$y = x^3 + 2/x^3$. Give an interval I of definition of each of these solutions. Graph the solution curves. Is there an initial-value problem whose solution is defined on $(-\infty, \infty)$?

(c) Is each IVP found in part (b) unique? That is, can there be more than one IVP for which, say, $y = x^3 - 1/x^3$, x in some interval I, is the solution?

44. In determining the integrating factor (5), we did not use a constant of integration in the evaluation of $\int P(x)\,dx$. Explain why using $\int P(x)\,dx + c$ has no effect on the solution of (2).

45. Suppose $P(x)$ is continuous on some interval I and a is a number in I. What can be said about the solution of the initial-value problem $y' + P(x)y = 0$, $y(a) = 0$?

Mathematical Models

46. Radioactive Decay Series The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements:

$$\frac{dx}{dt} = -\lambda_1 x$$

$$\frac{dy}{dt} = \lambda_1 x - \lambda_2 y,$$

where λ_1 and λ_2 are constants. Discuss how to solve this system subject to $x(0) = x_0$, $y(0) = y_0$. Carry out your ideas.

47. Heart Pacemaker A heart pacemaker consists of a switch, a battery of constant voltage E_0, a capacitor with constant capacitance C, and the heart as a resistor with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E.$$

Solve the DE subject to $E(4) = E_0$.

Computer Lab Assignments

48. (a) Express the solution of the initial-value problem $y' - 2xy = -1$, $y(0) = \sqrt{\pi}/2$, in terms of $\text{erfc}(x)$.

(b) Use tables or a CAS to find the value of $y(2)$. Use a CAS to graph the solution curve for the IVP on $(-\infty, \infty)$.

49. (a) The **sine integral function** is defined by $\text{Si}(x) = \int_0^x (\sin t/t)\,dt$, where the integrand is defined to be 1 at $t = 0$. Express the solution $y(x)$ of the initial-value problem $x^3y' + 2x^2y = 10\sin x$, $y(1) = 0$ in terms of $\text{Si}(x)$.

(b) Use a CAS to graph the solution curve for the IVP for $x > 0$.

(c) Use a CAS to find the value of the absolute maximum of the solution $y(x)$ for $x > 0$.

50. (a) The **Fresnel sine integral** is defined by $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$. Express the solution $y(x)$ of the initial-value problem $y' - (\sin x^2)y = 0$, $y(0) = 5$, in terms of $S(x)$.

(b) Use a CAS to graph the solution curve for the IVP on $(-\infty, \infty)$.

(c) It is known that $S(x) \to \frac{1}{2}$ as $x \to \infty$ and $S(x) \to -\frac{1}{2}$ as $x \to -\infty$. What does the solution $y(x)$ approach as $x \to \infty$? As $x \to -\infty$?

(d) Use a CAS to find the values of the absolute maximum and the absolute minimum of the solution $y(x)$.

2.4 EXACT EQUATIONS

REVIEW MATERIAL

- Multivariate calculus
- Partial differentiation and partial integration
- Differential of a function of two variables

INTRODUCTION Although the simple first-order equation

$$y\,dx + x\,dy = 0$$

is separable, we can solve the equation in an alternative manner by recognizing that the expression on the left-hand side of the equality is the differential of the function $f(x, y) = xy$; that is,

$$d(xy) = y\,dx + x\,dy.$$

In this section we examine first-order equations in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$. By applying a simple test to M and N, we can determine whether $M(x, y)\,dx + N(x, y)\,dy$ is a differential of a function $f(x, y)$. If the answer is yes, we can construct f by partial integration.

DIFFERENTIAL OF A FUNCTION OF TWO VARIABLES If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{1}$$

In the special case when $f(x, y) = c$, where c is a constant, then (1) implies

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0. \tag{2}$$

In other words, given a one-parameter family of functions $f(x, y) = c$, we can generate a first-order differential equation by computing the differential of both sides of the equality. For example, if $x^2 - 5xy + y^3 = c$, then (2) gives the first-order DE

$$(2x - 5y)\,dx + (-5x + 3y^2)\,dy = 0. \tag{3}$$

A DEFINITION Of course, not every first-order DE written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$ corresponds to a differential of $f(x, y) = c$. So for our purposes it is more important to turn the foregoing example around; namely, if we are given a first-order DE such as (3), is there some way we can recognize that the differential expression $(2x - 5y)\,dx + (-5x + 3y^2)\,dy$ is the differential $d(x^2 - 5xy + y^3)$? If there is, then an implicit solution of (3) is $x^2 - 5xy + y^3 = c$. We answer this question after the next definition.

DEFINITION 2.4.1 **Exact Equation**

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

For example, $x^2y^3\,dx + x^3y^2\,dy = 0$ is an exact equation, because its left-hand side is an exact differential:

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2\,dy.$$

Notice that if we make the identifications $M(x, y) = x^2y^3$ and $N(x, y) = x^3y^2$, then $\partial M/\partial y = 3x^2y^2 = \partial N/\partial x$. Theorem 2.4.1, given next, shows that the equality of the partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ is no coincidence.

THEOREM 2.4.1 **Criterion for an Exact Differential**

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

PROOF OF THE NECESSITY For simplicity let us assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives for all (x, y). Now if the expression $M(x, y)\,dx + N(x, y)\,dy$ is exact, there exists some function f such that for all x in R,

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$

Therefore $$M(x, y) = \frac{\partial f}{\partial x}, \qquad N(x, y) = \frac{\partial f}{\partial y},$$

and $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

The equality of the mixed partials is a consequence of the continuity of the first partial derivatives of $M(x, y)$ and $N(x, y)$. ■

The sufficiency part of Theorem 2.4.1 consists of showing that there exists a function f for which $\partial f/\partial x = M(x, y)$ and $\partial f/\partial y = N(x, y)$ whenever (4) holds. The construction of the function f actually reflects a basic procedure for solving exact equations.

METHOD OF SOLUTION Given an equation in the differential form $M(x, y)\,dx + N(x, y)\,dy = 0$, determine whether the equality in (4) holds. If it does, then there exists a function f for which

$$\frac{\partial f}{\partial x} = M(x, y).$$

We can find f by integrating $M(x, y)$ with respect to x while holding y constant:

$$f(x, y) = \int M(x, y)\,dx + g(y), \tag{5}$$

where the arbitrary function $g(y)$ is the “constant” of integration. Now differentiate (5) with respect to y and assume that $\partial f/\partial y = N(x, y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\,dx + g'(y) = N(x, y).$$

This gives $$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx. \tag{6}$$

Finally, integrate (6) with respect to y and substitute the result in (5). The implicit solution of the equation is $f(x, y) = c$.

Some observations are in order. First, it is important to realize that the expression $N(x, y) - (\partial/\partial y)\int M(x, y)\,dx$ in (6) is independent of x, because

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x, y)\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Second, we could just as well start the foregoing procedure with the assumption that $\partial f/\partial y = N(x, y)$. After integrating N with respect to y and then differentiating that result, we would find the analogues of (5) and (6) to be, respectively,

$$f(x, y) = \int N(x, y)\,dy + h(x) \qquad \text{and} \qquad h'(x) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)\,dy.$$

In either case *none of these formulas should be memorized.*

EXAMPLE 1 Solving an Exact DE

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so by Theorem 2.4.1 there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2y - y$, so the solution of the equation in implicit form is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$. ■

NOTE The solution of the DE in Example 1 is *not* $f(x, y) = x^2y - y$. Rather, it is $f(x, y) = c$; if a constant is used in the integration of $g'(y)$, we can then write the solution as $f(x, y) = 0$. Note, too, that the equation could be solved by separation of variables.

EXAMPLE 2 Solving an Exact DE

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

SOLUTION The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x, y)$ exists for which

$$M(x, y) = \frac{\partial f}{\partial x} \qquad \text{and} \qquad N(x, y) = \frac{\partial f}{\partial y}.$$

Now for variety we shall start with the assumption that $\partial f/\partial y = N(x, y)$; that is,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x, y) = 2x\int e^{2y}\,dy - x\int \cos xy\,dy + 2\int y\,dy.$$

Remember, the reason x can come out in front of the symbol $\int$ is that in the integration with respect to y, x is treated as an ordinary constant. It follows that

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy, \quad \leftarrow M(x, y)$$

and so $h'(x) = 0$ or $h(x) = c$. Hence a family of solutions is

$$xe^{2y} - \sin xy + y^2 + c = 0.$$

EXAMPLE 3 An Initial-Value Problem

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$

SOLUTION By writing the differential equation in the form

$$(\cos x \sin x - xy^2)\,dx + y(1 - x^2)\,dy = 0,$$

we recognize that the equation is exact because

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}.$$

Now

$$\frac{\partial f}{\partial y} = y(1 - x^2)$$

$$f(x, y) = \frac{y^2}{2}(1 - x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2.$$

The last equation implies that $h'(x) = \cos x \sin x$. Integrating gives

$$h(x) = -\int (\cos x)(-\sin x\,dx) = -\frac{1}{2}\cos^2 x.$$

Thus

$$\frac{y^2}{2}(1 - x^2) - \frac{1}{2}\cos^2 x = c_1 \quad \text{or} \quad y^2(1 - x^2) - \cos^2 x = c, \tag{7}$$

where $2c_1$ has been replaced by c. The initial condition $y = 2$ when $x = 0$ demands that $4(1) - \cos^2(0) = c$, and so $c = 3$. An implicit solution of the problem is then $y^2(1 - x^2) - \cos^2 x = 3$.

The solution curve of the IVP is the curve drawn in dark blue in Figure 2.4.1; it is part of an interesting family of curves. The graphs of the members of the one-parameter family of solutions given in (7) can be obtained in several ways, two of which are using software to graph level curves (as discussed in Section 2.2) and using a graphing utility to carefully graph the explicit functions obtained for various values of c by solving $y^2 = (c + \cos^2 x)/(1 - x^2)$ for y.

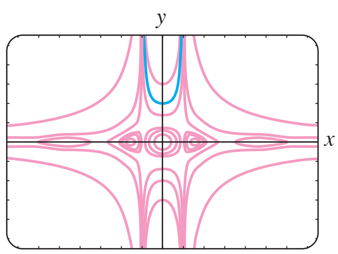

FIGURE 2.4.1 Some graphs of members of the family $y^2(1 - x^2) - \cos^2 x = c$

INTEGRATING FACTORS Recall from Section 2.3 that the left-hand side of the linear equation $y' + P(x)y = f(x)$ can be transformed into a derivative when we multiply the equation by an integrating factor. The same basic idea sometimes works for a nonexact differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$. That is, it is sometimes possible to find an **integrating factor** $\mu(x, y)$ so that after multiplying, the left-hand side of

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0 \tag{8}$$

is an exact differential. In an attempt to find μ, we turn to the criterion (4) for exactness. Equation (8) is exact if and only if $(\mu M)_y = (\mu N)_x$, where the subscripts denote partial derivatives. By the Product Rule of differentiation the last equation is the same as $\mu M_y + \mu_y M = \mu N_x + \mu_x N$ or

$$\mu_x N - \mu_y M = (M_y - N_x)\mu. \tag{9}$$

Although M, N, M_y, and N_x are known functions of x and y, the difficulty here in determining the unknown $\mu(x, y)$ from (9) is that we must solve a partial differential equation. Since we are not prepared to do that, we make a simplifying assumption. Suppose μ is a function of one variable; for example, say that μ depends only on x. In this case, $\mu_x = d\mu/dx$ and $\mu_y = 0$, so (9) can be written as

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{10}$$

We are still at an impasse if the quotient $(M_y - N_x)/N$ depends on both x and y. However, if after all obvious algebraic simplifications are made, the quotient $(M_y - N_x)/N$ turns out to depend solely on the variable x, then (10) is a first-order ordinary differential equation. We can finally determine μ because (10) is *separable* as well as *linear.* It follows from either Section 2.2 or Section 2.3 that $\mu(x) = e^{\int((M_y - N_x)/N)\,dx}$. In like manner, it follows from (9) that if μ depends only on the variable y, then

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu. \tag{11}$$

In this case, if $(N_x - M_y)/M$ is a function of y only, then we can solve (11) for μ.

We summarize the results for the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{12}$$

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for (12) is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}. \tag{13}$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for (12) is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}dy}. \tag{14}$$

EXAMPLE 4 A Nonexact DE Made Exact

The nonlinear first-order differential equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$, we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere, since

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

depends on x and y. However, (14) yields a quotient that depends only on y:

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}.$$

The integrating factor is then $e^{\int 3dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$. After we multiply the given DE by $\mu(y) = y^3$, the resulting equation is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0.$$

You should verify that the last equation is now exact as well as show, using the method of this section, that a family of solutions is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$. ■

REMARKS

(*i*) When testing an equation for exactness, make sure it is of the precise form $M(x, y)\,dx + N(x, y)\,dy = 0$. Sometimes a differential equation is written $G(x, y)\,dx = H(x, y)\,dy$. In this case, first rewrite it as $G(x, y)\,dx - H(x, y)\,dy = 0$ and then identify $M(x, y) = G(x, y)$ and $N(x, y) = -H(x, y)$ before using (4).

(*ii*) In some texts on differential equations the study of exact equations precedes that of linear DEs. Then the method for finding integrating factors just discussed can be used to derive an integrating factor for $y' + P(x)y = f(x)$. By rewriting the last equation in the differential form $(P(x)y - f(x))\,dx + dy = 0$, we see that

$$\frac{M_y - N_x}{N} = P(x).$$

From (13) we arrive at the already familiar integrating factor $e^{\int P(x)dx}$, used in Section 2.3.

EXERCISES 2.4

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$
2. $(2x + y)\,dx - (x + 6y)\,dy = 0$
3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$
4. $(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$
5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$
6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$
7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$
8. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)\,dy$
9. $(x - y^3 + y^2\sin x)\,dx = (3xy^2 + 2y\cos x)\,dy$
10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$
11. $(y\ln y - e^{-xy})\,dx + \left(\dfrac{1}{y} + x\ln y\right)dy = 0$
12. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$
13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$
14. $\left(1 - \dfrac{3}{y} + x\right)\dfrac{dy}{dx} + y = \dfrac{3}{x} - 1$
15. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$
16. $(5y - 2x)y' - 2y = 0$
17. $(\tan x - \sin x\sin y)\,dx + \cos x\cos y\,dy = 0$
18. $(2y\sin x\cos x - y + 2y^2e^{xy^2})\,dx = (x - \sin^2 x - 4xye^{xy^2})\,dy$
19. $(4t^3y - 15t^2 - y)\,dt + (t^4 + 3y^2 - t)\,dy = 0$
20. $\left(\dfrac{1}{t} + \dfrac{1}{t^2} - \dfrac{y}{t^2 + y^2}\right)dt + \left(ye^y + \dfrac{t}{t^2 + y^2}\right)dy = 0$

In Problems 21–26 solve the given initial-value problem.

21. $(x + y)^2\, dx + (2xy + x^2 - 1)\, dy = 0, \quad y(1) = 1$

22. $(e^x + y)\, dx + (2 + x + ye^y)\, dy = 0, \quad y(0) = 1$

23. $(4y + 2t - 5)\, dt + (6y + 4t - 1)\, dy = 0, \quad y(-1) = 2$

24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$

25. $(y^2 \cos x - 3x^2 y - 2x)\, dx$
$+ (2y \sin x - x^3 + \ln y)\, dy = 0, \quad y(0) = e$

26. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \; y(0) = 1$

In Problems 27 and 28 find the value of k so that the given differential equation is exact.

27. $(y^3 + kxy^4 - 2x)\, dx + (3xy^2 + 20x^2y^3)\, dy = 0$

28. $(6xy^3 + \cos y)\, dx + (2kx^2y^2 - x \sin y)\, dy = 0$

In Problems 29 and 30 verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor $\mu(x, y)$ and verify that the new equation is exact. Solve.

29. $(-xy \sin x + 2y \cos x)\, dx + 2x \cos x\, dy = 0;$
$\mu(x, y) = xy$

30. $(x^2 + 2xy - y^2)\, dx + (y^2 + 2xy - x^2)\, dy = 0;$
$\mu(x, y) = (x + y)^{-2}$

In Problems 31–36 solve the given differential equation by finding, as in Example 4, an appropriate integrating factor.

31. $(2y^2 + 3x)\, dx + 2xy\, dy = 0$

32. $y(x + y + 1)\, dx + (x + 2y)\, dy = 0$

33. $6xy\, dx + (4y + 9x^2)\, dy = 0$

34. $\cos x\, dx + \left(1 + \dfrac{2}{y}\right)\sin x\, dy = 0$

35. $(10 - 6y + e^{-3x})\, dx - 2\, dy = 0$

36. $(y^2 + xy^3)\, dx + (5y^2 - xy + y^3 \sin y)\, dy = 0$

In Problems 37 and 38 solve the given initial-value problem by finding, as in Example 4, an appropriate integrating factor.

37. $x\, dx + (x^2y + 4y)\, dy = 0, \quad y(4) = 0$

38. $(x^2 + y^2 - 5)\, dx = (y + xy)\, dy, \quad y(0) = 1$

39. **(a)** Show that a one-parameter family of solutions of the equation

$$(4xy + 3x^2)\, dx + (2y + 2x^2)\, dy = 0$$

is $x^3 + 2x^2y + y^2 = c$.

(b) Show that the initial conditions $y(0) = -2$ and $y(1) = 1$ determine the same implicit solution.

(c) Find explicit solutions $y_1(x)$ and $y_2(x)$ of the differential equation in part (a) such that $y_1(0) = -2$ and $y_2(1) = 1$. Use a graphing utility to graph $y_1(x)$ and $y_2(x)$.

Discussion Problems

40. Consider the concept of an integrating factor used in Problems 29–38. Are the two equations $M\, dx + N\, dy = 0$ and $\mu M\, dx + \mu N\, dy = 0$ necessarily equivalent in the sense that a solution of one is also a solution of the other? Discuss.

41. Reread Example 3 and then discuss why we can conclude that the interval of definition of the explicit solution of the IVP (the blue curve in Figure 2.4.1) is $(-1, 1)$.

42. Discuss how the functions $M(x, y)$ and $N(x, y)$ can be found so that each differential equation is exact. Carry out your ideas.

(a) $M(x, y)\, dx + \left(xe^{xy} + 2xy + \dfrac{1}{x}\right) dy = 0$

(b) $\left(x^{-1/2}y^{1/2} + \dfrac{x}{x^2 + y}\right) dx + N(x, y)\, dy = 0$

43. Differential equations are sometimes solved by having a clever idea. Here is a little exercise in cleverness: Although the differential equation $(x - \sqrt{x^2 + y^2})\, dx + y\, dy = 0$ is not exact, show how the rearrangement $(x\, dx + y\, dy)/\sqrt{x^2 + y^2} = dx$ and the observation $\frac{1}{2}d(x^2 + y^2) = x\, dx + y\, dy$ can lead to a solution.

44. True or False: Every separable first-order equation $dy/dx = g(x)h(y)$ is exact.

Mathematical Model

45. Falling Chain A portion of a uniform chain of length 8 ft is loosely coiled around a peg at the edge of a high horizontal platform, and the remaining portion of the chain hangs at rest over the edge of the platform. See Figure 2.4.2. Suppose that the length of the overhanging chain is 3 ft, that the chain weighs 2 lb/ft, and that the positive direction is downward. Starting at $t = 0$ seconds, the weight of the overhanging portion causes the chain on the table to uncoil smoothly and to fall to the floor. If $x(t)$ denotes the length of the chain overhanging the table at time $t > 0$, then $v = dx/dt$ is its velocity. When all resistive forces are ignored, it can be shown that a mathematical model relating v to x is given by

$$xv\frac{dv}{dx} + v^2 = 32x.$$

(a) Rewrite this model in differential form. Proceed as in Problems 31–36 and solve the DE for v in terms of x by finding an appropriate integrating factor. Find an explicit solution $v(x)$.

(b) Determine the velocity with which the chain leaves the platform.

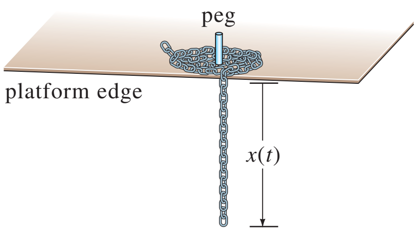

FIGURE 2.4.2 Uncoiling chain in Problem 45

Computer Lab Assignments

46. Streamlines

(a) The solution of the differential equation

$$\frac{2xy}{(x^2+y^2)^2}\,dx + \left[1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right]dy = 0$$

is a family of curves that can be interpreted as streamlines of a fluid flow around a circular object whose boundary is described by the equation $x^2 + y^2 = 1$. Solve this DE and note the solution $f(x, y) = c$ for $c = 0$.

(b) Use a CAS to plot the streamlines for $c = 0, \pm 0.2, \pm 0.4, \pm 0.6$, and ± 0.8 in three different ways. First, use the *contourplot* of a CAS. Second, solve for x in terms of the variable y. Plot the resulting two functions of y for the given values of c, and then combine the graphs. Third, use the CAS to solve a cubic equation for y in terms of x.

2.5 SOLUTIONS BY SUBSTITUTIONS

REVIEW MATERIAL

- Techniques of integration
- Separation of variables
- Solution of linear DEs

INTRODUCTION We usually solve a differential equation by recognizing it as a certain kind of equation (say, separable, linear, or exact) and then carrying out a procedure, consisting of *equation-specific mathematical steps,* that yields a solution of the equation. But it is not uncommon to be stumped by a differential equation because it does not fall into one of the classes of equations that we know how to solve. The procedures that are discussed in this section may be helpful in this situation.

SUBSTITUTIONS Often the first step in solving a differential equation consists of transforming it into another differential equation by means of a **substitution.** For example, suppose we wish to transform the first-order differential equation $dy/dx = f(x, y)$ by the substitution $y = g(x, u)$, where u is regarded as a function of the variable x. If g possesses first-partial derivatives, then the Chain Rule

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx} \quad \text{gives} \quad \frac{dy}{dx} = g_x(x, u) + g_u(x, u)\frac{du}{dx}.$$

If we replace dy/dx by the foregoing derivative and replace y in $f(x, y)$ by $g(x, u)$, then the DE $dy/dx = f(x, y)$ becomes $g_x(x, u) + g_u(x, u)\dfrac{du}{dx} = f(x, g(x, u))$, which, solved for du/dx, has the form $\dfrac{du}{dx} = F(x, u)$. If we can determine a solution $u = \phi(x)$ of this last equation, then a solution of the original differential equation is $y = g(x, \phi(x))$.

In the discussion that follows we examine three different kinds of first-order differential equations that are solvable by means of a substitution.