve for v in terms of x. Solve for x in terms of t. Express v in terms of t.

(b) Determine how long it takes for the chain to fall completely to the ground.

(c) What velocity does the model in part (a) predict for the upper end of the chain as it hits the ground?

Miscellaneous Mathematical Models

17. **Pursuit Curve** In a naval exercise a ship S_1 is pursued by a submarine S_2 as shown in Figure 5.3.8. Ship S_1 departs point (0, 0) at $t = 0$ and proceeds along a straight-line course (the y-axis) at a constant speed v_1. The submarine S_2 keeps ship S_1 in visual contact, indicated by the straight dashed line L in the figure, while traveling at a constant speed v_2 along a curve C. Assume that ship S_2 starts at the point $(a, 0)$, $a > 0$, at $t = 0$ and that L is tangent to C.

(a) Determine a mathematical model that describes the curve C.

(b) Find an explicit solution of the differential equation. For convenience define $r = v_1/v_2$.

(c) Determine whether the paths of S_1 and S_2 will ever intersect by considering the cases $r > 1$, $r < 1$, and $r = 1$.

[*Hint*: $\dfrac{dt}{dx} = \dfrac{dt}{ds}\dfrac{ds}{dx}$, where s is arc length measured along C.]

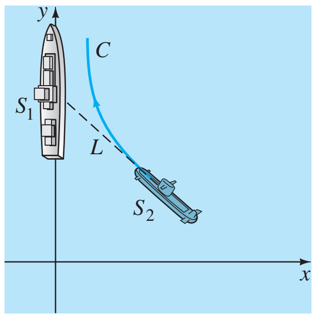


FIGURE 5.3.8 Pursuit curve in Problem 17

18. **Pursuit Curve** In another naval exercise a destroyer S_1 pursues a submerged submarine S_2. Suppose that S_1 at (9, 0) on the x-axis detects S_2 at (0, 0) and that S_2 simultaneously detects S_1. The captain of the destroyer S_1 assumes that the submarine will take immediate evasive action and conjectures that its likely new course is the straight line indicated in Figure 5.3.9. When S_1 is at (3, 0), it changes from its straight-line course toward the origin to a pursuit curve C. Assume that the speed of the destroyer is, at all times, a constant 30 mi/h and that the submarine's speed is a constant 15 mi/h.

(a) Explain why the captain waits until S_1 reaches (3, 0) before ordering a course change to C.

(b) Using polar coordinates, find an equation $r = f(\theta)$ for the curve C.

(c) Let T denote the time, measured from the initial detection, at which the destroyer intercepts the submarine. Find an upper bound for T.

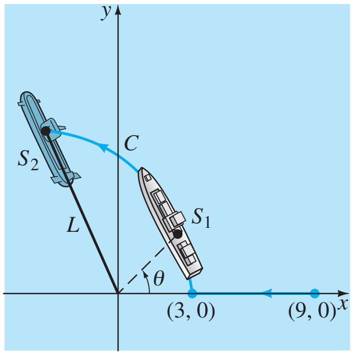


FIGURE 5.3.9 Pursuit curve in Problem 18

Discussion Problems

19. Discuss why the damping term in equation (3) is written as

$$\beta\left|\frac{dx}{dt}\right|\frac{dx}{dt} \quad \text{instead of} \quad \beta\left(\frac{dx}{dt}\right)^2.$$

20. **(a)** Experiment with a calculator to find an interval $0 \le \theta < \theta_1$, where θ is measured in radians, for which you think $\sin\theta \approx \theta$ is a fairly good estimate. Then use a graphing utility to plot the graphs of $y = x$ and $y = \sin x$ on the same coordinate axes for $0 \le x \le \pi/2$. Do the graphs confirm your observations with the calculator?

(b) Use a numerical solver to plot the solution curves of the initial-value problems

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = 0$$

and
$$\frac{d^2\theta}{dt^2} + \theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = 0$$

for several values of θ_0 in the interval $0 \le \theta < \theta_1$ found in part (a). Then plot solution curves of the initial-value problems for several values of θ_0 for which $\theta_0 > \theta_1$.

21. **(a)** Consider the nonlinear pendulum whose oscillations are defined by (6). Use a numerical solver as an aid to determine whether a pendulum of length l will oscillate faster on the Earth or on the Moon. Use the same initial conditions, but choose these initial conditions so that the pendulum oscillates back and forth.

(b) For which location in part (a) does the pendulum have greater amplitude?

(c) Are the conclusions in parts (a) and (b) the same when the linear model (7) is used?

Computer Lab Assignments

22. Consider the initial-value problem

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \frac{\pi}{12}, \quad \theta'(0) = -\frac{1}{3}$$

for a nonlinear pendulum. Since we cannot solve the differential equation, we can find no explicit solution of this problem. But suppose we wish to determine the first time $t_1 > 0$ for which the pendulum in Figure 5.3.3, starting from its initial position to the right, reaches the position OP—that is, the first positive root of $\theta(t) = 0$. In this problem and the next we examine several ways to proceed.

(a) Approximate t_1 by solving the linear problem $d^2\theta/dt^2 + \theta = 0$, $\theta(0) = \pi/12$, $\theta'(0) = -\frac{1}{3}$.

(b) Use the method illustrated in Example 3 of Section 4.9 to find the first four nonzero terms of a Taylor series solution $\theta(t)$ centered at 0 for the nonlinear initial-value problem. Give the exact values of all coefficients.

(c) Use the first two terms of the Taylor series in part (b) to approximate t_1.

(d) Use the first three terms of the Taylor series in part (b) to approximate t_1.

(e) Use a root-finding application of a CAS (or a graphic calculator) and the first four terms of the Taylor series in part (b) to approximate t_1.

(f) In this part of the problem you are led through the commands in *Mathematica* that enable you to approximate the root t_1. The procedure is easily modified so that any root of $\theta(t) = 0$ can be approximated. (*If you do not have Mathematica, adapt the given procedure by finding the corresponding syntax for the CAS you have on hand.*) Precisely reproduce and then, in turn, execute each line in the given sequence of commands.

```
sol = NDSolve[{y″[t] + Sin[y[t]] == 0,
        y[0] == Pi/12, y′[0] == −1/3},
        y, {t, 0, 5}]//Flatten
solution = y[t]/.sol
Clear[y]
y[t_]: = Evaluate[solution]
y[t]
gr1 = Plot[y[t], {t, 0, 5}]
root = FindRoot[y[t] == 0, {t, 1}]
```

(g) Appropriately modify the syntax in part (f) and find the next two positive roots of $\theta(t) = 0$.

23. Consider a pendulum that is released from rest from an initial displacement of θ_0 radians. Solving the linear model (7) subject to the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$ gives $\theta(t) = \theta_0 \cos\sqrt{g/l}\,t$. The period of oscillations predicted by this model is given by the familiar formula $T = 2\pi/\sqrt{g/l} = 2\pi\sqrt{l/g}$. The interesting thing about this formula for T is that it does not depend on the magnitude of the initial displacement θ_0. In other words, the linear model predicts that the time it would take the pendulum to swing from an initial displacement of, say, $\theta_0 = \pi/2$ $(= 90°)$ to $-\pi/2$ and back again would be exactly the same as the time it would take to cycle from, say, $\theta_0 = \pi/360$ $(= 0.5°)$ to $-\pi/360$. This is intuitively unreasonable; the actual period must depend on θ_0.

If we assume that $g = 32$ ft/s^2 and $l = 32$ ft, then the period of oscillation of the linear model is $T = 2\pi$ s. Let us compare this last number with the period predicted by the nonlinear model when $\theta_0 = \pi/4$. Using a numerical solver that is capable of generating hard data, approximate the solution of

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \frac{\pi}{4}, \quad \theta'(0) = 0$$

on the interval $0 \le t \le 2$. As in Problem 22, if t_1 denotes the first time the pendulum reaches the position OP in Figure 5.3.3, then the period of the nonlinear pendulum is $4t_1$. Here is another way of solving the equation $\theta(t) = 0$. Experiment with small step sizes and advance the time, starting at $t = 0$ and ending at $t = 2$. From your hard data observe the time t_1 when $\theta(t)$ changes, for the first time, from positive to negative. Use the value t_1 to determine the true value of the period of the nonlinear pendulum. Compute the percentage relative error in the period estimated by $T = 2\pi$.

Contributed Problem

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24. The Ballistic Pendulum Historically, to maintain quality control over munitions (bullets) produced by an assembly line, the manufacturer would use a **ballistic pendulum** to determine the muzzle velocity of a gun, that is, the speed of a bullet as it leaves the barrel. The ballistic pendulum (invented in 1742) is simply a plane pendulum consisting of a rod of negligible mass to which a block of wood of mass m_w is attached. The system is set in motion by the impact of a bullet that is moving horizontally at the unknown velocity v_b; at the time of the impact, which we take as $t = 0$, the combined mass is $m_w + m_b$, where m_b is the mass of the bullet imbedded in the wood. In (7) we saw that in the case of small oscillations, the angular displacement $\theta(t)$ of a plane pendulum shown in Figure 5.3.3 is given by the linear DE $\theta'' + (g/l)\theta = 0$, where $\theta > 0$ corresponds to motion to the right of vertical. The velocity v_b can be found by measuring the height h of the mass $m_w + m_b$ at the maximum displacement angle θ_{max} shown in Figure 5.3.10.

Intuitively, the horizontal velocity V of the combined mass (wood plus bullet) after impact is only a fraction of the velocity v_b of the bullet, that is,

$$V = \left(\frac{m_b}{m_w + m_b}\right) v_b.$$

Now, recall that a distance s traveled by a particle moving along a circular path is related to the radius l and central angle θ by the formula $s = l\theta$. By differentiating the last formula with respect to time t, it follows that the angular velocity ω of the mass and its linear velocity v are related by $v = l\omega$. Thus the initial angular velocity ω_0 at the time t at which the bullet impacts the wood block is related to V by $V = l\omega_0$ or

$$\omega_0 = \left(\frac{m_b}{m_w + m_b}\right)\frac{v_b}{l}.$$

(a) Solve the initial-value problem

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0, \quad \theta(0) = 0, \quad \theta'(0) = \omega_0.$$

(b) Use the result from part (a) to show that

$$v_b = \left(\frac{m_w + m_b}{m_b}\right)\sqrt{lg}\,\theta_{max}.$$

(c) Use Figure 5.3.10 to express $\cos\theta_{max}$ in terms of l and h. Then use the first two terms of the Maclaurin series for $\cos\theta$ to express θ_{max} in terms of l and h. Finally, show that v_b is given (approximately) by

$$v_b = \left(\frac{m_w + m_b}{m_b}\right)\sqrt{2gh}.$$

(d) Use the result in part (c) to find v_b when $m_b = 5$ g, $m_w = 1$ kg, and $h = 6$ cm.

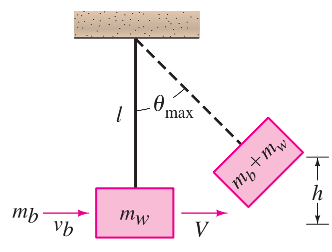

FIGURE 5.3.10 Ballistic pendulum

CHAPTER 5 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-8.

Answer Problems 1–8 without referring back to the text. Fill in the blank or answer true/false.

1. If a mass weighing 10 pounds stretches a spring 2.5 feet, a mass weighing 32 pounds will stretch it __________ feet.

2. The period of simple harmonic motion of mass weighing 8 pounds attached to a spring whose constant is 6.25 lb/ft is __________ seconds.

3. The differential equation of a spring/mass system is $x'' + 16x = 0$. If the mass is initially released from a