1.2 INITIAL-VALUE PROBLEMS

REVIEW MATERIAL

- Normal form of a DE
- Solution of a DE
- Family of solutions

INTRODUCTION We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ satisfies prescribed side conditions—that is, conditions imposed on the unknown $y(x)$ or its derivatives. On some interval I containing x_0 the problem

$$\begin{aligned} &\textit{Solve:} && \frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \\ &\textit{Subject to:} && y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}, \end{aligned} \tag{1}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrarily specified real constants, is called an **initial-value problem (IVP).** The values of $y(x)$ and its first $n - 1$ derivatives at a single point x_0, $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$, are called **initial conditions.**

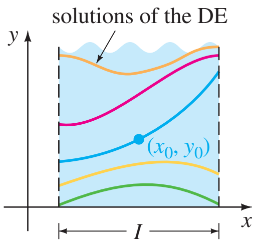


FIGURE 1.2.1 Solution of first-order IVP

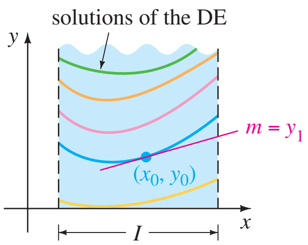


FIGURE 1.2.2 Solution of second-order IVP

FIRST- AND SECOND-ORDER IVPS The problem given in (1) is also called an ***n*th-order initial-value problem.** For example,

$$\begin{aligned} &\textit{Solve:} && \frac{dy}{dx} = f(x, y) \\ &\textit{Subject to:} && y(x_0) = y_0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\textit{Solve:} && \frac{d^2y}{dx^2} = f(x, y, y') \\ &\textit{Subject to:} && y(x_0) = y_0, y'(x_0) = y_1 \end{aligned} \tag{3}$$

are **first-** and **second-order** initial-value problems, respectively. These two problems are easy to interpret in geometric terms. For (2) we are seeking a solution $y(x)$ of the differential equation $y' = f(x, y)$ on an interval I containing x_0 so that its graph passes through the specified point (x_0, y_0). A solution curve is shown in blue in Figure 1.2.1. For (3) we want to find a solution $y(x)$ of the differential equation $y'' = f(x, y, y')$ on an interval I containing x_0 so that its graph not only passes through (x_0, y_0) but the slope of the curve at this point is the number y_1. A solution curve is shown in blue in Figure 1.2.2. The words *initial conditions* derive from physical systems where the independent variable is time t and where $y(t_0) = y_0$ and $y'(t_0) = y_1$ represent the position and velocity, respectively, of an object at some beginning, or initial, time t_0.

Solving an nth-order initial-value problem such as (1) frequently entails first finding an n-parameter family of solutions of the given differential equation and then using the n initial conditions at x_0 to determine numerical values of the n constants in the family. The resulting particular solution is defined on some interval I containing the initial point x_0.

EXAMPLE 1 **Two First-Order IVPs**

In Problem 41 in Exercises 1.1 you were asked to deduce that $y = ce^x$ is a one-parameter family of solutions of the simple first-order equation $y' = y$. All the solutions in this family are defined on the interval $(-\infty, \infty)$. If we impose an initial condition, say, $y(0) = 3$, then substituting $x = 0$, $y = 3$ in the family determines the

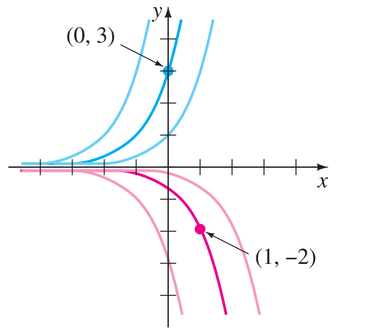
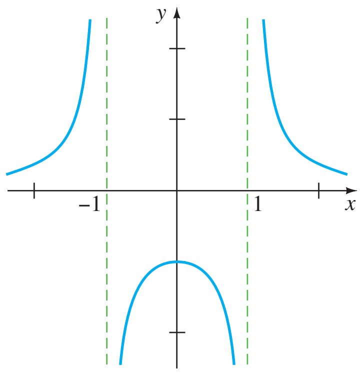
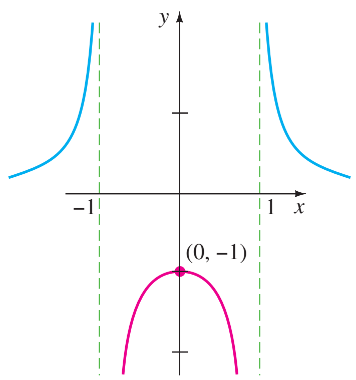


FIGURE 1.2.3 Solutions of two IVPs



(a) function defined for all x except $x = \pm 1$



(b) solution defined on interval containing $x = 0$

FIGURE 1.2.4 Graphs of function and solution of IVP in Example 2

constant $3 = ce^0 = c$. Thus $y = 3e^x$ is a solution of the IVP

$$y' = y, \quad y(0) = 3.$$

Now if we demand that a solution curve pass through the point $(1, -2)$ rather than $(0, 3)$, then $y(1) = -2$ will yield $-2 = ce$ or $c = -2e^{-1}$. In this case $y = -2e^{x-1}$ is a solution of the IVP

$$y' = y, \quad y(1) = -2.$$

The two solution curves are shown in dark blue and dark red in Figure 1.2.3. ■

The next example illustrates another first-order initial-value problem. In this example notice how the interval I of definition of the solution $y(x)$ depends on the initial condition $y(x_0) = y_0$.

EXAMPLE 2 Interval *I* of Definition of a Solution

In Problem 6 of Exercises 2.2 you will be asked to show that a one-parameter family of solutions of the first-order differential equation $y' + 2xy^2 = 0$ is $y = 1/(x^2 + c)$. If we impose the initial condition $y(0) = -1$, then substituting $x = 0$ and $y = -1$ into the family of solutions gives $-1 = 1/c$ or $c = -1$. Thus $y = 1/(x^2 - 1)$. We now emphasize the following three distinctions:

- Considered as a *function*, the domain of $y = 1/(x^2 - 1)$ is the set of real numbers x for which $y(x)$ is defined; this is the set of all real numbers except $x = -1$ and $x = 1$. See Figure 1.2.4(a).
- Considered as a *solution of the differential equation* $y' + 2xy^2 = 0$, the interval I of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined and differentiable. As can be seen in Figure 1.2.4(a), the largest intervals on which $y = 1/(x^2 - 1)$ is a solution are $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.
- Considered as a *solution of the initial-value problem* $y' + 2xy^2 = 0$, $y(0) = -1$, the interval I of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined, differentiable, *and* contains the initial point $x = 0$; the largest interval for which this is true is $(-1, 1)$. See the red curve in Figure 1.2.4(b). ■

See Problems 3–6 in Exercises 1.2 for a continuation of Example 2.

EXAMPLE 3 Second-Order IVP

In Example 4 of Section 1.1 we saw that $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of $x'' + 16x = 0$. Find a solution of the initial-value problem

$$x'' + 16x = 0, \quad x\left(\frac{\pi}{2}\right) = -2, \quad x'\left(\frac{\pi}{2}\right) = 1. \tag{4}$$

SOLUTION We first apply $x(\pi/2) = -2$ to the given family of solutions: $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$. Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we find that $c_1 = -2$. We next apply $x'(\pi/2) = 1$ to the one-parameter family $x(t) = -2 \cos 4t + c_2 \sin 4t$. Differentiating and then setting $t = \pi/2$ and $x' = 1$ gives $8 \sin 2\pi + 4c_2 \cos 2\pi = 1$, from which we see that $c_2 = \frac{1}{4}$. Hence $x = -2 \cos 4t + \frac{1}{4} \sin 4t$ is a solution of (4). ■

EXISTENCE AND UNIQUENESS Two fundamental questions arise in considering an initial-value problem:

Does a solution of the problem exist?
If a solution exists, is it unique?

For the first-order initial-value problem (2) we ask:

Existence $\begin{cases} \textit{Does the differential equation } dy/dx = f(x, y) \textit{ possess solutions?} \\ \textit{Do any of the solution curves pass through the point } (x_0, y_0)\textit{?} \end{cases}$

Uniqueness $\begin{cases} \textit{When can we be certain that there is precisely one solution curve} \\ \textit{passing through the point } (x_0, y_0)\textit{?} \end{cases}$

Note that in Examples 1 and 3 the phrase "*a* solution" is used rather than "*the* solution" of the problem. The indefinite article "a" is used deliberately to suggest the possibility that other solutions may exist. At this point it has not been demonstrated that there is a single solution of each problem. The next example illustrates an initial-value problem with two solutions.

EXAMPLE 4 An IVP Can Have Several Solutions

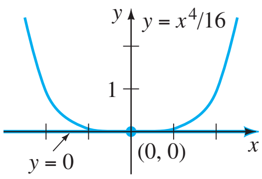


FIGURE 1.2.5 Two solutions of the same IVP

Each of the functions $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the differential equation $dy/dx = xy^{1/2}$ and the initial condition $y(0) = 0$, so the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

has at least two solutions. As illustrated in Figure 1.2.5, the graphs of both functions pass through the same point (0, 0). ■

Within the safe confines of a formal course in differential equations one can be fairly confident that *most* differential equations will have solutions and that solutions of initial-value problems will *probably* be unique. Real life, however, is not so idyllic. Therefore it is desirable to know in advance of trying to solve an initial-value problem whether a solution exists and, when it does, whether it is the only solution of the problem. Since we are going to consider first-order differential equations in the next two chapters, we state here without proof a straightforward theorem that gives conditions that are sufficient to guarantee the existence and uniqueness of a solution of a first-order initial-value problem of the form given in (2). We shall wait until Chapter 4 to address the question of existence and uniqueness of a second-order initial-value problem.

THEOREM 1.2.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem (2).

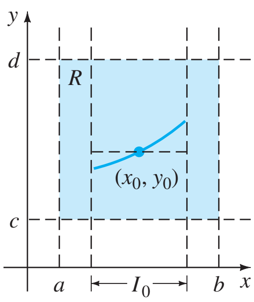


FIGURE 1.2.6 Rectangular region R

The foregoing result is one of the most popular existence and uniqueness theorems for first-order differential equations because the criteria of continuity of $f(x, y)$ and $\partial f/\partial y$ are relatively easy to check. The geometry of Theorem 1.2.1 is illustrated in Figure 1.2.6.

EXAMPLE 5 Example 4 Revisited

We saw in Example 4 that the differential equation $dy/dx = xy^{1/2}$ possesses at least two solutions whose graphs pass through (0, 0). Inspection of the functions

$$f(x, y) = xy^{1/2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

shows that they are continuous in the upper half-plane defined by $y > 0$. Hence Theorem 1.2.1 enables us to conclude that through any point (x_0, y_0), $y_0 > 0$ in the upper half-plane there is some interval centered at x_0 on which the given differential equation has a unique solution. Thus, for example, even without solving it, we know that there exists some interval centered at 2 on which the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ has a unique solution. ■

In Example 1, Theorem 1.2.1 guarantees that there are no other solutions of the initial-value problems $y' = y$, $y(0) = 3$ and $y' = y$, $y(1) = -2$ other than $y = 3e^x$ and $y = -2e^{x-1}$, respectively. This follows from the fact that $f(x, y) = y$ and $\partial f/\partial y = 1$ are continuous throughout the entire xy-plane. It can be further shown that the interval I on which each solution is defined is $(-\infty, \infty)$.

INTERVAL OF EXISTENCE/UNIQUENESS Suppose $y(x)$ represents a solution of the initial-value problem (2). The following three sets on the real x-axis may not be the same: the domain of the function $y(x)$, the interval I over which the solution $y(x)$ is defined or exists, and the interval I_0 of existence *and* uniqueness. Example 2 of Section 1.1 illustrated the difference between the domain of a function and the interval I of definition. Now suppose (x_0, y_0) is a point in the interior of the rectangular region R in Theorem 1.2.1. It turns out that the continuity of the function $f(x, y)$ on R by itself is sufficient to guarantee the existence of at least one solution of $dy/dx = f(x, y)$, $y(x_0) = y_0$, defined on some interval I. The interval I of definition for this initial-value problem is usually taken to be the largest interval containing x_0 over which the solution $y(x)$ is defined and differentiable. The interval I depends on both $f(x, y)$ and the initial condition $y(x_0) = y_0$. See Problems 31–34 in Exercises 1.2. The extra condition of continuity of the first partial derivative $\partial f/\partial y$ on R enables us to say that not only does a solution exist on some interval I_0 containing x_0, but it is the *only* solution satisfying $y(x_0) = y_0$. However, Theorem 1.2.1 does not give any indication of the sizes of intervals I and I_0; *the interval I of definition need not be as wide as the region R, and the interval I_0 of existence and uniqueness may not be as large as I.* The number $h > 0$ that defines the interval I_0: $(x_0 - h, x_0 + h)$ could be very small, so it is best to think that the solution $y(x)$ is *unique in a local sense*—that is, a solution defined near the point (x_0, y_0). See Problem 44 in Exercises 1.2.

REMARKS

(*i*) The conditions in Theorem 1.2.1 are sufficient but not necessary. This means that when $f(x, y)$ and $\partial f/\partial y$ are continuous on a rectangular region R, it must always follow that a solution of (2) exists and is unique whenever (x_0, y_0) is a point interior to R. However, if the conditions stated in the hypothesis of Theorem 1.2.1 do not hold, then anything could happen: Problem (2) *may* still have a solution and this solution *may* be unique, or (2) may have several solutions, or it may have no solution at all. A rereading of Example 5 reveals that the hypotheses of Theorem 1.2.1 do not hold on the line $y = 0$ for the differential equation $dy/dx = xy^{1/2}$, so it is not surprising, as we saw in Example 4 of this section, that there are two solutions defined on a common interval $-h < x < h$ satisfying $y(0) = 0$. On the other hand, the hypotheses of Theorem 1.2.1 do not hold on the line $y = 1$ for the differential equation $dy/dx = |y - 1|$. Nevertheless it can be proved that the solution of the initial-value problem $dy/dx = |y - 1|$, $y(0) = 1$, is unique. Can you guess this solution?

(*ii*) You are encouraged to read, think about, work, and then keep in mind Problem 43 in Exercises 1.2.

EXERCISES 1.2

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1 and 2, $y = 1/(1 + c_1e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(0) = -\frac{1}{3}$ **2.** $y(-1) = 2$

In Problems 3–6, $y = 1/(x^2 + c)$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition. Give the largest interval I over which the solution is defined.

3. $y(2) = \frac{1}{3}$ **4.** $y(-2) = \frac{1}{2}$

5. $y(0) = 1$ **6.** $y\left(\frac{1}{2}\right) = -4$

In Problems 7–10, $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

7. $x(0) = -1, \quad x'(0) = 8$

8. $x(\pi/2) = 0, \quad x'(\pi/2) = 1$

9. $x(\pi/6) = \frac{1}{2}, \quad x'(\pi/6) = 0$

10. $x(\pi/4) = \sqrt{2}, \quad x'(\pi/4) = 2\sqrt{2}$

In Problems 11–14, $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

11. $y(0) = 1, \quad y'(0) = 2$

12. $y(1) = 0, \quad y'(1) = e$

13. $y(-1) = 5, \quad y'(-1) = -5$

14. $y(0) = 0, \quad y'(0) = 0$

In Problems 15 and 16 determine by inspection at least two solutions of the given first-order IVP.

15. $y' = 3y^{2/3}, \quad y(0) = 0$

16. $xy' = 2y, \quad y(0) = 0$

In Problems 17–24 determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

17. $\dfrac{dy}{dx} = y^{2/3}$ **18.** $\dfrac{dy}{dx} = \sqrt{xy}$

19. $x\dfrac{dy}{dx} = y$ **20.** $\dfrac{dy}{dx} - y = x$

21. $(4 - y^2)y' = x^2$ **22.** $(1 + y^3)y' = x^2$

23. $(x^2 + y^2)y' = y^2$ **24.** $(y - x)y' = y + x$

In Problems 25–28 determine whether Theorem 1.2.1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

25. $(1, 4)$ **26.** $(5, 3)$

27. $(2, -3)$ **28.** $(-1, 1)$

29. **(a)** By inspection find a one-parameter family of solutions of the differential equation $xy' = y$. Verify that each member of the family is a solution of the initial-value problem $xy' = y$, $y(0) = 0$.

(b) Explain part (a) by determining a region R in the xy-plane for which the differential equation $xy' = y$ would have a unique solution through a point (x_0, y_0) in R.

(c) Verify that the piecewise-defined function

$$y = \begin{cases} 0, & x < 0 \\ x, & x \ge 0 \end{cases}$$

satisfies the condition $y(0) = 0$. Determine whether this function is also a solution of the initial-value problem in part (a).

30. **(a)** Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.

(b) Since $f(x, y) = 1 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Use the family of solutions in part (a) to find an explicit solution of the first-order initial-value problem $y' = 1 + y^2$, $y(0) = 0$. Even though $x_0 = 0$ is in the interval $(-2, 2)$, explain why the solution is not defined on this interval.

(c) Determine the largest interval I of definition for the solution of the initial-value problem in part (b).

31. **(a)** Verify that $y = -1/(x + c)$ is a one-parameter family of solutions of the differential equation $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of definition for the solution of each initial-value problem.

(c) Determine the largest interval I of definition for the solution of the first-order initial-value problem $y' = y^2$, $y(0) = 0$. [*Hint*: The solution is not a member of the family of solutions in part (a).]

32. **(a)** Show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(1) = 1$, is $y = 1/(2 - x)$.

(b) Then show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(3) = -1$, is $y = 1/(2 - x)$.

(c) Are the solutions in parts (a) and (b) the same?

33. **(a)** Verify that $3x^2 - y^2 = c$ is a one-parameter family of solutions of the differential equation $y\,dy/dx = 3x$.

(b) By hand, sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all explicit solutions $y = \phi(x)$ of the DE in part (a) defined by this relation. Give the interval I of definition of each explicit solution.

(c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$, but which of the explicit solutions in part (b) satisfies $y(-2) = 3$?

34. **(a)** Use the family of solutions in part (a) of Problem 33 to find an implicit solution of the initial-value problem $y\,dy/dx = 3x$, $y(2) = -4$. Then, by hand, sketch the graph of the explicit solution of this problem and give its interval I of definition.

(b) Are there any explicit solutions of $y\,dy/dx = 3x$ that pass through the origin?

In Problems 35–38 the graph of a member of a family of solutions of a second-order differential equation $d^2y/dx^2 = f(x, y, y')$ is given. Match the solution curve with at least one pair of the following initial conditions.

(a) $y(1) = 1, \quad y'(1) = -2$

(b) $y(-1) = 0, \quad y'(-1) = -4$

(c) $y(1) = 1, \quad y'(1) = 2$

(d) $y(0) = -1, \quad y'(0) = 2$

(e) $y(0) = -1, \quad y'(0) = 0$

(f) $y(0) = -4, \quad y'(0) = -2$

35.

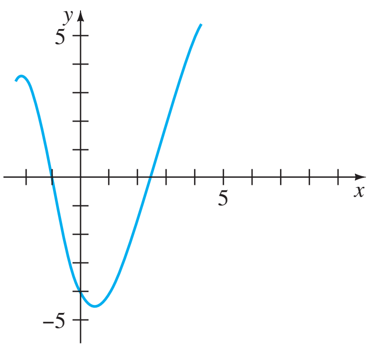


FIGURE 1.2.7 Graph for Problem 35

36.

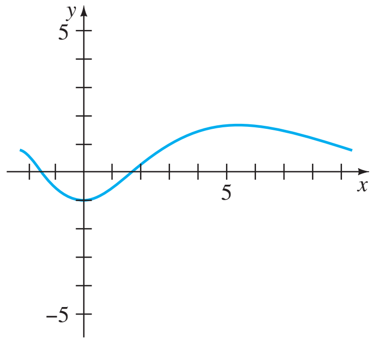


FIGURE 1.2.8 Graph for Problem 36

37.

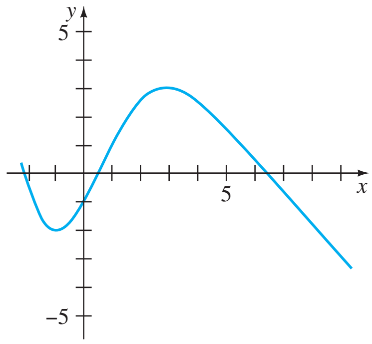


FIGURE 1.2.9 Graph for Problem 37

38.

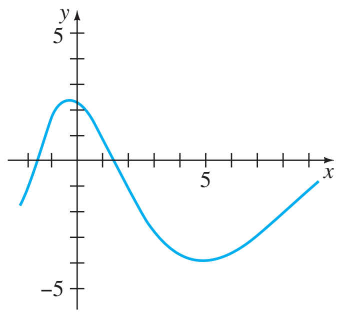


FIGURE 1.2.10 Graph for Problem 38

Discussion Problems

In Problems 39 and 40 use Problem 51 in Exercises 1.1 and (2) and (3) of this section.

39. Find a function $y = f(x)$ whose graph at each point (x, y) has the slope given by $8e^{2x} + 6x$ and has the y-intercept $(0, 9)$.

40. Find a function $y = f(x)$ whose second derivative is $y'' = 12x - 2$ at each point (x, y) on its graph and $y = -x + 5$ is tangent to the graph at the point corresponding to $x = 1$.

41. Consider the initial-value problem $y' = x - 2y$, $y(0) = \frac{1}{2}$. Determine which of the two curves shown in Figure 1.2.11 is the only plausible solution curve. Explain your reasoning.

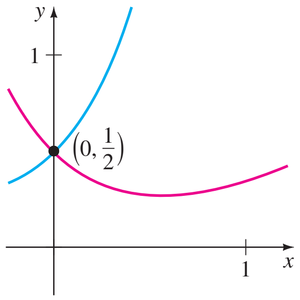


FIGURE 1.2.11 Graphs for Problem 41

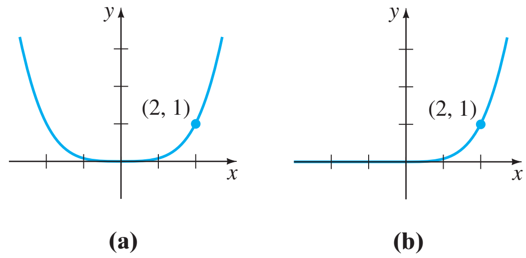


FIGURE 1.2.12 Two solutions of the IVP in Problem 44

42. Determine a plausible value of x_0 for which the graph of the solution of the initial-value problem $y' + 2y = 3x - 6$, $y(x_0) = 0$ is tangent to the x-axis at $(x_0, 0)$. Explain your reasoning.

43. Suppose that the first-order differential equation $dy/dx = f(x, y)$ possesses a one-parameter family of solutions and that $f(x, y)$ satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R of the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R.

44. The functions $y(x) = \frac{1}{16}x^4$, $\quad -\infty < x < \infty$ and

$$y(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

have the same domain but are clearly different. See Figures 1.2.12(a) and 1.2.12(b), respectively. Show that both functions are solutions of the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ on the interval $(-\infty, \infty)$. Resolve the apparent contradiction between this fact and the last sentence in Example 5.

Mathematical Model

45. Population Growth Beginning in the next section we will see that differential equations can be used to describe or *model* many different physical systems. In this problem suppose that a model of the growing population of a small community is given by the initial-value problem

$$\frac{dP}{dt} = 0.15P(t) + 20, \quad P(0) = 100,$$

where P is the number of individuals in the community and time t is measured in years. How fast—that is, at what *rate*—is the population increasing at $t = 0$? How fast is the population increasing when the population is 500?

1.3 DIFFERENTIAL EQUATIONS AS MATHEMATICAL MODELS

REVIEW MATERIAL

- Units of measurement for weight, mass, and density
- Newton's second law of motion
- Hooke's law
- Kirchhoff's laws
- Archimedes' principle

INTRODUCTION In this section we introduce the notion of a differential equation as a mathematical model and discuss some specific models in biology, chemistry, and physics. Once we have studied some methods for solving DEs in Chapters 2 and 4, we return to, and solve, some of these models in Chapters 3 and 5.

MATHEMATICAL MODELS It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system of phenomenon is called a **mathematical model** and is constructed with certain goals in mind. For example, we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system, or we may wish to date fossils by analyzing the decay of a radioactive substance either in the fossil or in the stratum in which it was discovered.

4.5 UNDETERMINED COEFFICIENTS—ANNIHILATOR APPROACH

REVIEW MATERIAL

- Review Theorems 4.1.6 and 4.1.7 (Section 4.1)

INTRODUCTION We saw in Section 4.1 that an nth-order differential equation can be written

$$a_nD^ny + a_{n-1}D^{n-1}y + \cdots + a_1Dy + a_0y = g(x), \tag{1}$$

where $D^ky = d^ky/dx^k$, $k = 0, 1, \ldots, n$. When it suits our purpose, (1) is also written as $L(y) = g(x)$, where L denotes the linear nth-order differential, or polynomial, operator

$$a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0. \tag{2}$$

Not only is the operator notation a helpful shorthand, but also on a very practical level the application of differential operators enables us to justify the somewhat mind-numbing rules for determining the form of particular solution y_p that were presented in the preceding section. In this section there are no special rules; the form of y_p follows almost automatically once we have found an appropriate linear differential operator that *annihilates* $g(x)$ in (1). Before investigating how this is done, we need to examine two concepts.

FACTORING OPERATORS When the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants, a linear differential operator (1) can be factored whenever the characteristic polynomial $a_nm^n + a_{n-1}m^{n-1} + \cdots + a_1m + a_0$ factors. In other words, if r_1 is a root of the auxiliary equation

$$a_nm^n + a_{n-1}m^{n-1} + \cdots + a_1m + a_0 = 0,$$

then $L = (D - r_1)\,P(D)$, where the polynomial expression $P(D)$ is a linear differential operator of order $n - 1$. For example, if we treat D as an algebraic quantity, then the operator $D^2 + 5D + 6$ can be factored as $(D + 2)(D + 3)$ or as $(D + 3)(D + 2)$. Thus if a function $y = f(x)$ possesses a second derivative, then

$$(D^2 + 5D + 6)y = (D + 2)(D + 3)y = (D + 3)(D + 2)y.$$

This illustrates a general property:

Factors of a linear differential operator with constant coefficients commute.

A differential equation such as $y'' + 4y' + 4y = 0$ can be written as

$$(D^2 + 4D + 4)y = 0 \quad \text{or} \quad (D + 2)(D + 2)y = 0 \quad \text{or} \quad (D + 2)^2y = 0.$$

ANNIHILATOR OPERATOR If L is a linear differential operator with constant coefficients and f is a sufficiently differentiable function such that

$$L(f(x)) = 0,$$

then L is said to be an **annihilator** of the function. For example, a constant function $y = k$ is annihilated by D, since $Dk = 0$. The function $y = x$ is annihilated by the differential operator D^2 since the first and second derivatives of x are 1 and 0, respectively. Similarly, $D^3x^2 = 0$, and so on.

The differential operator D^n annihilates each of the functions

$$1, \quad x, \quad x^2, \quad \ldots, \quad x^{n-1}. \tag{3}$$

As an immediate consequence of (3) and the fact that differentiation can be done term by term, a polynomial

$$c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} \tag{4}$$

can be annihilated by finding an operator that annihilates the highest power of x.

The functions that are annihilated by a linear nth-order differential operator L are simply those functions that can be obtained from the general solution of the homogeneous differential equation $L(y) = 0$.

The differential operator $(D - \alpha)^n$ annihilates each of the functions

$$e^{\alpha x}, \quad xe^{\alpha x}, \quad x^2e^{\alpha x}, \quad \ldots, \quad x^{n-1}e^{\alpha x}. \tag{5}$$

To see this, note that the auxiliary equation of the homogeneous equation $(D - \alpha)^n y = 0$ is $(m - \alpha)^n = 0$. Since α is a root of multiplicity n, the general solution is

$$y = c_1e^{\alpha x} + c_2xe^{\alpha x} + \cdots + c_nx^{n-1}e^{\alpha x}. \tag{6}$$

EXAMPLE 1 Annihilator Operators

Find a differential operator that annihilates the given function.

(a) $1 - 5x^2 + 8x^3$ **(b)** e^{-3x} **(c)** $4e^{2x} - 10xe^{2x}$

SOLUTION **(a)** From (3) we know that $D^4x^3 = 0$, so it follows from (4) that

$$D^4(1 - 5x^2 + 8x^3) = 0.$$

(b) From (5), with $\alpha = -3$ and $n = 1$, we see that

$$(D + 3)e^{-3x} = 0.$$

(c) From (5) and (6), with $\alpha = 2$ and $n = 2$, we have

$$(D - 2)^2(4e^{2x} - 10xe^{2x}) = 0.$$

■

When α and β, $\beta > 0$ are real numbers, the quadratic formula reveals that $[m^2 - 2\alpha m + (\alpha^2 + \beta^2)]^n = 0$ has complex roots $\alpha + i\beta$, $\alpha - i\beta$, both of multiplicity n. From the discussion at the end of Section 4.3 we have the next result.

The differential operator $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$ annihilates each of the functions

$$\begin{aligned} &e^{\alpha x}\cos\beta x, \quad xe^{\alpha x}\cos\beta x, \quad x^2e^{\alpha x}\cos\beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x}\cos\beta x, \\ &e^{\alpha x}\sin\beta x, \quad xe^{\alpha x}\sin\beta x, \quad x^2e^{\alpha x}\sin\beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x}\sin\beta x. \end{aligned} \tag{7}$$

EXAMPLE 2 Annihilator Operator

Find a differential operator that annihilates $5e^{-x}\cos 2x - 9e^{-x}\sin 2x$.

SOLUTION Inspection of the functions $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$ shows that $\alpha = -1$ and $\beta = 2$. Hence from (7) we conclude that $D^2 + 2D + 5$ will annihilate each function. Since $D^2 + 2D + 5$ is a linear operator, it will annihilate *any* linear combination of these functions such as $5e^{-x}\cos 2x - 9e^{-x}\sin 2x$. ■

When $\alpha = 0$ and $n = 1$, a special case of (7) is

$$(D^2 + \beta^2)\begin{cases}\cos \beta x \\ \sin \beta x\end{cases} = 0. \tag{8}$$

For example, $D^2 + 16$ will annihilate any linear combination of $\sin 4x$ and $\cos 4x$.

We are often interested in annihilating the sum of two or more functions. As we have just seen in Examples 1 and 2, if L is a linear differential operator such that $L(y_1) = 0$ and $L(y_2) = 0$, then L will annihilate the linear combination $c_1y_1(x) + c_2y_2(x)$. This is a direct consequence of Theorem 4.1.2. Let us now suppose that L_1 and L_2 are linear differential operators with constant coefficients such that L_1 annihilates $y_1(x)$ and L_2 annihilates $y_2(x)$, but $L_1(y_2) \neq 0$ and $L_2(y_1) \neq 0$. Then the *product* of differential operators L_1L_2 annihilates the sum $c_1y_1(x) + c_2y_2(x)$. We can easily demonstrate this, using linearity and the fact that $L_1L_2 = L_2L_1$:

$$\begin{aligned} L_1L_2(y_1 + y_2) &= L_1L_2(y_1) + L_1L_2(y_2) \\ &= L_2L_1(y_1) + L_1L_2(y_2) \\ &= L_2[\underbrace{L_1(y_1)}_{\text{zero}}] + L_1[\underbrace{L_2(y_2)}_{\text{zero}}] = 0. \end{aligned}$$

For example, we know from (3) that D^2 annihilates $7 - x$ and from (8) that $D^2 + 16$ annihilates $\sin 4x$. Therefore the product of operators $D^2(D^2 + 16)$ will annihilate the linear combination $7 - x + 6 \sin 4x$.

NOTE The differential operator that annihilates a function is not unique. We saw in part (b) of Example 1 that $D + 3$ will annihilate e^{-3x}, but so will differential operators of higher order as long as $D + 3$ is one of the factors of the operator. For example, $(D + 3)(D + 1)$, $(D + 3)^2$, and $D^3(D + 3)$ all annihilate e^{-3x}. (Verify this.) As a matter of course, when we seek a differential annihilator for a function $y = f(x)$, we want the operator of *lowest possible order* that does the job.

UNDETERMINED COEFFICIENTS This brings us to the point of the preceding discussion. Suppose that $L(y) = g(x)$ is a linear differential equation with constant coefficients and that the input $g(x)$ consists of finite sums and products of the functions listed in (3), (5), and (7)—that is, $g(x)$ is a linear combination of functions of the form

$$k \text{ (constant)}, \quad x^m, \quad x^m e^{\alpha x}, \quad x^m e^{\alpha x} \cos \beta x, \quad \text{and} \quad x^m e^{\alpha x} \sin \beta x,$$

where m is a nonnegative integer and α and β are real numbers. We now know that such a function $g(x)$ can be annihilated by a differential operator L_1 of lowest order, consisting of a product of the operators D^n, $(D - \alpha)^n$, and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^n$. Applying L_1 to both sides of the equation $L(y) = g(x)$ yields $L_1L(y) = L_1(g(x)) = 0$. By solving the *homogeneous higher-order* equation $L_1L(y) = 0$, we can discover the *form* of a particular solution y_p for the original *nonhomogeneous* equation $L(y) = g(x)$. We then substitute this assumed form into $L(y) = g(x)$ to find an explicit particular solution. This procedure for determining y_p, called the **method of undetermined coefficients,** is illustrated in the next several examples.

Before proceeding, recall that the general solution of a nonhomogeneous linear differential equation $L(y) = g(x)$ is $y = y_c + y_p$, where y_c is the complementary function—that is, the general solution of the associated homogeneous equation $L(y) = 0$. The general solution of each equation $L(y) = g(x)$ is defined on the interval $(-\infty, \infty)$.

EXAMPLE 3 General Solution Using Undetermined Coefficients

Solve $y'' + 3y' + 2y = 4x^2$. (9)

SOLUTION **Step 1.** First, we solve the homogeneous equation $y'' + 3y' + 2y = 0$. Then, from the auxiliary equation $m^2 + 3m + 2 = (m + 1)(m + 2) = 0$ we find $m_1 = -1$ and $m_2 = -2$, and so the complementary function is

$$y_c = c_1e^{-x} + c_2e^{-2x}.$$

Step 2. Now, since $4x^2$ is annihilated by the differential operator D^3, we see that $D^3(D^2 + 3D + 2)y = 4D^3x^2$ is the same as

$$D^3(D^2 + 3D + 2)y = 0. \tag{10}$$

The auxiliary equation of the fifth-order equation in (10),

$$m^3(m^2 + 3m + 2) = 0 \quad \text{or} \quad m^3(m + 1)(m + 2) = 0,$$

has roots $m_1 = m_2 = m_3 = 0$, $m_4 = -1$, and $m_5 = -2$. Thus its general solution must be

$$y = c_1 + c_2x + c_3x^2 + \boxed{c_4e^{-x} + c_5e^{-2x}}. \tag{11}$$

The terms in the shaded box in (11) constitute the complementary function of the original equation (9). We can then argue that a particular solution y_p of (9) should also satisfy equation (10). This means that the terms remaining in (11) must be the basic form of y_p:

$$y_p = A + Bx + Cx^2, \tag{12}$$

where, for convenience, we have replaced c_1, c_2, and c_3 by A, B, and C, respectively. For (12) to be a particular solution of (9), it is necessary to find *specific* coefficients A, B, and C. Differentiating (12), we have

$$y_p' = B + 2Cx, \qquad y_p'' = 2C,$$

and substitution into (9) then gives

$$y_p'' + 3y_p' + 2y_p = 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2 = 4x^2.$$

Because the last equation is supposed to be an identity, the coefficients of like powers of x must be equal:

equal

$$\boxed{2C}\,x^2 + \boxed{2B + 6C}\,x + \boxed{2A + 3B + 2C} = 4x^2 + 0x + 0.$$

That is $\qquad 2C = 4, \qquad 2B + 6C = 0, \qquad 2A + 3B + 2C = 0.$ (13)

Solving the equations in (13) gives $A = 7$, $B = -6$, and $C = 2$. Thus $y_p = 7 - 6x + 2x^2$.

Step 3. The general solution of the equation in (9) is $y = y_c + y_p$ or

$$y = c_1e^{-x} + c_2e^{-2x} + 7 - 6x + 2x^2.$$ ■

EXAMPLE 4 General Solution Using Undetermined Coefficients

Solve $y'' - 3y' = 8e^{3x} + 4\sin x$. (14)

SOLUTION **Step 1.** The auxiliary equation for the associated homogeneous equation $y'' - 3y' = 0$ is $m^2 - 3m = m(m - 3) = 0$, so $y_c = c_1 + c_2e^{3x}$.

Step 2. Now, since $(D - 3)e^{3x} = 0$ and $(D^2 + 1)\sin x = 0$, we apply the differential operator $(D - 3)(D^2 + 1)$ to both sides of (14):

$$(D - 3)(D^2 + 1)(D^2 - 3D)y = 0. \tag{15}$$

The auxiliary equation of (15) is

$$(m - 3)(m^2 + 1)(m^2 - 3m) = 0 \qquad \text{or} \qquad m(m - 3)^2(m^2 + 1) = 0.$$

Thus $$y = \boxed{c_1 + c_2e^{3x}} + c_3xe^{3x} + c_4\cos x + c_5\sin x.$$

After excluding the linear combination of terms in the box that corresponds to y_c, we arrive at the form of y_p:

$$y_p = Axe^{3x} + B\cos x + C\sin x.$$

Substituting y_p in (14) and simplifying yield

$$y_p'' - 3y_p' = 3Ae^{3x} + (-B - 3C)\cos x + (3B - C)\sin x = 8e^{3x} + 4\sin x.$$

Equating coefficients gives $3A = 8$, $-B - 3C = 0$, and $3B - C = 4$. We find $A = \frac{8}{3}$, $B = \frac{6}{5}$, and $C = -\frac{2}{5}$, and consequently,

$$y_p = \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

Step 3. The general solution of (14) is then

$$y = c_1 + c_2e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$ ■

EXAMPLE 5 General Solution Using Undetermined Coefficients

Solve $y'' + y = x\cos x - \cos x$. (16)

SOLUTION The complementary function is $y_c = c_1\cos x + c_2\sin x$. Now by comparing $\cos x$ and $x\cos x$ with the functions in the first row of (7), we see that $\alpha = 0$ and $n = 1$, and so $(D^2 + 1)^2$ is an annihilator for the right-hand member of the equation in (16). Applying this operator to the differential equation gives

$$(D^2 + 1)^2(D^2 + 1)y = 0 \qquad \text{or} \qquad (D^2 + 1)^3y = 0.$$

Since i and $-i$ are both complex roots of multiplicity 3 of the auxiliary equation of the last differential equation, we conclude that

$$y = \boxed{c_1\cos x + c_2\sin x} + c_3x\cos x + c_4x\sin x + c_5x^2\cos x + c_6x^2\sin x.$$

We substitute

$$y_p = Ax\cos x + Bx\sin x + Cx^2\cos x + Ex^2\sin x$$

into (16) and simplify:

$$\begin{aligned} y_p'' + y_p &= 4Ex\cos x - 4Cx\sin x + (2B + 2C)\cos x + (-2A + 2E)\sin x \\ &= x\cos x - \cos x. \end{aligned}$$

Equating coefficients gives the equations $4E = 1$, $-4C = 0$, $2B + 2C = -1$, and $-2A + 2E = 0$, from which we find $A = \frac{1}{4}$, $B = -\frac{1}{2}$, $C = 0$, and $E = \frac{1}{4}$. Hence the general solution of (16) is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \cos x - \frac{1}{2} x \sin x + \frac{1}{4} x^2 \sin x.$$

EXAMPLE 6 Form of a Particular Solution

Determine the form of a particular solution for

$$y'' - 2y' + y = 10e^{-2x} \cos x. \tag{17}$$

SOLUTION The complementary function for the given equation is $y_c = c_1e^x + c_2xe^x$.

Now from (7), with $\alpha = -2$, $\beta = 1$, and $n = 1$, we know that

$$(D^2 + 4D + 5)e^{-2x} \cos x = 0.$$

Applying the operator $D^2 + 4D + 5$ to (17) gives

$$(D^2 + 4D + 5)(D^2 - 2D + 1)y = 0. \tag{18}$$

Since the roots of the auxiliary equation of (18) are $-2 - i$, $-2 + i$, 1, and 1, we see from

$$y = c_1e^x + c_2xe^x + c_3e^{-2x} \cos x + c_4e^{-2x} \sin x$$

that a particular solution of (17) can be found with the form

$$y_p = Ae^{-2x} \cos x + Be^{-2x} \sin x.$$

EXAMPLE 7 Form of a Particular Solution

Determine the form of a particular solution for

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}. \tag{19}$$

SOLUTION Observe that

$$D^3(5x^2 - 6x) = 0, \qquad (D - 2)^3x^2e^{2x} = 0, \qquad \text{and} \qquad (D - 5)e^{5x} = 0.$$

Therefore $D^3(D - 2)^3(D - 5)$ applied to (19) gives

$$D^3(D - 2)^3(D - 5)(D^3 - 4D^2 + 4D)y = 0$$

or

$$D^4(D - 2)^5(D - 5)y = 0.$$

The roots of the auxiliary equation for the last differential equation are easily seen to be 0, 0, 0, 0, 2, 2, 2, 2, 2, and 5. Hence

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{2x} + c_6xe^{2x} + c_7x^2e^{2x} + c_8x^3e^{2x} + c_9x^4e^{2x} + c_{10}e^{5x}. \tag{20}$$

Because the linear combination $c_1 + c_5e^{2x} + c_6xe^{2x}$ corresponds to the complementary function of (19), the remaining terms in (20) give the form of a particular solution of the differential equation:

$$y_p = Ax + Bx^2 + Cx^3 + Ex^2e^{2x} + Fx^3e^{2x} + Gx^4e^{2x} + He^{5x}.$$

SUMMARY OF THE METHOD For your convenience the method of undetermined coefficients is summarized as follows.

UNDETERMINED COEFFICIENTS—ANNIHILATOR APPROACH

The differential equation $L(y) = g(x)$ has constant coefficients, and the function $g(x)$ consists of finite sums and products of constants, polynomials, exponential functions $e^{\alpha x}$, sines, and cosines.

(*i*) Find the complementary solution y_c for the homogeneous equation $L(y) = 0$.

(*ii*) Operate on both sides of the nonhomogeneous equation $L(y) = g(x)$ with a differential operator L_1 that annihilates the function $g(x)$.

(*iii*) Find the general solution of the higher-order homogeneous differential equation $L_1L(y) = 0$.

(*iv*) Delete from the solution in step (*iii*) all those terms that are duplicated in the complementary solution y_c found in step (*i*). Form a linear combination y_p of the terms that remain. This is the form of a particular solution of $L(y) = g(x)$.

(*v*) Substitute y_p found in step (*iv*) into $L(y) = g(x)$. Match coefficients of the various functions on each side of the equality, and solve the resulting system of equations for the unknown coefficients in y_p.

(*vi*) With the particular solution found in step (*v*), form the general solution $y = y_c + y_p$ of the given differential equation.

REMARKS

The method of undetermined coefficients is not applicable to linear differential equations with variable coefficients nor is it applicable to linear equations with constant coefficients when $g(x)$ is a function such as

$$g(x) = \ln x, \qquad g(x) = \frac{1}{x}, \qquad g(x) = \tan x, \qquad g(x) = \sin^{-1} x,$$

and so on. Differential equations in which the input $g(x)$ is a function of this last kind will be considered in the next section.

EXERCISES 4.5

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–10 write the given differential equation in the form $L(y) = g(x)$, where L is a linear differential operator with constant coefficients. If possible, factor L.

1. $9y'' - 4y = \sin x$

2. $y'' - 5y = x^2 - 2x$

3. $y'' - 4y' - 12y = x - 6$

4. $2y'' - 3y' - 2y = 1$

5. $y''' + 10y'' + 25y' = e^x$

6. $y''' + 4y' = e^x \cos 2x$

7. $y''' + 2y'' - 13y' + 10y = xe^{-x}$

8. $y''' + 4y'' + 3y' = x^2 \cos x - 3x$

9. $y^{(4)} + 8y' = 4$

10. $y^{(4)} - 8y'' + 16y = (x^3 - 2x)e^{4x}$

In Problems 11–14 verify that the given differential operator annihilates the indicated functions.

11. $D^4; \quad y = 10x^3 - 2x$

12. $2D - 1; \quad y = 4e^{x/2}$

13. $(D - 2)(D + 5); \quad y = e^{2x} + 3e^{-5x}$

14. $D^2 + 64; \quad y = 2 \cos 8x - 5 \sin 8x$

In Problems 15–26 find a linear differential operator that annihilates the given function.

15. $1 + 6x - 2x^3$

16. $x^3(1 - 5x)$

17. $1 + 7e^{2x}$

18. $x + 3xe^{6x}$

19. $\cos 2x$

20. $1 + \sin x$

21. $13x + 9x^2 - \sin 4x$

22. $8x - \sin x + 10 \cos 5x$

23. $e^{-x} + 2xe^x - x^2e^x$

24. $(2 - e^x)^2$

25. $3 + e^x \cos 2x$

26. $e^{-x} \sin x - e^{2x} \cos x$

In Problems 27–34 find linearly independent functions that are annihilated by the given differential operator.

27. D^5

28. $D^2 + 4D$

29. $(D - 6)(2D + 3)$

30. $D^2 - 9D - 36$

31. $D^2 + 5$

32. $D^2 - 6D + 10$

33. $D^3 - 10D^2 + 25D$

34. $D^2(D - 5)(D - 7)$

In Problems 35–64 solve the given differential equation by undetermined coefficients.

35. $y'' - 9y = 54$

36. $2y'' - 7y' + 5y = -29$

37. $y'' + y' = 3$

38. $y''' + 2y'' + y' = 10$

39. $y'' + 4y' + 4y = 2x + 6$

40. $y'' + 3y' = 4x - 5$

41. $y''' + y'' = 8x^2$

42. $y'' - 2y' + y = x^3 + 4x$

43. $y'' - y' - 12y = e^{4x}$

44. $y'' + 2y' + 2y = 5e^{6x}$

45. $y'' - 2y' - 3y = 4e^x - 9$

46. $y'' + 6y' + 8y = 3e^{-2x} + 2x$

47. $y'' + 25y = 6 \sin x$

48. $y'' + 4y = 4 \cos x + 3 \sin x - 8$

49. $y'' + 6y' + 9y = -xe^{4x}$

50. $y'' + 3y' - 10y = x(e^x + 1)$

51. $y'' - y = x^2e^x + 5$

52. $y'' + 2y' + y = x^2e^{-x}$

53. $y'' - 2y' + 5y = e^x \sin x$

54. $y'' + y' + \frac{1}{4}y = e^x(\sin 3x - \cos 3x)$

55. $y'' + 25y = 20 \sin 5x$

56. $y'' + y = 4 \cos x - \sin x$

57. $y'' + y' + y = x \sin x$

58. $y'' + 4y = \cos^2 x$

59. $y''' + 8y'' = -6x^2 + 9x + 2$

60. $y''' - y'' + y' - y = xe^x - e^{-x} + 7$

61. $y''' - 3y'' + 3y' - y = e^x - x + 16$

62. $2y''' - 3y'' - 3y' + 2y = (e^x + e^{-x})^2$

63. $y^{(4)} - 2y''' + y'' = e^x + 1$

64. $y^{(4)} - 4y'' = 5x^2 - e^{2x}$

In Problems 65–72 solve the given initial-value problem.

65. $y'' - 64y = 16, \quad y(0) = 1, y'(0) = 0$

66. $y'' + y' = x, \quad y(0) = 1, y'(0) = 0$

67. $y'' - 5y' = x - 2, \quad y(0) = 0, y'(0) = 2$

68. $y'' + 5y' - 6y = 10e^{2x}, \quad y(0) = 1, y'(0) = 1$

69. $y'' + y = 8 \cos 2x - 4 \sin x, \quad y\left(\frac{\pi}{2}\right) = -1, y'\left(\frac{\pi}{2}\right) = 0$

70. $y''' - 2y'' + y' = xe^x + 5, \quad y(0) = 2, y'(0) = 2, y''(0) = -1$

71. $y'' - 4y' + 8y = x^3, \quad y(0) = 2, y'(0) = 4$

72. $y^{(4)} - y''' = x + e^x, \quad y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0$

Discussion Problems

73. Suppose L is a linear differential operator that factors but has variable coefficients. Do the factors of L commute? Defend your answer.

4.6 VARIATION OF PARAMETERS

REVIEW MATERIAL

- Variation of parameters was first introduced in Section 2.3 and used again in Section 4.2. A review of those sections is recommended.

INTRODUCTION The procedure that we used to find a particular solution y_p of a linear first-order differential equation on an interval is applicable to linear higher-order DEs as well. To adapt the method of **variation of parameters** to a linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{1}$$

we begin by putting the equation into the standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

by dividing through by the lead coefficient $a_2(x)$. Equation (2) is the second-order analogue of the standard form of a linear first-order equation: $dy/dx + P(x)y = f(x)$. In (2) we suppose that $P(x)$, $Q(x)$, and $f(x)$ are continuous on some common interval I. As we have already seen in Section 4.3, there is no difficulty in obtaining the complementary function y_c, the general solution of the associated homogeneous equation of (2), when the coefficients are constant.