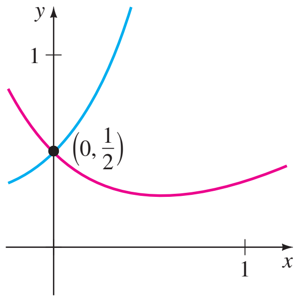


FIGURE 1.2.11 Graphs for Problem 41

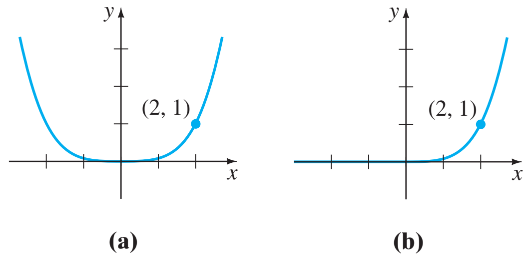


FIGURE 1.2.12 Two solutions of the IVP in Problem 44

42. Determine a plausible value of x_0 for which the graph of the solution of the initial-value problem $y' + 2y = 3x - 6$, $y(x_0) = 0$ is tangent to the x-axis at $(x_0, 0)$. Explain your reasoning.

43. Suppose that the first-order differential equation $dy/dx = f(x, y)$ possesses a one-parameter family of solutions and that $f(x, y)$ satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R of the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R.

44. The functions $y(x) = \frac{1}{16}x^4$, $\quad -\infty < x < \infty$ and

$$y(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

have the same domain but are clearly different. See Figures 1.2.12(a) and 1.2.12(b), respectively. Show that both functions are solutions of the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ on the interval $(-\infty, \infty)$. Resolve the apparent contradiction between this fact and the last sentence in Example 5.

Mathematical Model

45. Population Growth Beginning in the next section we will see that differential equations can be used to describe or *model* many different physical systems. In this problem suppose that a model of the growing population of a small community is given by the initial-value problem

$$\frac{dP}{dt} = 0.15P(t) + 20, \quad P(0) = 100,$$

where P is the number of individuals in the community and time t is measured in years. How fast—that is, at what *rate*—is the population increasing at $t = 0$? How fast is the population increasing when the population is 500?

1.3 DIFFERENTIAL EQUATIONS AS MATHEMATICAL MODELS

REVIEW MATERIAL

- Units of measurement for weight, mass, and density
- Newton's second law of motion
- Hooke's law
- Kirchhoff's laws
- Archimedes' principle

INTRODUCTION In this section we introduce the notion of a differential equation as a mathematical model and discuss some specific models in biology, chemistry, and physics. Once we have studied some methods for solving DEs in Chapters 2 and 4, we return to, and solve, some of these models in Chapters 3 and 5.

MATHEMATICAL MODELS It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system of phenomenon is called a **mathematical model** and is constructed with certain goals in mind. For example, we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system, or we may wish to date fossils by analyzing the decay of a radioactive substance either in the fossil or in the stratum in which it was discovered.

Construction of a mathematical model of a system starts with

(*i*) identification of the variables that are responsible for changing the system. We may choose not to incorporate all these variables into the model at first. In this step we are specifying the **level of resolution** of the model.

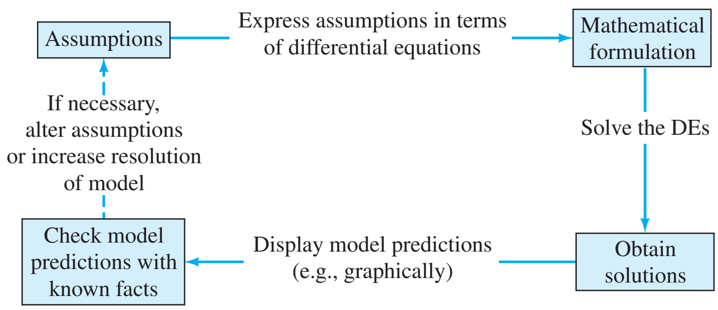
Next

(*ii*) we make a set of reasonable assumptions, or hypotheses, about the system we are trying to describe. These assumptions will also include any empirical laws that may be applicable to the system.

For some purposes it may be perfectly within reason to be content with low-resolution models. For example, you may already be aware that in beginning physics courses, the retarding force of air friction is sometimes ignored in modeling the motion of a body falling near the surface of the Earth, but if you are a scientist whose job it is to accurately predict the flight path of a long-range projectile, you have to take into account air resistance and other factors such as the curvature of the Earth.

Since the assumptions made about a system frequently involve *a rate of change* of one or more of the variables, the mathematical depiction of all these assumptions may be one or more equations involving *derivatives.* In other words, the mathematical model may be a differential equation or a system of differential equations.

Once we have formulated a mathematical model that is either a differential equation or a system of differential equations, we are faced with the not insignificant problem of trying to solve it. *If* we can solve it, then we deem the model to be reasonable if its solution is consistent with either experimental data or known facts about the behavior of the system. But if the predictions produced by the solution are poor, we can either increase the level of resolution of the model or make alternative assumptions about the mechanisms for change in the system. The steps of the modeling process are then repeated, as shown in the following diagram:



Of course, by increasing the resolution, we add to the complexity of the mathematical model and increase the likelihood that we cannot obtain an explicit solution.

A mathematical model of a physical system will often involve the variable time t. A solution of the model then gives the **state of the system;** in other words, the values of the dependent variable (or variables) for appropriate values of t describe the system in the past, present, and future.

POPULATION DYNAMICS One of the earliest attempts to model human **population growth** by means of mathematics was by the English economist Thomas Malthus in 1798. Basically, the idea behind the Malthusian model is the assumption that the rate at which the population of a country grows at a certain time is proportional* to the total population of the country at that time. In other words, the more people there are at time t, the more there are going to be in the future. In mathematical terms, if $P(t)$ denotes the

*If two quantities u and v are proportional, we write $u \propto v$. This means that one quantity is a constant multiple of the other: $u = kv$.

total population at time t, then this assumption can be expressed as

$$\frac{dP}{dt} \propto P \qquad \text{or} \qquad \frac{dP}{dt} = kP, \tag{1}$$

where k is a constant of proportionality. This simple model, which fails to take into account many factors that can influence human populations to either grow or decline (immigration and emigration, for example), nevertheless turned out to be fairly accurate in predicting the population of the United States during the years 1790–1860. Populations that grow at a rate described by (1) are rare; nevertheless, (1) is still used to model *growth of small populations over short intervals of time* (bacteria growing in a petri dish, for example).

RADIOACTIVE DECAY The nucleus of an atom consists of combinations of protons and neutrons. Many of these combinations of protons and neutrons are unstable—that is, the atoms decay or transmute into atoms of another substance. Such nuclei are said to be radioactive. For example, over time the highly radioactive radium, Ra-226, transmutes into the radioactive gas radon, Rn-222. To model the phenomenon of **radioactive decay,** it is assumed that the rate dA/dt at which the nuclei of a substance decay is proportional to the amount (more precisely, the number of nuclei) $A(t)$ of the substance remaining at time t:

$$\frac{dA}{dt} \propto A \qquad \text{or} \qquad \frac{dA}{dt} = kA. \tag{2}$$

Of course, equations (1) and (2) are exactly the same; the difference is only in the interpretation of the symbols and the constants of proportionality. For growth, as we expect in (1), $k > 0$, and for decay, as in (2), $k < 0$.

The model (1) for growth can also be seen as the equation $dS/dt = rS$, which describes the growth of capital S when an annual rate of interest r is compounded continuously. The model (2) for decay also occurs in biological applications such as determining the half-life of a drug—the time that it takes for 50% of a drug to be eliminated from a body by excretion or metabolism. In chemistry the decay model (2) appears in the mathematical description of a first-order chemical reaction. The point is this:

> *A single differential equation can serve as a mathematical model for many different phenomena.*

Mathematical models are often accompanied by certain side conditions. For example, in (1) and (2) we would expect to know, in turn, the initial population P_0 and the initial amount of radioactive substance A_0 on hand. If the initial point in time is taken to be $t = 0$, then we know that $P(0) = P_0$ and $A(0) = A_0$. In other words, a mathematical model can consist of either an initial-value problem or, as we shall see later on in Section 5.2, a boundary-value problem.

NEWTON'S LAW OF COOLING/WARMING According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and dT/dt the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m \qquad \text{or} \qquad \frac{dT}{dt} = k(T - T_m), \tag{3}$$

where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it stands to reason that $k < 0$.

SPREAD OF A DISEASE A contagious disease—for example, a flu virus—is spread throughout a community by people coming into contact with other people. Let $x(t)$ denote the number of people who have contracted the disease and $y(t)$ denote the number of people who have not yet been exposed. It seems reasonable to assume that the rate dx/dt at which the disease spreads is proportional to the number of encounters, or *interactions,* between these two groups of people. If we assume that the number of interactions is jointly proportional to $x(t)$ and $y(t)$—that is, proportional to the product xy—then

$$\frac{dx}{dt} = kxy, \tag{4}$$

where k is the usual constant of proportionality. Suppose a small community has a fixed population of n people. If one infected person is introduced into this community, then it could be argued that $x(t)$ and $y(t)$ are related by $x + y = n + 1$. Using this last equation to eliminate y in (4) gives us the model

$$\frac{dx}{dt} = kx(n + 1 - x). \tag{5}$$

An obvious initial condition accompanying equation (5) is $x(0) = 1$.

CHEMICAL REACTIONS The disintegration of a radioactive substance, governed by the differential equation (1), is said to be a **first-order reaction.** In chemistry a few reactions follow this same empirical law: If the molecules of substance A decompose into smaller molecules, it is a natural assumption that the rate at which this decomposition takes place is proportional to the amount of the first substance that has not undergone conversion; that is, if $X(t)$ is the amount of substance A remaining at any time, then $dX/dt = kX$, where k is a negative constant since X is decreasing. An example of a first-order chemical reaction is the conversion of *t*-butyl chloride, $(CH_3)_3CCl$, into *t*-butyl alcohol, $(CH_3)_3COH$:

$$(CH_3)_3CCl + NaOH \rightarrow (CH_3)_3COH + NaCl.$$

Only the concentration of the *t*-butyl chloride controls the rate of reaction. But in the reaction

$$CH_3Cl + NaOH \rightarrow CH_3OH + NaCl$$

one molecule of sodium hydroxide, NaOH, is consumed for every molecule of methyl chloride, CH_3Cl, thus forming one molecule of methyl alcohol, CH_3OH, and one molecule of sodium chloride, NaCl. In this case the rate at which the reaction proceeds is proportional to the product of the remaining concentrations of CH_3Cl and NaOH. To describe this second reaction in general, let us suppose *one* molecule of a substance A combines with *one* molecule of a substance B to form *one* molecule of a substance C. If X denotes the amount of chemical C formed at time t and if α and β are, in turn, the amounts of the two chemicals A and B at $t = 0$ (the initial amounts), then the instantaneous amounts of A and B not converted to chemical C are $\alpha - X$ and $\beta - X$, respectively. Hence the rate of formation of C is given by

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \tag{6}$$

where k is a constant of proportionality. A reaction whose model is equation (6) is said to be a **second-order reaction.**

MIXTURES The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture. Let us suppose that a large mixing tank initially holds 300 gallons of brine (that is, water in which a certain number of pounds of salt has been dissolved). Another

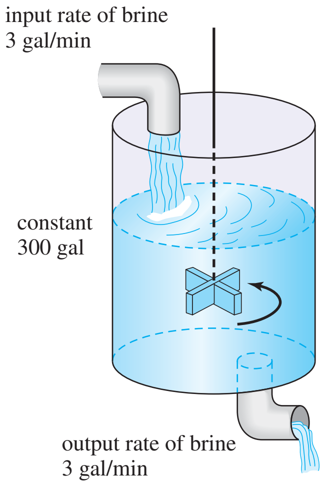


FIGURE 1.3.1 Mixing tank

brine solution is pumped into the large tank at a rate of 3 gallons per minute; the concentration of the salt in this inflow is 2 pounds per gallon. When the solution in the tank is well stirred, it is pumped out at the same rate as the entering solution. See Figure 1.3.1. If $A(t)$ denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} = \left(\begin{matrix}\textit{input rate}\\ \textit{of salt}\end{matrix}\right) - \left(\begin{matrix}\textit{output rate}\\ \textit{of salt}\end{matrix}\right) = R_{in} - R_{out}. \tag{7}$$

The input rate R_{in} at which salt enters the tank is the product of the inflow concentration of salt and the inflow rate of fluid. Note that R_{in} is measured in pounds per minute:

$$R_{in} = \underset{\substack{\uparrow\\ \text{concentration of salt in inflow}}}{(2\ \text{lb/gal})} \cdot \underset{\substack{\uparrow\\ \text{input rate of brine}}}{(3\ \text{gal/min})} = \underset{\substack{\uparrow\\ \text{input rate of salt}}}{(6\ \text{lb/min})}.$$

Now, since the solution is being pumped out of the tank at the same rate that it is pumped in, the number of gallons of brine in the tank at time t is a constant 300 gallons. Hence the concentration of the salt in the tank as well as in the outflow is $c(t) = A(t)/300$ lb/gal, so the output rate R_{out} of salt is

$$R_{out} = \underset{\substack{\uparrow\\ \text{concentration of salt in outflow}}}{\left(\frac{A(t)}{300}\ \text{lb/gal}\right)} \cdot \underset{\substack{\uparrow\\ \text{output rate of brine}}}{(3\ \text{gal/min})} = \underset{\substack{\uparrow\\ \text{output rate of salt}}}{\frac{A(t)}{100}\ \text{lb/min}}.$$

The net rate (7) then becomes

$$\frac{dA}{dt} = 6 - \frac{A}{100} \quad \text{or} \quad \frac{dA}{dt} + \frac{1}{100}A = 6. \tag{8}$$

If r_{in} and r_{out} denote general input and output rates of the brine solutions,* then there are three possibilities: $r_{in} = r_{out}$, $r_{in} > r_{out}$, and $r_{in} < r_{out}$. In the analysis leading to (8) we have assumed that $r_{in} = r_{out}$. In the latter two cases the number of gallons of brine in the tank is either increasing $(r_{in} > r_{out})$ or decreasing $(r_{in} < r_{out})$ at the net rate $r_{in} - r_{out}$. See Problems 10–12 in Exercises 1.3.

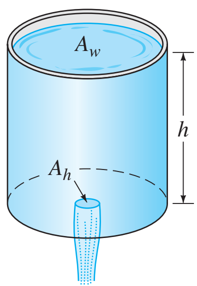


FIGURE 1.3.2 Draining tank

DRAINING A TANK In hydrodynamics **Torricelli's law** states that the speed v of efflux of water though a sharp-edged hole at the bottom of a tank filled to a depth h is the same as the speed that a body (in this case a drop of water) would acquire in falling freely from a height h—that is, $v = \sqrt{2gh}$, where g is the acceleration due to gravity. This last expression comes from equating the kinetic energy $\frac{1}{2}mv^2$ with the potential energy mgh and solving for v. Suppose a tank filled with water is allowed to drain through a hole under the influence of gravity. We would like to find the depth h of water remaining in the tank at time t. Consider the tank shown in Figure 1.3.2. If the area of the hole is A_h (in ft^2) and the speed of the water leaving the tank is $v = \sqrt{2gh}$ (in ft/s), then the volume of water leaving the tank per second is $A_h\sqrt{2gh}$ (in ft^3/s). Thus if $V(t)$ denotes the volume of water in the tank at time t, then

$$\frac{dV}{dt} = -A_h\sqrt{2gh}, \tag{9}$$

*Don't confuse these symbols with R_{in} and R_{out}, which are input and output rates of *salt*.

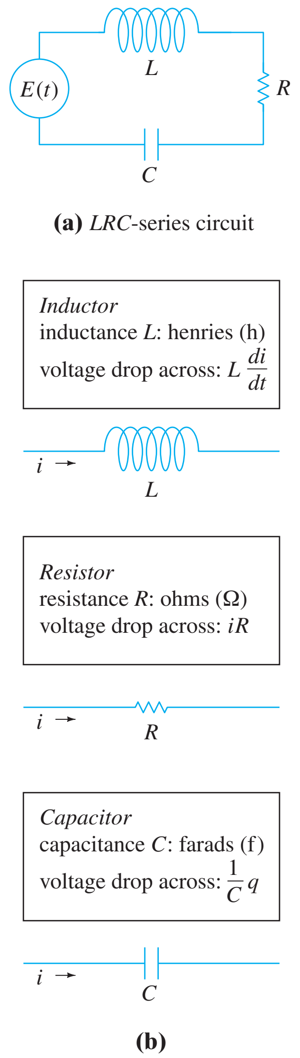


FIGURE 1.3.3 Symbols, units, and voltages. Current $i(t)$ and charge $q(t)$ are measured in amperes (A) and coulombs (C), respectively

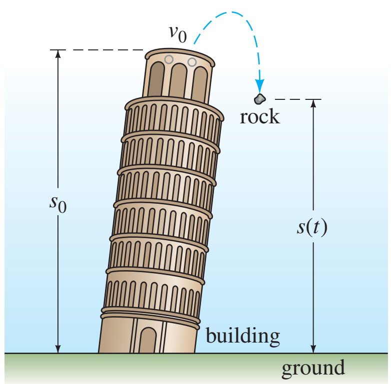


FIGURE 1.3.4 Position of rock measured from ground level

where the minus sign indicates that V is decreasing. Note here that we are ignoring the possibility of friction at the hole that might cause a reduction of the rate of flow there. Now if the tank is such that the volume of water in it at time t can be written $V(t) = A_w h$, where A_w (in ft^2) is the *constant* area of the upper surface of the water (see Figure 1.3.2), then $dV/dt = A_w\, dh/dt$. Substituting this last expression into (9) gives us the desired differential equation for the height of the water at time t:

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}. \tag{10}$$

It is interesting to note that (10) remains valid even when A_w is not constant. In this case we must express the upper surface area of the water as a function of h—that is, $A_w = A(h)$. See Problem 14 in Exercises 1.3.

SERIES CIRCUITS Consider the single-loop series circuit shown in Figure 1.3.3(a), containing an inductor, resistor, and capacitor. The current in a circuit after a switch is closed is denoted by $i(t)$; the charge on a capacitor at time t is denoted by $q(t)$. The letters L, R, and C are known as inductance, resistance, and capacitance, respectively, and are generally constants. Now according to **Kirchhoff's second law,** the impressed voltage $E(t)$ on a closed loop must equal the sum of the voltage drops in the loop. Figure 1.3.3(b) shows the symbols and the formulas for the respective voltage drops across an inductor, a capacitor, and a resistor. Since current $i(t)$ is related to charge $q(t)$ on the capacitor by $i = dq/dt$, adding the three voltages

$$\overset{\text{inductor}}{L\frac{di}{dt} = L\frac{d^2q}{dt^2}}, \qquad \overset{\text{resistor}}{iR = R\frac{dq}{dt}}, \qquad \text{and} \qquad \overset{\text{capacitor}}{\frac{1}{C}q}$$

and equating the sum to the impressed voltage yields a second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{11}$$

We will examine a differential equation analogous to (11) in great detail in Section 5.1.

FALLING BODIES To construct a mathematical model of the motion of a body moving in a force field, one often starts with Newton's second law of motion. Recall from elementary physics that **Newton's first law of motion** states that a body either will remain at rest or will continue to move with a constant velocity unless acted on by an external force. In each case this is equivalent to saying that when the sum of the forces $F = \sum F_k$—that is, the *net* or resultant force—acting on the body is zero, then the acceleration a of the body is zero. **Newton's second law of motion** indicates that when the net force acting on a body is not zero, then the net force is proportional to its acceleration a or, more precisely, $F = ma$, where m is the mass of the body.

Now suppose a rock is tossed upward from the roof of a building as illustrated in Figure 1.3.4. What is the position $s(t)$ of the rock relative to the ground at time t? The acceleration of the rock is the second derivative d^2s/dt^2. If we assume that the upward direction is positive and that no force acts on the rock other than the force of gravity, then Newton's second law gives

$$m\frac{d^2s}{dt^2} = -mg \qquad \text{or} \qquad \frac{d^2s}{dt^2} = -g. \tag{12}$$

In other words, the net force is simply the weight $F = F_1 = -W$ of the rock near the surface of the Earth. Recall that the magnitude of the weight is $W = mg$, where m is the mass of the body and g is the acceleration due to gravity. The minus sign in (12) is used because the weight of the rock is a force directed downward, which is opposite to the positive direction. If the height of the building is s_0 and the initial velocity of the rock is v_0, then s is determined from the second-order initial-value problem

$$\frac{d^2s}{dt^2} = -g, \quad s(0) = s_0, \quad s'(0) = v_0. \tag{13}$$

Although we have not been stressing solutions of the equations we have constructed, note that (13) can be solved by integrating the constant $-g$ twice with respect to t. The initial conditions determine the two constants of integration. From elementary physics you might recognize the solution of (13) as the formula $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$.

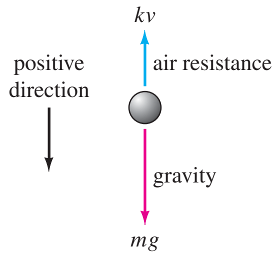


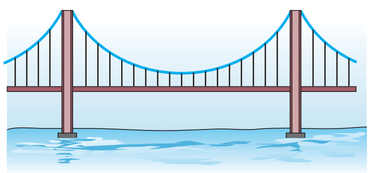
FIGURE 1.3.5 Falling body of mass m

FALLING BODIES AND AIR RESISTANCE Before Galileo's famous experiment from the leaning tower of Pisa, it was generally believed that heavier objects in free fall, such as a cannonball, fell with a greater acceleration than lighter objects, such as a feather. Obviously, a cannonball and a feather when dropped simultaneously from the same height *do* fall at different rates, but it is not because a cannonball is heavier. The difference in rates is due to air resistance. The resistive force of air was ignored in the model given in (13). Under some circumstances a falling body of mass m, such as a feather with low density and irregular shape, encounters air resistance proportional to its instantaneous velocity v. If we take, in this circumstance, the positive direction to be oriented downward, then the net force acting on the mass is given by $F = F_1 + F_2 = mg - kv$, where the weight $F_1 = mg$ of the body is force acting in the positive direction and air resistance $F_2 = -kv$ is a force, called **viscous damping,** acting in the opposite or upward direction. See Figure 1.3.5. Now since v is related to acceleration a by $a = dv/dt$, Newton's second law becomes $F = ma = m\,dv/dt$. By equating the net force to this form of Newton's second law, we obtain a first-order differential equation for the velocity $v(t)$ of the body at time t,

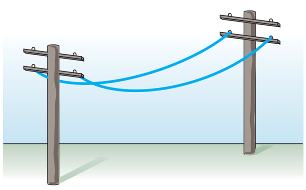
$$m\frac{dv}{dt} = mg - kv. \tag{14}$$

Here k is a positive constant of proportionality. If $s(t)$ is the distance the body falls in time t from its initial point of release, then $v = ds/dt$ and $a = dv/dt = d^2s/dt^2$. In terms of s, (14) is a second-order differential equation

$$m\frac{d^2s}{dt^2} = mg - k\frac{ds}{dt} \qquad \text{or} \qquad m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg. \tag{15}$$



(a) suspension bridge cable



(b) telephone wires

FIGURE 1.3.6 Cables suspended between vertical supports

SUSPENDED CABLES Suppose a flexible cable, wire, or heavy rope is suspended between two vertical supports. Physical examples of this could be one of the two cables supporting the roadbed of a suspension bridge as shown in Figure 1.3.6(a) or a long telephone wire strung between two posts as shown in Figure 1.3.6(b). Our goal is to construct a mathematical model that describes the shape that such a cable assumes.

To begin, let's agree to examine only a portion or element of the cable between its lowest point P_1 and any arbitrary point P_2. As drawn in blue in Figure 1.3.7, this element of the cable is the curve in a rectangular coordinate system with y-axis chosen to pass through the lowest point P_1 on the curve and the x-axis chosen a units below P_1. Three forces are acting on the cable: the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the cable that are tangent to the cable at P_1 and P_2, respectively, and the portion $\mathbf{W}$ of the total vertical load between the points P_1 and P_2. Let $T_1 = |\mathbf{T}_1|$, $T_2 = |\mathbf{T}_2|$, and $W = |\mathbf{W}|$ denote the magnitudes of these vectors. Now the tension $\mathbf{T}_2$ resolves into horizontal and vertical components (scalar quantities) $T_2 \cos\theta$ and $T_2 \sin\theta$.

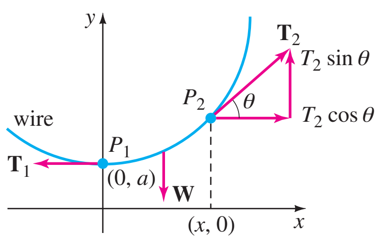


FIGURE 1.3.7 Element of cable

Because of static equilibrium we can write

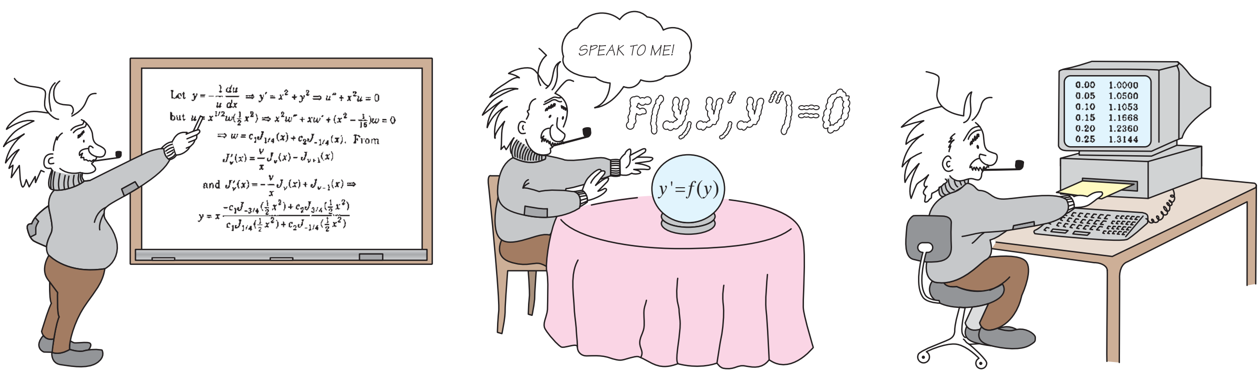
$$T_1 = T_2 \cos\theta \qquad \text{and} \qquad W = T_2 \sin\theta.$$

By dividing the last equation by the first, we eliminate T_2 and get $\tan\theta = W/T_1$. But because $dy/dx = \tan\theta$, we arrive at

$$\frac{dy}{dx} = \frac{W}{T_1}. \tag{16}$$

This simple first-order differential equation serves as a model for both the shape of a flexible wire such as a telephone wire hanging under its own weight and the shape of the cables that support the roadbed of a suspension bridge. We will come back to equation (16) in Exercises 2.2 and Section 5.3.

WHAT LIES AHEAD Throughout this text you will see three different types of approaches to, or analyses of, differential equations. Over the centuries differential equations would often spring from the efforts of a scientist or engineer to describe some physical phenomenon or to translate an empirical or experimental law into mathematical terms. As a consequence a scientist, engineer, or mathematician would often spend many years of his or her life trying to find the solutions of a DE. With a solution in hand, the study of its properties then followed. This quest for solutions is called by some the *analytical approach* to differential equations. Once they realized that explicit solutions are at best difficult to obtain and at worst impossible to obtain, mathematicians learned that a differential equation itself could be a font of valuable information. It is possible, in some instances, to glean directly from the differential equation answers to questions such as *Does the DE actually have solutions? If a solution of the DE exists and satisfies an initial condition, is it the only such solution? What are some of the properties of the unknown solutions? What can we say about the geometry of the solution curves?* Such an approach is *qualitative analysis.* Finally, if a differential equation cannot be solved by analytical methods, yet we can prove that a solution exists, the next logical query is *Can we somehow approximate the values of an unknown solution?* Here we enter the realm of *numerical analysis.* An affirmative answer to the last question stems from the fact that a differential equation can be used as a cornerstone for constructing very accurate approximation algorithms. In Chapter 2 we start with qualitative considerations of first-order ODEs, then examine analytical stratagems for solving some special first-order equations, and conclude with an introduction to an elementary numerical method. See Figure 1.3.8.



(a) analytical **(b)** qualitative **(c)** numerical

FIGURE 1.3.8 Different approaches to the study of differential equations

REMARKS

Each example in this section has described a dynamical system—a system that changes or evolves with the flow of time t. Since the study of dynamical systems is a branch of mathematics currently in vogue, we shall occasionally relate the terminology of that field to the discussion at hand.

In more precise terms, a **dynamical system** consists of a set of time-dependent variables, called **state variables,** together with a rule that enables us to determine (without ambiguity) the state of the system (this may be a past, present, or future state) in terms of a state prescribed at some time t_0. Dynamical systems are classified as either discrete-time systems or continuous-time systems. In this course we shall be concerned only with continuous-time systems—systems in which *all* variables are defined over a continuous range of time. The rule, or mathematical model, in a continuous-time dynamical system is a differential equation or a system of differential equations. The **state of the system** at a time t is the value of the state variables at that time; the specified state of the system at a time t_0 is simply the initial conditions that accompany the mathematical model. The solution of the initial-value problem is referred to as the **response of the system.** For example, in the case of radioactive decay, the rule is $dA/dt = kA$. Now if the quantity of a radioactive substance at some time t_0 is known, say $A(t_0) = A_0$, then by solving the rule we find that the response of the system for $t \geq t_0$ is $A(t) = A_0e^{(t-t_0)}$ (see Section 3.1). The response $A(t)$ is the single state variable for this system. In the case of the rock tossed from the roof of a building, the response of the system—the solution of the differential equation $d^2s/dt^2 = -g$, subject to the initial state $s(0) = s_0$, $s'(0) = v_0$, is the function $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$, $0 \leq t \leq T$, where T represents the time when the rock hits the ground. The state variables are $s(t)$ and $s'(t)$, which are the vertical position of the rock above ground and its velocity at time t, respectively. The acceleration $s''(t)$ is *not* a state variable, since we have to know only any initial position and initial velocity at a time t_0 to uniquely determine the rock's position $s(t)$ and velocity $s'(t) = v(t)$ for any time in the interval $t_0 \leq t \leq T$. The acceleration $s''(t) = a(t)$ is, of course, given by the differential equation $s''(t) = -g$, $0 < t < T$.

One last point: Not every system studied in this text is a dynamical system. We shall also examine some static systems in which the model is a differential equation.

EXERCISES 1.3

Answers to selected odd-numbered problems begin on page ANS-1.

Population Dynamics

1. Under the same assumptions that underlie the model in (1), determine a differential equation for the population $P(t)$ of a country when individuals are allowed to immigrate into the country at a constant rate $r > 0$. What is the differential equation for the population $P(t)$ of the country when individuals are allowed to emigrate from the country at a constant rate $r > 0$?

2. The population model given in (1) fails to take death into consideration; the growth rate equals the birth rate. In another model of a changing population of a community it is assumed that the rate at which the population changes is a *net* rate—that is, the difference between the rate of births and the rate of deaths in the community. Determine a model for the population $P(t)$ if both the birth rate and the death rate are proportional to the population present at time t.

3. Using the concept of net rate introduced in Problem 2, determine a model for a population $P(t)$ if the birth rate is proportional to the population present at time t but the death rate is proportional to the square of the population present at time t.

4. Modify the model in Problem 3 for net rate at which the population $P(t)$ of a certain kind of fish changes by also assuming that the fish are harvested at a constant rate $h > 0$.

Newton's Law of Cooling/Warming

5. A cup of coffee cools according to Newton's law of cooling (3). Use data from the graph of the temperature $T(t)$ in Figure 1.3.9 to estimate the constants T_m, T_0, and k in a model of the form of a first-order initial-value problem: $dT/dt = k(T - T_m)$, $T(0) = T_0$.

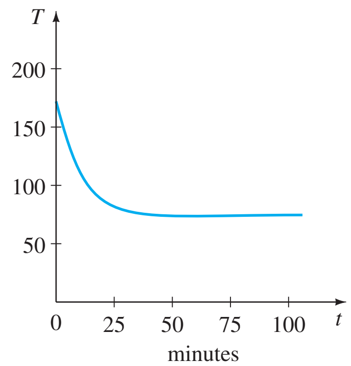


FIGURE 1.3.9 Cooling curve in Problem 5

6. The ambient temperature T_m in (3) could be a function of time t. Suppose that in an artificially controlled environment, $T_m(t)$ is periodic with a 24-hour period, as illustrated in Figure 1.3.10. Devise a mathematical model for the temperature $T(t)$ of a body within this environment.

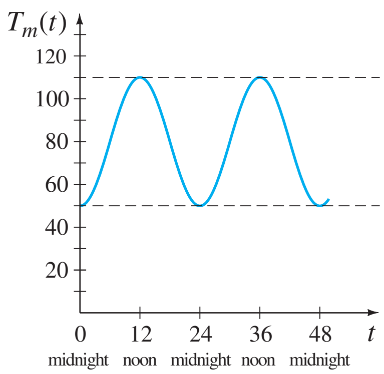


FIGURE 1.3.10 Ambient temperature in Problem 6

Spread of a Disease/Technology

7. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine a differential equation for the number of people $x(t)$ who have contracted the flu if the rate at which the disease spreads is proportional to the number of interactions between the number of students who have the flu and the number of students who have not yet been exposed to it.

8. At a time denoted as $t = 0$ a technological innovation is introduced into a community that has a fixed population of n people. Determine a differential equation for the number of people $x(t)$ who have adopted the innovation at time t if it is assumed that the rate at which the innovations spread through the community is jointly proportional to the number of people who have adopted it and the number of people who have not adopted it.

Mixtures

9. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at the same rate. Determine a differential equation for the amount of salt $A(t)$ in the tank at time t. What is $A(0)$?

10. Suppose that a large mixing tank initially holds 300 gallons of water is which 50 pounds of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at a *slower* rate of 2 gal/min. If the concentration of the solution entering is 2 lb/gal, determine a differential equation for the amount of salt $A(t)$ in the tank at time t.

11. What is the differential equation in Problem 10, if the well-stirred solution is pumped out at a *faster* rate of 3.5 gal/min?

12. Generalize the model given in equation (8) on page 23 by assuming that the large tank initially contains N_0 number of gallons of brine, r_{in} and r_{out} are the input and output rates of the brine, respectively (measured in gallons per minute), c_{in} is the concentration of the salt in the inflow, $c(t)$ the concentration of the salt in the tank as well as in the outflow at time t (measured in pounds of salt per gallon), and $A(t)$ is the amount of salt in the tank at time t.

Draining a Tank

13. Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of water leaving the tank per second to $cA_h\sqrt{2gh}$, where c $(0 < c < 1)$ is an empirical constant. Determine a differential equation for the height h of water at time t for the cubical tank shown in Figure 1.3.11. The radius of the hole is 2 in., and $g = 32$ ft/s^2.

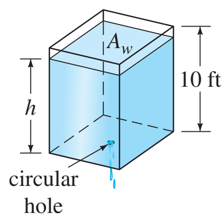


FIGURE 1.3.11 Cubical tank in Problem 13

14. The right-circular conical tank shown in Figure 1.3.12 loses water out of a circular hole at its bottom. Determine a differential equation for the height of the water h at time t. The radius of the hole is 2 in., $g = 32$ ft/s^2, and the friction/contraction factor introduced in Problem 13 is $c = 0.6$.

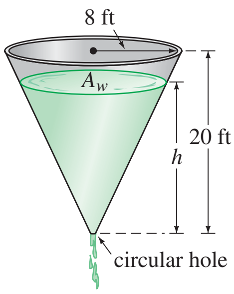


FIGURE 1.3.12 Conical tank in Problem 14

Series Circuits

15. A series circuit contains a resistor and an inductor as shown in Figure 1.3.13. Determine a differential equation for the current $i(t)$ if the resistance is R, the inductance is L, and the impressed voltage is $E(t)$.

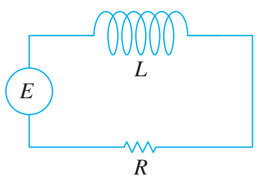


FIGURE 1.3.13 *LR* series circuit in Problem 15

16. A series circuit contains a resistor and a capacitor as shown in Figure 1.3.14. Determine a differential equation for the charge $q(t)$ on the capacitor if the resistance is R, the capacitance is C, and the impressed voltage is $E(t)$.

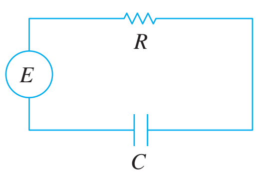


FIGURE 1.3.14 *RC* series circuit in Problem 16

Falling Bodies and Air Resistance

17. For high-speed motion through the air—such as the skydiver shown in Figure 1.3.15, falling before the parachute is opened—air resistance is closer to a power of the instantaneous velocity $v(t)$. Determine a differential equation for the velocity $v(t)$ of a falling body of mass m if air resistance is proportional to the square of the instantaneous velocity.

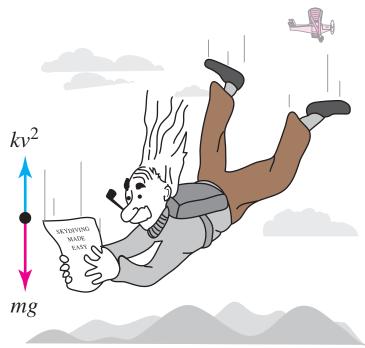


FIGURE 1.3.15 Air resistance proportional to square of velocity in Problem 17

Newton's Second Law and Archimedes' Principle

18. A cylindrical barrel s feet in diameter of weight w lb is floating in water as shown in Figure 1.3.16(a). After an initial depression the barrel exhibits an up-and-down bobbing motion along a vertical line. Using Figure 1.3.16(b), determine a differential equation for the vertical displacement $y(t)$ if the origin is taken to be on the vertical axis at the surface of the water when the barrel is at rest. Use **Archimedes' principle:** Buoyancy, or upward force of the water on the barrel, is equal to the weight of the water displaced. Assume that the downward direction is positive, that the weight density of water is 62.4 lb/ft^3, and that there is no resistance between the barrel and the water.

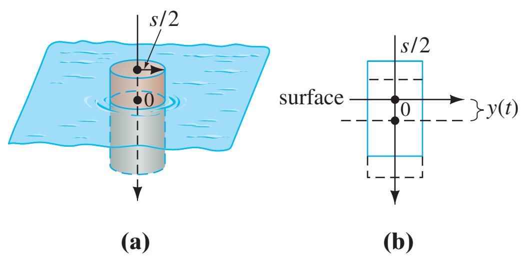


FIGURE 1.3.16 Bobbing motion of floating barrel in Problem 18

Newton's Second Law and Hooke's Law

19. After a mass m is attached to a spring, it stretches it s units and then hangs at rest in the equilibrium position as shown in Figure 1.3.17(b). After the spring/mass

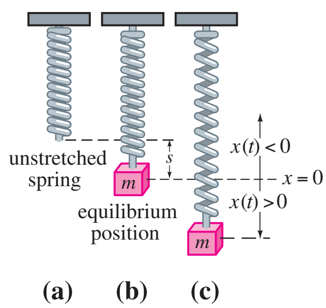


FIGURE 1.3.17 Spring/mass system in Problem 19

system has been set in motion, let $x(t)$ denote the directed distance of the mass beyond the equilibrium position. As indicated in Figure 1.3.17(c), assume that the downward direction is positive, that the motion takes place in a vertical straight line through the center of gravity of the mass, and that the only forces acting on the system are the weight of the mass and the restoring force of the stretched spring. Use **Hooke's law:** The restoring force of a spring is proportional to its total elongation. Determine a differential equation for the displacement $x(t)$ at time t.

20. In Problem 19, what is a differential equation for the displacement $x(t)$ if the motion takes place in a medium that imparts a damping force on the spring/mass system that is proportional to the instantaneous velocity of the mass and acts in a direction opposite to that of motion?

Newton's Second Law and the Law of Universal Gravitation

21. By **Newton's universal law of gravitation** the free-fall acceleration a of a body, such as the satellite shown in Figure 1.3.18, falling a great distance to the surface is *not* the constant g. Rather, the acceleration a is inversely proportional to the square of the distance from the center of the Earth, $a = k/r^2$, where k is the constant of proportionality. Use the fact that at the surface of the Earth $r = R$ and $a = g$ to determine k. If the positive direction is upward, use Newton's second law and his universal law of gravitation to find a differential equation for the distance r.

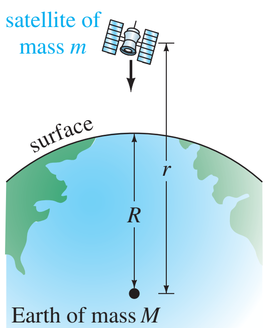


FIGURE 1.3.18 Satellite in Problem 21

22. Suppose a hole is drilled through the center of the Earth and a bowling ball of mass m is dropped into the hole, as shown in Figure 1.3.19. Construct a mathematical model that describes the motion of the ball. At time t let r denote the distance from the center of the Earth to the mass m, M denote the mass of the Earth, M_r denote the mass of that portion of the Earth within a sphere of radius r, and δ denote the constant density of the Earth.

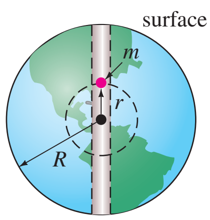


FIGURE 1.3.19 Hole through Earth in Problem 22

Additional Mathematical Models

23. Learning Theory In the theory of learning, the rate at which a subject is memorized is assumed to be proportional to the amount that is left to be memorized. Suppose M denotes the total amount of a subject to be memorized and $A(t)$ is the amount memorized in time t. Determine a differential equation for the amount $A(t)$.

24. Forgetfulness In Problem 23 assume that the rate at which material is *forgotten* is proportional to the amount memorized in time t. Determine a differential equation for the amount $A(t)$ when forgetfulness is taken into account.

25. Infusion of a Drug A drug is infused into a patient's bloodstream at a constant rate of r grams per second. Simultaneously, the drug is removed at a rate proportional to the amount $x(t)$ of the drug present at time t. Determine a differential equation for the amount $x(t)$.

26. Tractrix A person P, starting at the origin, moves in the direction of the positive x-axis, pulling a weight along the curve C, called a **tractrix,** as shown in Figure 1.3.20. The weight, initially located on the y-axis at $(0, s)$, is pulled by a rope of constant length s, which is kept taut throughout the motion. Determine a differential equation for the path C of motion. Assume that the rope is always tangent to C.

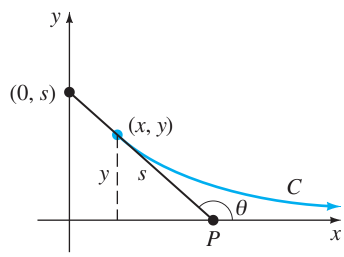


FIGURE 1.3.20 Tractrix curve in Problem 26

27. Reflecting Surface Assume that when the plane curve C shown in Figure 1.3.21 is revolved about the x-axis, it generates a surface of revolution with the property that all light rays L parallel to the x-axis striking the surface are reflected to a single point O (the origin). Use the fact that the angle of incidence is equal to the angle of reflection to determine a differential equation that

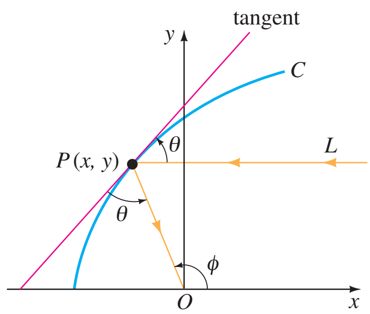


FIGURE 1.3.21 Reflecting surface in Problem 27

describes the shape of the curve C. Such a curve C is important in applications ranging from construction of telescopes to satellite antennas, automobile headlights, and solar collectors. [*Hint*: Inspection of the figure shows that we can write $\phi = 2\theta$. Why? Now use an appropriate trigonometric identity.]

Discussion Problems

28. Reread Problem 41 in Exercises 1.1 and then give an explicit solution $P(t)$ for equation (1). Find a one-parameter family of solutions of (1).

29. Reread the sentence following equation (3) and assume that T_m is a positive constant. Discuss why we would expect $k < 0$ in (3) in both cases of cooling and warming. You might start by interpreting, say, $T(t) > T_m$ in a graphical manner.

30. Reread the discussion leading up to equation (8). If we assume that initially the tank holds, say, 50 lb of salt, it stands to reason that because salt is being added to the tank continuously for $t > 0$, $A(t)$ should be an increasing function. Discuss how you might determine from the DE, without actually solving it, the number of pounds of salt in the tank after a long period of time.

31. Population Model The differential equation $\dfrac{dP}{dt} = (k \cos t)P$, where k is a positive constant, is a model of human population $P(t)$ of a certain community. Discuss an interpretation for the solution of this equation. In other words, what kind of population do you think the differential equation describes?

32. Rotating Fluid As shown in Figure 1.3.22(a), a right-circular cylinder partially filled with fluid is rotated with a constant angular velocity ω about a vertical y-axis through its center. The rotating fluid forms a surface of revolution S. To identify S, we first establish a coordinate system consisting of a vertical plane determined by the y-axis and an x-axis drawn perpendicular to the y-axis such that the point of intersection of the axes (the origin) is located at the lowest point on the surface S. We then seek a function $y = f(x)$ that represents the curve C of intersection of the surface S and the vertical coordinate plane. Let the point $P(x, y)$ denote the position of a particle of the rotating fluid of mass m in the coordinate plane. See Figure 1.3.22(b).

(a) At P there is a reaction force of magnitude F due to the other particles of the fluid which is normal to the surface S. By Newton's second law the magnitude of the net force acting on the particle is $m\omega^2 x$. What is this force? Use Figure 1.3.22(b) to discuss the nature and origin of the equations

$$F \cos \theta = mg, \qquad F \sin \theta = m\omega^2 x.$$

(b) Use part (a) to find a first-order differential equation that defines the function $y = f(x)$.

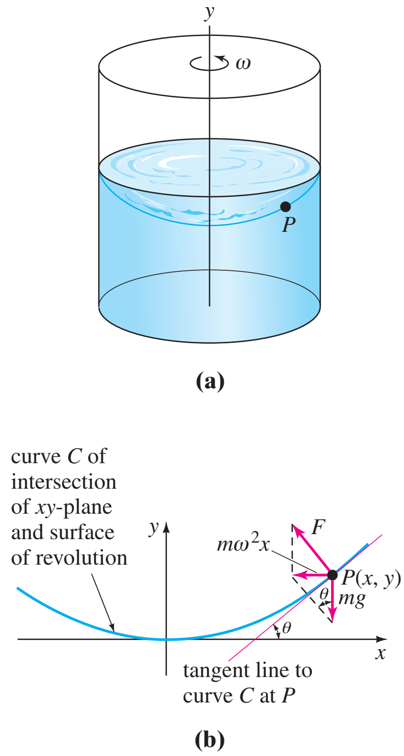


FIGURE 1.3.22 Rotating fluid in Problem 32

33. Falling Body In Problem 21, suppose $r = R + s$, where s is the distance from the surface of the Earth to the falling body. What does the differential equation obtained in Problem 21 become when s is very small in comparison to R? [*Hint*: Think binomial series for

$$(R + s)^{-2} = R^{-2}(1 + s/R)^{-2}.]$$

34. Raindrops Keep Falling In meteorology the term *virga* refers to falling raindrops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical. Starting at some time, which we can designate as $t = 0$, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate.

(a) If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates—that is, the rate at which it loses mass—is proportional to its surface area. Show that this latter assumption implies that the rate at which the radius r of the raindrop decreases is a constant. Find $r(t)$. [*Hint*: See Problem 51 in Exercises 1.1.]

(b) If the positive direction is downward, construct a mathematical model for the velocity v of the falling raindrop at time t. Ignore air resistance. [*Hint*: When the mass m of an object is changing with time, Newton's second law becomes $F = \dfrac{d}{dt}(mv)$, where F is the net force acting on the body and mv is its momentum.]

35. Let It Snow The "snowplow problem" is a classic and appears in many differential equations texts but was probably made famous by Ralph Palmer Agnew:

> *"One day it started snowing at a heavy and steady rate. A snowplow started out at noon, going 2 miles the first hour and 1 mile the second hour. What time did it start snowing?"*

Find the text *Differential Equations,* Ralph Palmer Agnew, McGraw-Hill Book Co., and then discuss the construction and solution of the mathematical model.

36. Reread this section and classify each mathematical model as linear or nonlinear.

CHAPTER 1 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1 and 2 fill in the blank and then write this result as a linear first-order differential equation that is free of the symbol c_1 and has the form $dy/dx = f(x, y)$. The symbol c_1 represents a constant.

1. $\dfrac{d}{dx}c_1e^{10x} =$ __________

2. $\dfrac{d}{dx}(5 + c_1e^{-2x}) =$ __________

In Problems 3 and 4 fill in the blank and then write this result as a linear second-order differential equation that is free of the symbols c_1 and c_2 and has the form $F(y, y'') = 0$. The symbols c_1, c_2, and k represent constants.

3. $\dfrac{d^2}{dx^2}(c_1 \cos kx + c_2 \sin kx) =$ __________

4. $\dfrac{d^2}{dx^2}(c_1 \cosh kx + c_2 \sinh kx) =$ __________

In Problems 5 and 6 compute y' and y'' and then combine these derivatives with y as a linear second-order differential equation that is free of the symbols c_1 and c_2 and has the form $F(y, y'\ y'') = 0$. The symbols c_1 and c_2 represent constants.

5. $y = c_1e^x + c_2xe^x$ **6.** $y = c_1e^x \cos x + c_2e^x \sin x$

In Problems 7–12 match each of the given differential equations with one or more of these solutions:

(a) $y = 0$, **(b)** $y = 2$, **(c)** $y = 2x$, **(d)** $y = 2x^2$.

7. $xy' = 2y$ **8.** $y' = 2$

9. $y' = 2y - 4$ **10.** $xy' = y$

11. $y'' + 9y = 18$ **12.** $xy'' - y' = 0$

In Problems 13 and 14 determine by inspection at least one solution of the given differential equation.

13. $y'' = y'$ **14.** $y' = y(y - 3)$

In Problems 15 and 16 interpret each statement as a differential equation.

15. On the graph of $y = \phi(x)$ the slope of the tangent line at a point $P(x, y)$ is the square of the distance from $P(x, y)$ to the origin.

16. On the graph of $y = \phi(x)$ the rate at which the slope changes with respect to x at a point $P(x, y)$ is the negative of the slope of the tangent line at $P(x, y)$.

17. (a) Give the domain of the function $y = x^{2/3}$.

(b) Give the largest interval I of definition over which $y = x^{2/3}$ is solution of the differential equation $3xy' - 2y = 0$.

18. (a) Verify that the one-parameter family $y^2 - 2y = x^2 - x + c$ is an implicit solution of the differential equation $(2y - 2)y' = 2x - 1$.

(b) Find a member of the one-parameter family in part (a) that satisfies the initial condition $y(0) = 1$.

(c) Use your result in part (b) to find an explicit *function* $y = \phi(x)$ that satisfies $y(0) = 1$. Give the domain of the function ϕ. Is $y = \phi(x)$ a *solution* of the initial-value problem? If so, give its interval I of definition; if not, explain.

19. Given that $y = x - 2/x$ is a solution of the DE $xy' + y = 2x$. Find x_0 and the largest interval I for which $y(x)$ is a solution of the first-order IVP $xy' + y = 2x$, $y(x_0) = 1$.

20. Suppose that $y(x)$ denotes a solution of the first-order IVP $y' = x^2 + y^2$, $y(1) = -1$ and that $y(x)$ possesses at least a second derivative at $x = 1$. In some neighborhood of $x = 1$ use the DE to determine whether $y(x)$ is increasing or decreasing and whether the graph $y(x)$ is concave up or concave down.

21. A differential equation may possess more than one family of solutions.

(a) Plot different members of the families $y = \phi_1(x) = x^2 + c_1$ and $y = \phi_2(x) = -x^2 + c_2$.

(b) Verify that $y = \phi_1(x)$ and $y = \phi_2(x)$ are two solutions of the nonlinear first-order differential equation $(y')^2 = 4x^2$.

(c) Construct a piecewise-defined function that is a solution of the nonlinear DE in part (b) but is not a member of either family of solutions in part (a).

22. What is the slope of the tangent line to the graph of a solution of $y' = 6\sqrt{y} + 5x^3$ that passes through $(-1, 4)$?

In Problems 23–26 verify that the indicated function is a particular solution of the given differential equation. Give an interval of definition I for each solution.

23. $y'' + y = 2 \cos x - 2 \sin x$; $y = x \sin x + x \cos x$

24. $y'' + y = \sec x$; $y = x \sin x + (\cos x)\ln(\cos x)$