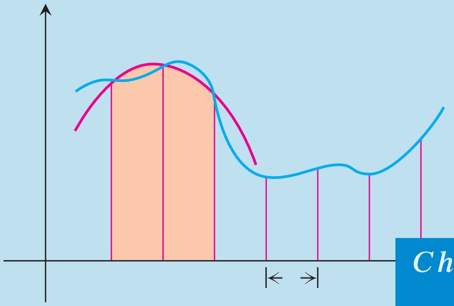


Chapter

8 TECHNIQUES OF INTEGRATION

OVERVIEW The Fundamental Theorem connects antiderivatives and the definite integral. Evaluating the indefinite integral

$$\int f(x)\,dx$$

is equivalent to finding a function F such that $F'(x) = f(x)$, and then adding an arbitrary constant C:

$$\int f(x)\,dx = F(x) + C.$$

In this chapter we study a number of important techniques for finding indefinite integrals of more complicated functions than those seen before. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer. We also extend the idea of the definite integral to *improper integrals* for which the integrand may be unbounded over the interval of integration, or the interval itself may no longer be finite.

8.1 Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

Recall the Substitution Rule from Section 5.5:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I. Success in integration often hinges on the ability to spot what part of the integrand should be called u in order that one will also have du, so that a known formula can be applied. This means that the first requirement for skill in integration is a thorough mastery of the formulas for differentiation.

Table 8.1 shows the basic forms of integrals we have evaluated so far. In this section we present several algebraic or substitution methods to help us use this table. There is a more extensive table at the back of the book; we discuss its use in Section 8.6.

TABLE 8.1 Basic integration formulas

1. $\displaystyle\int du = u + C$
2. $\displaystyle\int k\,du = ku + C \qquad (\text{any number } k)$
3. $\displaystyle\int (du + dv) = \int du + \int dv$
4. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
6. $\displaystyle\int \sin u\,du = -\cos u + C$
7. $\displaystyle\int \cos u\,du = \sin u + C$
8. $\displaystyle\int \sec^2 u\,du = \tan u + C$
9. $\displaystyle\int \csc^2 u\,du = -\cot u + C$
10. $\displaystyle\int \sec u \tan u\,du = \sec u + C$
11. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$
12. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\displaystyle\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\displaystyle\int e^u\,du = e^u + C$
15. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\displaystyle\int \sinh u\,du = \cosh u + C$
17. $\displaystyle\int \cosh u\,du = \sinh u + C$
18. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$
22. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

We often have to rewrite an integral to match it to a standard formula.

EXAMPLE 1 Making a Simplifying Substitution

Video

You Try It

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx.$$

Solution

$$\begin{aligned}
\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx &= \int \frac{du}{\sqrt{u}} && u = x^2-9x+1,\; du = (2x-9)\,dx. \\
&= \int u^{-1/2}\,du \\
&= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C && \text{Table 8.1 Formula 4, with } n = -1/2 \\
&= 2u^{1/2} + C \\
&= 2\sqrt{x^2-9x+1} + C
\end{aligned}$$

■

EXAMPLE 2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x-x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned}
8x-x^2 &= -(x^2-8x) = -(x^2-8x+16-16) \\
&= -(x^2-8x+16)+16 = 16-(x-4)^2.
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{8x-x^2}} &= \int \frac{dx}{\sqrt{16-(x-4)^2}} \\
&= \int \frac{du}{\sqrt{a^2-u^2}} && a = 4,\; u = (x-4),\; du = dx \\
&= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\
&= \sin^{-1}\left(\frac{x-4}{4}\right) + C.
\end{aligned}$$

■

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2\,dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}
\int (\sec x + \tan x)^2\,dx &= \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\,dx \\
&= 2\int \sec^2 x\,dx + 2\int \sec x \tan x\,dx - \int 1\,dx \\
&= 2\tan x + 2\sec x - x + C.
\end{aligned}$$

■

EXAMPLE 4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx.$$

Solution We use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx &= \int_0^{\pi/4} \sqrt{2}\,\sqrt{\cos^2 2x}\,dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx && \sqrt{u^2} = |u| \\
&= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx && \text{On } [0, \pi/4],\ \cos 2x \ge 0, \text{ so } |\cos 2x| = \cos 2x. \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \\
&= \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.
\end{aligned}$$

■

EXAMPLE 5 Reducing an Improper Fraction

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx.$$

$$\begin{array}{r}
x - 3 \\
3x + 2\,\big)\,\overline{3x^2 - 7x} \\
\underline{3x^2 + 2x} \\
-9x \\
\underline{-9x - 6} \\
+\,6
\end{array}$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{3x + 2}\right)dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$ ■

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

You Try It

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{and} \qquad x\,dx = -\frac{1}{2}\,du.$$

$$3\int \frac{x\,dx}{\sqrt{1-x^2}} = 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du$$

$$= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1-x^2} + C_1$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1}x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = -3\sqrt{1-x^2} + 2\sin^{-1}x + C.$$ ■

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

HISTORICAL BIOGRAPHY

George David Birkhoff (1884–1944)

Biography

You Try It

EXAMPLE 7 Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate

$$\int \sec x\,dx.$$

Solution

$$\int \sec x\,dx = \int (\sec x)(1)\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} \qquad \begin{array}{l} u = \tan x + \sec x, \\ du = (\sec^2 x + \sec x \tan x)\,dx \end{array}$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C.$$ ■

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

TABLE 8.2 The secant and cosecant integrals

1. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

2. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

Procedures for Matching Integrals to Basic Formulas

PROCEDURE	EXAMPLE
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$\begin{aligned}(\sec x + \tan x)^2 &= \sec^2 x + 2\sec x\tan x + \tan^2 x\\ &= \sec^2 x + 2\sec x\tan x\\ &\quad + (\sec^2 x - 1)\\ &= 2\sec^2 x + 2\sec x\tan x - 1\end{aligned}$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\begin{aligned}\sec x &= \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}\\ &= \dfrac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\end{aligned}$

EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$

2. $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$

4. $\displaystyle\int \cot^3 y\csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

Exercises

7. $\displaystyle\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

8. $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

9. $\displaystyle\int \cot(3-7x)\,dx$

10. $\displaystyle\int \csc(\pi x-1)\,dx$

11. $\displaystyle\int e^{\theta}\csc(e^{\theta}+1)\,d\theta$

12. $\displaystyle\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\displaystyle\int \sec\frac{t}{3}\,dt$

14. $\displaystyle\int x\sec(x^2-5)\,dx$

15. $\displaystyle\int \csc(s-\pi)\,ds$

16. $\displaystyle\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\displaystyle\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}\,dx$

18. $\displaystyle\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\,dy$

19. $\displaystyle\int e^{\tan v}\sec^2 v\,dv$

20. $\displaystyle\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\displaystyle\int 3^{x+1}\,dx$

22. $\displaystyle\int \frac{2^{\ln x}}{x}\,dx$

23. $\displaystyle\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\displaystyle\int 10^{2\theta}\,d\theta$

25. $\displaystyle\int \frac{9\,du}{1+9u^2}$

26. $\displaystyle\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\displaystyle\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$

28. $\displaystyle\int_0^{1} \frac{dt}{\sqrt{4-t^2}}$

29. $\displaystyle\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\displaystyle\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\displaystyle\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

34. $\displaystyle\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\displaystyle\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$

36. $\displaystyle\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

Exercises

37. $\displaystyle\int_1^2 \frac{8\,dx}{x^2-2x+2}$

38. $\displaystyle\int_2^4 \frac{2\,dx}{x^2-6x+10}$

39. $\displaystyle\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\displaystyle\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

Exercises

43. $\displaystyle\int (\sec x+\cot x)^2\,dx$

44. $\displaystyle\int (\csc x-\tan x)^2\,dx$

45. $\displaystyle\int \csc x\sin 3x\,dx$

46. $\displaystyle\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

Exercises

47. $\displaystyle\int \frac{x}{x+1}\,dx$

48. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

49. $\displaystyle\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx$

50. $\displaystyle\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx$

51. $\displaystyle\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$

52. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

Exercises

53. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

54. $\displaystyle\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

55. $\displaystyle\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$

56. $\displaystyle\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

Exercises

57. $\displaystyle\int \frac{1}{1+\sin x}\,dx$

58. $\displaystyle\int \frac{1}{1+\cos x}\,dx$

59. $\displaystyle\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

60. $\displaystyle\int \frac{1}{\csc\theta+\cot\theta}\,d\theta$

61. $\displaystyle\int \frac{1}{1-\sec x}\,dx$

62. $\displaystyle\int \frac{1}{1-\csc x}\,dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

Exercises

63. $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$

64. $\displaystyle\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

65. $\int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t}\, dt$ **66.** $\int_{-\pi}^{0} \sqrt{1 + \cos t}\, dt$

67. $\int_{-\pi}^{0} \sqrt{1 - \cos^2 \theta}\, d\theta$ **68.** $\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta}\, d\theta$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 y}\, dy$ **70.** $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy$

Assorted Integrations

Evaluate each integral in Exercises 71–82 by using any technique you think is appropriate.

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx$ **72.** $\int_{0}^{\pi/4} (\sec x + 4\cos x)^2\, dx$

73. $\int \cos\theta \csc(\sin\theta)\, d\theta$ **74.** $\int \left(1 + \frac{1}{x}\right) \cot(x + \ln x)\, dx$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

76. $\int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx$

77. $\int \frac{6\, dy}{\sqrt{y}(1 + y)}$ **78.** $\int \frac{dx}{x\sqrt{4x^2 - 1}}$

79. $\int \frac{7\, dx}{(x - 1)\sqrt{x^2 - 2x - 48}}$ **80.** $\int \frac{dx}{(2x + 1)\sqrt{4x^2 + 4x}}$

81. $\int \sec^2 t \tan(\tan t)\, dt$ **82.** $\int \frac{dx}{x\sqrt{3 + x^2}}$

Trigonometric Powers

83. **a.** Evaluate $\int \cos^3 \theta\, d\theta$. (*Hint:* $\cos^2\theta = 1 - \sin^2\theta$.)

b. Evaluate $\int \cos^5 \theta\, d\theta$.

c. Without actually evaluating the integral, explain how you would evaluate $\int \cos^9 \theta\, d\theta$.

84. **a.** Evaluate $\int \sin^3 \theta\, d\theta$. (*Hint:* $\sin^2\theta = 1 - \cos^2\theta$.)

b. Evaluate $\int \sin^5 \theta\, d\theta$.

c. Evaluate $\int \sin^7 \theta\, d\theta$.

d. Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13} \theta\, d\theta$.

85. **a.** Express $\int \tan^3 \theta\, d\theta$ in terms of $\int \tan\theta\, d\theta$. Then evaluate $\int \tan^3 \theta\, d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta - 1$.)

b. Express $\int \tan^5 \theta\, d\theta$ in terms of $\int \tan^3 \theta\, d\theta$.

c. Express $\int \tan^7 \theta\, d\theta$ in terms of $\int \tan^5 \theta\, d\theta$.

d. Express $\int \tan^{2k+1} \theta\, d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1} \theta\, d\theta$.

86. **a.** Express $\int \cot^3 \theta\, d\theta$ in terms of $\int \cot\theta\, d\theta$. Then evaluate $\int \cot^3 \theta\, d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta - 1$.)

b. Express $\int \cot^5 \theta\, d\theta$ in terms of $\int \cot^3 \theta\, d\theta$.

c. Express $\int \cot^7 \theta\, d\theta$ in terms of $\int \cot^5 \theta\, d\theta$.

d. Express $\int \cot^{2k+1} \theta\, d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1} \theta\, d\theta$.

Theory and Examples

87. Area Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

88. Area Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

89. Volume Find the volume of the solid generated by revolving the region in Exercise 87 about the x-axis.

90. Volume Find the volume of the solid generated by revolving the region in Exercise 88 about the x-axis.

91. Arc length Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

92. Arc length Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

93. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

94. Centroid Find the centroid of the region that is bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

95. The integral of csc x Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

96. Using different substitutions Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3}\, dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x + 1)$

b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$ **e.** $u = \tan^{-1}((x - 1)/2)$

f. $u = \cos^{-1} x$ **g.** $u = \cosh^{-1} x$

What is the value of the integral? (*Source:* "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)

8.2 Integration by Parts

Since

$$\int x\,dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx.$$

In other words, the integral of a product is generally *not* the product of the individual-integrals:

$$\int f(x)g(x)\,dx \text{ is not equal to } \int f(x)\,dx \cdot \int g(x)\,dx.$$

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{1}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique.

EXAMPLE 1 Using Integration by Parts

Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$\begin{aligned} u &= x, & dv &= \cos x\,dx,\\ du &= dx, & v &= \sin x. \end{aligned}$$

Simplest antiderivative of $\cos x$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Let us examine the choices available for u and dv in Example 1.

EXAMPLE 2 Example 1 Revisited

To apply integration by parts to

$$\int x\cos x\,dx = \int u\,dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x \, dx$ to get v.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x \cos x, & dv &= dx, \\ du &= (\cos x - x \sin x) \, dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v \, du = \int (x \cos x - x^2 \sin x) \, dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x \, dx, \\ du &= -\sin x \, dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v \, du = -\int \frac{x^2}{2} \sin x \, dx.$$

This, too, is worse. ■

The goal of integration by parts is to go from an integral $\int u \, dv$ that we don't see how to evaluate to an integral $\int v \, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. Keep in mind that integration by parts does not always work.

EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\qquad v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C.$$ ■

Sometimes we have to use integration by parts more than once.

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x \, dx .$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx .$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C .$$

Hence,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C . \end{aligned}$$

■

The technique of Example 4 works for any integral $\int x^n e^x \, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy. We say more about this later in this section when we discuss *tabular integration.*

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 5 Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x \, dx .$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx .$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx .$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx . \end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\, dx \qquad (3)$$

In applying Equation (3), we normally use the u and v notation from Equation (2) because it is easier to remember. Here is an example.

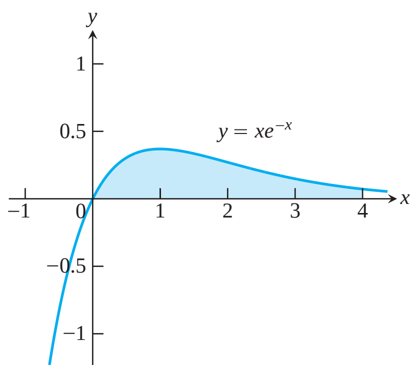


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Finding Area

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\, dx.$$

Let $u = x$, $dv = e^{-x}\, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}
\int_0^4 xe^{-x}\, dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\, dx \\
&= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\, dx \\
&= -4e^{-4} - e^{-x}\Big]_0^4 \\
&= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.
\end{aligned}$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\, dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

***f*(*x*) and its derivatives**		***g*(*x*) and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 4. ■

EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

***f*(*x*) and its derivatives**		***g*(*x*) and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \quad ■$$

The Additional Exercises at the end of this chapter show how tabular integration can be used when neither function f nor g can be differentiated repeatedly to become zero.

Summary

When substitution doesn't work, try integration by parts. Start with an integral in which the integrand is the product of two functions,

$$\int f(x)g(x)\,dx.$$

(Remember that g may be the constant function 1, as in Example 3.) Match the integral with the form

$$\int u\,dv$$

by choosing dv to be part of the integrand including dx and either $f(x)$ or $g(x)$. Remember that we must be able to readily integrate dv to get v in order to obtain the right side of the formula

$$\int u\,dv = uv - \int v\,du.$$

If the new integral on the right side is more complex than the original one, try a different choice for u and dv.

EXAMPLE 9 A Reduction Formula

Obtain a "reduction" formula that expresses the integral

$$\int \cos^n x\,dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\,dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\,dx) \qquad \text{and} \qquad v = \sin x.$$

Hence

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx, \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\,dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

This allows us to reduce the exponent on $\cos x$ by 2 and is a very useful formula. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is either

$$\int \cos x\,dx = \sin x + C \qquad \text{or} \qquad \int \cos^0 x\,dx = \int dx = x + C.$$ ■

EXAMPLE 10 Using a Reduction Formula

Evaluate

$$\int \cos^3 x\,dx.$$

Solution From the result in Example 9,

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$ ■

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

3. $\int t^2 \cos t \, dt$

4. $\int x^2 \sin x \, dx$

5. $\int_1^2 x \ln x \, dx$

6. $\int_1^e x^3 \ln x \, dx$

7. $\int \tan^{-1} y \, dy$

8. $\int \sin^{-1} y \, dy$

9. $\int x \sec^2 x \, dx$

10. $\int 4x \sec^2 2x \, dx$

11. $\int x^3 e^x \, dx$

12. $\int p^4 e^{-p} \, dp$

13. $\int (x^2 - 5x)e^x \, dx$

14. $\int (r^2 + r + 1)e^r \, dr$

15. $\int x^5 e^x \, dx$

16. $\int t^2 e^{4t} \, dt$

17. $\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta$

18. $\int_0^{\pi/2} x^3 \cos 2x \, dx$

19. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt$

20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2) \, dx$

21. $\int e^\theta \sin \theta \, d\theta$

22. $\int e^{-y} \cos y \, dy$

23. $\int e^{2x} \cos 3x \, dx$

24. $\int e^{-2x} \sin 2x \, dx$

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$ **26.** $\int_0^1 x\sqrt{1-x}\, dx$

27. $\int_0^{\pi/3} x \tan^2 x\, dx$ **28.** $\int \ln(x + x^2)\, dx$

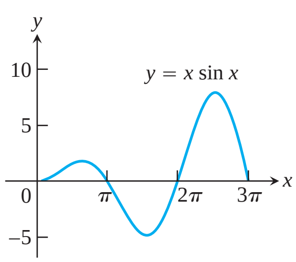
29. $\int \sin(\ln x)\, dx$ **30.** $\int z(\ln z)^2\, dz$

Theory and Examples

31. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$ **b.** $\pi \le x \le 2\pi$ **c.** $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



32. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for

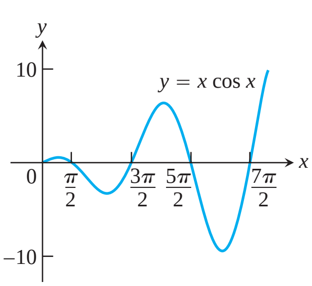
a. $\pi/2 \le x \le 3\pi/2$ **b.** $3\pi/2 \le x \le 5\pi/2$

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



33. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis. **b.** about the line $x = 1$.

35. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis. **b.** the line $x = \pi/2$.

36. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about

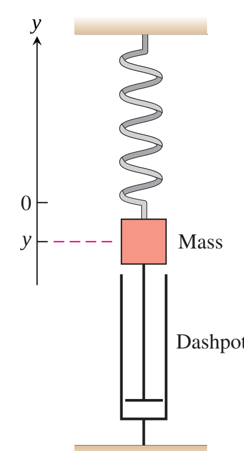
a. the y-axis. **b.** the line $x = \pi$.

(See Exercise 31 for a graph.)

37. Average value A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



38. Average value In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 39–42, use integration by parts to establish the *reduction formula*.

39. $\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$

40. $\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$

41. $\int x^n e^{ax}\, dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1} e^{ax}\, dx, \quad a \neq 0$

42. $\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned}\int f^{-1}(x)\, dx &= \int y f'(y)\, dy && y = f^{-1}(x), \quad x = f(y) \\ & && dx = f'(y)\, dy \\ &= yf(y) - \int f(y)\, dy && \text{Integration by parts with} \\ & && u = y,\ dv = f'(y)\, dy \\ &= xf^{-1}(x) - \int f(y)\, dy\end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned}\int \ln x\, dx &= \int ye^y\, dy && y = \ln x, \quad x = e^y \\ & && dx = e^y\, dy \\ &= ye^y - e^y + C \\ &= x\ln x - x + C.\end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned}\int \cos^{-1} x\, dx &= x\cos^{-1} x - \int \cos y\, dy && y = \cos^{-1} x \\ &= x\cos^{-1} x - \sin y + C \\ &= x\cos^{-1} x - \sin(\cos^{-1} x) + C.\end{aligned}$$

Use the formula

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int f(y)\, dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x.

43. $\int \sin^{-1} x\, dx$

44. $\int \tan^{-1} x\, dx$

45. $\int \sec^{-1} x\, dx$

46. $\int \log_2 x\, dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int x\left(\frac{d}{dx} f^{-1}(x)\right) dx. \tag{5}$$

Exercises 47 and 48 compare the results of using Equations (4) and (5).

47. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\, dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C \qquad$ Eq. (4)

b. $\int \cos^{-1} x\, dx = x\cos^{-1} x - \sqrt{1 - x^2} + C \qquad$ Eq. (5)

Can both integrations be correct? Explain.

48. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\, dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C \qquad$ Eq. (4)

b. $\int \tan^{-1} x\, dx = x\tan^{-1} x - \ln\sqrt{1 + x^2} + C \qquad$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with **(a)** Eq. (4) and **(b)** Eq. (5). In each case, check your work by differentiating your answer with respect to x.

49. $\int \sinh^{-1} x\, dx$

50. $\int \tanh^{-1} x\, dx$

8.3 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3},$$

which can be verified algebraically by placing the fractions on the right side over a common denominator $(x+1)(x-3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x-3)/(x+1)(x-3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the above example, it consists of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \tag{1}$$

(Pretend for a moment that we do not know that $A=2$ and $B=3$ will work.) We call the fractions $A/(x+1)$ and $B/(x-3)$ **partial fractions** because their denominators are only part of the original denominator x^2-2x-3. We call A and B **undetermined coefficients** until proper values for them have been found.

To find A and B, we first clear Equation (1) of fractions, obtaining

$$5x-3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

You Try It

$$A+B=5, \qquad -3A+B=-3.$$

Solving these equations simultaneously gives $A=2$ and $B=3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. See Example 3 of this section.
- *We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$
Do this for each distinct linear factor of $g(x)$.
2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Distinct Linear Factors

Evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx$$

using partial fractions.

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A, B, and C we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B \qquad = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

There are several ways for solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables, or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$
$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K,$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2 A Repeated Linear Factor

Evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$
$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$
$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$
$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$
$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$ ■

EXAMPLE 3 Integrating an Improper Fraction

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r@{}l} & 2x \\ x^2 - 2x - 3 \,\big)\! & \overline{2x^3 - 4x^2 - x - 3} \\ & \underline{2x^3 - 4x^2 - 6x} \\ & \qquad\qquad 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned}\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx &= \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx \\ &= \int 2x\,dx + \int \frac{2}{x + 1}\,dx + \int \frac{3}{x - 3}\,dx \\ &= x^2 + 2\ln|x + 1| + 3\ln|x - 3| + C.\end{aligned}$$

■

A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.

EXAMPLE 4 Integrating with an Irreducible Quadratic Factor in the Denominator

You Try It

Evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx$$

using partial fractions.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned}-2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D).\end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned}&\text{Coefficients of } x^3: & 0 &= A + C \\ &\text{Coefficients of } x^2: & 0 &= -2A + B - C + D \\ &\text{Coefficients of } x^1: & -2 &= A - 2B + C \\ &\text{Coefficients of } x^0: & 4 &= B - C + D\end{aligned}$$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$\begin{aligned}&-4 = -2A, \quad A = 2 && \text{Subtract fourth equation from second.} \\ &C = -A = -2 && \text{From the first equation} \\ &B = 1 && A = 2 \text{ and } C = -2 \text{ in third equation.} \\ &D = 4 - B + C = 1. && \text{From the fourth equation}\end{aligned}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \quad \blacksquare\end{aligned}$$

You Try It

EXAMPLE 5 A Repeated Irreducible Quadratic Factor

Evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\ &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad u = x^2+1,\ du = 2x\,dx \\ &= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\ &= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \quad \blacksquare\end{aligned}$$

HISTORICAL BIOGRAPHY

Oliver Heaviside
(1850–1925)

Biography

The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

EXAMPLE 6 Using the Heaviside Method

Find A, B, and C in the partial-fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \tag{3}$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,$$

$$A = 1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} \tag{4}$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x - 1)}}\,(1 - 2)(1 - 3)} = \frac{2}{(-1)(-2)} = 1.$$

Similarly, we find the value of B in Equation (3) by covering the factor $(x - 2)$ in Equation (4) and evaluating the rest at $x = 2$:

$$B = \frac{(2)^2 + 1}{(2 - 1)\,\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x - 2)}}\,(2 - 3)} = \frac{5}{(1)(-1)} = -5.$$

Finally, C is found by covering the $(x - 3)$ in Equation (4) and evaluating the rest at $x = 3$:

$$C = \frac{(3)^2 + 1}{(3 - 1)(3 - 2)\,\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x - 3)}}} = \frac{10}{(2)(1)} = 5. \qquad \blacksquare$$

Heaviside Method

1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

EXAMPLE 7 Integrating with the Heaviside Method

Evaluate

$$\int \frac{x + 4}{x^3 + 3x^2 - 10x}\,dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x + 4}{x^3 + 3x^2 - 10x} = \frac{x + 4}{x(x - 2)(x + 5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0 + 4}{\underbrace{\boxed{x}}_{\text{Cover}}(0 - 2)(0 + 5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

$$A_2 = \frac{2 + 4}{2\underbrace{\boxed{(x - 2)}}_{\text{Cover}}(2 + 5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

$$A_3 = \frac{-5 + 4}{(-5)(-5 - 2)\underbrace{\boxed{(x + 5)}}_{\text{Cover}}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C.$$

Other Ways to Determine the Coefficients

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

EXAMPLE 8 Using Differentiation

Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$

In some problems, assigning small values to x such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

EXAMPLE 9 Assigning Numerical Values to x

Find A, B, and C in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: \qquad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\
2 &= 2A \\
A &= 1 \\
x = 2: \qquad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\
5 &= -B \\
B &= -5 \\
x = 3: \qquad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\
10 &= 2C \\
C &= 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{1}{x - 1} - \frac{5}{x - 2} + \frac{5}{x - 3}.$$

■

EXERCISES 8.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x - 13}{(x - 3)(x - 2)}$

2. $\dfrac{5x - 7}{x^2 - 3x + 2}$

3. $\dfrac{x + 4}{(x + 1)^2}$

4. $\dfrac{2x + 2}{x^2 - 2x + 1}$

5. $\dfrac{z + 1}{z^2(z - 1)}$

6. $\dfrac{z}{z^3 - z^2 - 6z}$

7. $\dfrac{t^2 + 8}{t^2 - 5t + 6}$

8. $\dfrac{t^4 + 9}{t^4 + 9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1 - x^2}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x}$

11. $\displaystyle\int \frac{x + 4}{x^2 + 5x - 6}\,dx$

12. $\displaystyle\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

14. $\displaystyle\int_{1/2}^1 \frac{y + 4}{y^2 + y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3 + t^2 - 2t}$

16. $\displaystyle\int \frac{x + 3}{2x^3 - 8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2 - 2x + 1}$

19. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

Irreducible Quadratic Factors

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\,dt$

23. $\displaystyle\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}\,dx$

25. $\displaystyle\int \frac{2s + 2}{(s^2 + 1)(s - 1)^3}\,ds$

26. $\displaystyle\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

27. $\displaystyle\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\,d\theta$

28. $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}\,d\theta$

Improper Fractions

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

29. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

30. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

31. $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

32. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

33. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

34. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 35–40.

35. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

36. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

37. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

38. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

39. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$

40. $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 41–44 for x as a function of t.

41. $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

42. $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

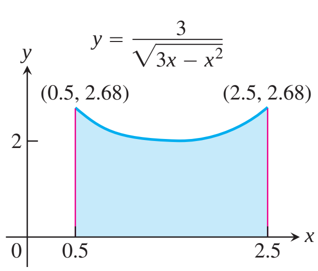
43. $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

44. $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = \pi/4$

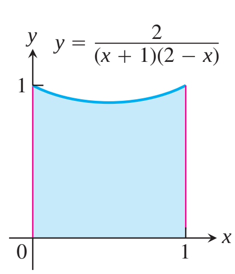
Applications and Examples

In Exercises 45 and 46, find the volume of the solid generated by revolving the shaded region about the indicated axis.

45. The x-axis

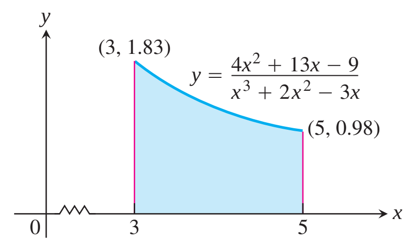


46. The y-axis



47. Find, to two decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

48. Find the x-coordinate of the centroid of this region to two decimal places.



49. Social diffusion Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a. Find x as a function of t.

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

50. Second-order chemical reactions Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

51. An integral connecting π to the approximation 22/7

a. Evaluate $\displaystyle\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx$.

b. How good is the approximation $\pi \approx 22/7$? Find out by expressing $\left(\dfrac{22}{7} - \pi\right)$ as a percentage of π.

c. Graph the function $y = \dfrac{x^4(x-1)^4}{x^2+1}$ for $0 \le x \le 1$. Experiment with the range on the y-axis set between 0 and 1, then between 0 and 0.5, and then decreasing the range until the graph can be seen. What do you conclude about the area under the curve?

52. Find the second-degree polynomial $P(x)$ such that $P(0) = 1$, $P'(0) = 0$, and

$$\int \frac{P(x)}{x^3(x-1)^2}\,dx$$

is a rational function.

8.4 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 m is Odd

Evaluate

$$\int \sin^3 x \cos^2 x\, dx.$$

Solution

$$
\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d\,(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2) \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}
$$

EXAMPLE 2 m is Even and n is Odd

Evaluate

$$\int \cos^5 x \, dx.$$

Solution

$$
\begin{aligned}
\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx &= \int (1 - \sin^2 x)^2 \, d(\sin x) && m = 0 \\
&= \int (1 - u^2)^2 \, du && u = \sin x \\
&= \int (1 - 2u^2 + u^4) \, du \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.
\end{aligned}
$$

EXAMPLE 3 m and n are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution

$$
\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx \\
&= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx\right].
\end{aligned}
$$

For the term involving $\cos^2 2x$ we use

$$\int \cos^2 2x\,dx = \frac{1}{2}\int (1 + \cos 4x)\,dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)\cos 2x\,dx$$

$u = \sin 2x,$
$du = 2\cos 2x\,dx$

$$= \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x\,dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx.$$

Solution To eliminate the square root we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\,dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx$$

$\cos 2x \geq 0$ on $[0, \pi/4]$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.$$

■

Integrals of Powers of tan *x* and sec *x*

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x \, dx .$$

Solution

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx .\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1 .$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C .$$

■

EXAMPLE 6 Evaluate

$$\int \sec^3 x \, dx .$$

Solution We integrate by parts, using

$$u = \sec x, \qquad dv = \sec^2 x \, dx, \qquad v = \tan x, \qquad du = \sec x \tan x \, dx .$$

Then

$$\begin{aligned}\int \sec^3 x \, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x \, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx && \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx .\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C. \quad \blacksquare$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \qquad \int \sin mx \cos nx\, dx, \qquad \text{and} \qquad \int \cos mx \cos nx\, dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos (m - n)x - \cos (m + n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin (m - n)x + \sin (m + n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos (m - n)x + \cos (m + n)x]. \tag{5}$$

These come from the angle sum formulas for the sine and cosine functions (Section 1.6). They give functions whose antiderivatives are easily found.

You Try It

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x\, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned}
\int \sin 3x \cos 5x\, dx &= \frac{1}{2}\int [\sin (-2x) + \sin 8x]\, dx \\
&= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\
&= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \quad \blacksquare
\end{aligned}$$

EXERCISES 8.4

Products of Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int_0^{\pi/2} \sin^5 x\, dx$

2. $\displaystyle\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$

3. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx$

4. $\displaystyle\int_0^{\pi/6} 3\cos^5 3x\, dx$

5. $\displaystyle\int_0^{\pi/2} \sin^7 y\, dy$

6. $\displaystyle\int_0^{\pi/2} 7\cos^7 t\, dt$

7. $\int_0^{\pi} 8\sin^4 x\,dx$

8. $\int_0^{1} 8\cos^4 2\pi x\,dx$

9. $\int_{-\pi/4}^{\pi/4} 16\sin^2 x\cos^2 x\,dx$

10. $\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy$

11. $\int_0^{\pi/2} 35\sin^4 x\cos^3 x\,dx$

12. $\int_0^{\pi} \sin 2x\cos^2 2x\,dx$

13. $\int_0^{\pi/4} 8\cos^3 2\theta\sin 2\theta\,d\theta$

14. $\int_0^{\pi/2} \sin^2 2\theta\cos^3 2\theta\,d\theta$

Integrals with Square Roots

Evaluate the integrals in Exercises 15–22.

15. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$

16. $\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

17. $\int_0^{\pi} \sqrt{1-\sin^2 t}\,dt$

18. $\int_0^{\pi} \sqrt{1-\cos^2\theta}\,d\theta$

19. $\int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2 x}\,dx$

20. $\int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x-1}\,dx$

21. $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta$

22. $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\,dt$

Powers of Tan x and Sec x

Evaluate the integrals in Exercises 23–32.

23. $\int_{-\pi/3}^{0} 2\sec^3 x\,dx$

24. $\int e^x\sec^3 e^x\,dx$

25. $\int_0^{\pi/4} \sec^4\theta\,d\theta$

26. $\int_0^{\pi/12} 3\sec^4 3x\,dx$

27. $\int_{\pi/4}^{\pi/2} \csc^4\theta\,d\theta$

28. $\int_{\pi/2}^{\pi} 3\csc^4\frac{\theta}{2}\,d\theta$

29. $\int_0^{\pi/4} 4\tan^3 x\,dx$

30. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\,dx$

31. $\int_{\pi/6}^{\pi/3} \cot^3 x\,dx$

32. $\int_{\pi/4}^{\pi/2} 8\cot^4 t\,dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 33–38.

33. $\int_{-\pi}^{0} \sin 3x\cos 2x\,dx$

34. $\int_0^{\pi/2} \sin 2x\cos 3x\,dx$

35. $\int_{-\pi}^{\pi} \sin 3x\sin 3x\,dx$

36. $\int_0^{\pi/2} \sin x\cos x\,dx$

37. $\int_0^{\pi} \cos 3x\cos 4x\,dx$

38. $\int_{-\pi/2}^{\pi/2} \cos x\cos 7x\,dx$

Theory and Examples

39. **Surface area** Find the area of the surface generated by revolving the arc

$$x = t^{2/3},\quad y = t^2/2,\quad 0 \le t \le 2,$$

about the x-axis.

40. **Arc length** Find the length of the curve

$$y = \ln(\cos x),\quad 0 \le x \le \pi/3.$$

41. **Arc length** Find the length of the curve

$$y = \ln(\sec x),\quad 0 \le x \le \pi/4.$$

42. **Center of gravity** Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

43. **Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

44. **Area** Find the area between the x-axis and the curve $y = \sqrt{1+\cos 4x}$, $0 \le x \le \pi$.

45. **Orthogonal functions** Two functions f and g are said to be **orthogonal** on an interval $a \le x \le b$ if $\int_a^b f(x)g(x)\,dx = 0$.

a. Prove that $\sin mx$ and $\sin nx$ are orthogonal on any interval of length 2π provided m and n are integers such that $m^2 \ne n^2$.

b. Prove the same for $\cos mx$ and $\cos nx$.

c. Prove the same for $\sin mx$ and $\cos nx$ even if $m = n$.

46. **Fourier series** A finite Fourier series is given by the sum

$$f(x) = \sum_{n=1}^{N} a_n\sin nx = a_1\sin x + a_2\sin 2x + \cdots + a_N\sin Nx$$

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

8.5 Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$. They come from the reference right triangles in Figure 8.2.

With $x = a\tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

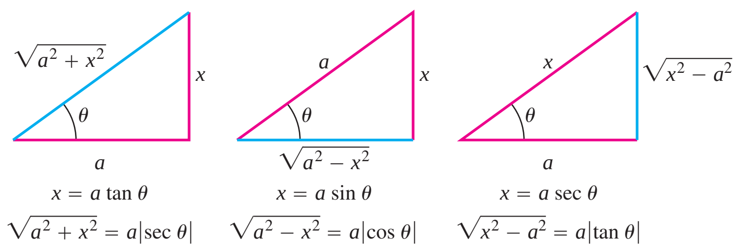


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

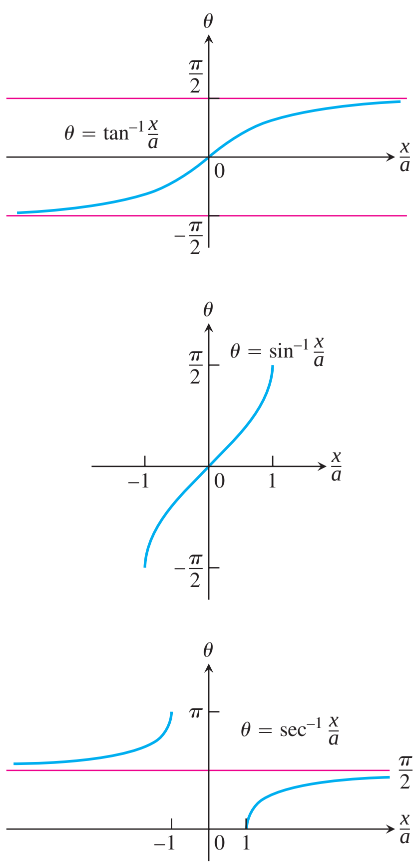


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

As we know from Section 7.7, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\ \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

EXAMPLE 1 Using the Substitution $x = a\tan\theta$

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

You Try It

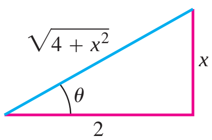


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$
$$4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta.$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} && \sqrt{\sec^2\theta} = |\sec\theta| \\
&= \int \sec\theta\, d\theta && \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C && \text{From Fig. 8.4} \\
&= \ln\left|\sqrt{4+x^2} + x\right| + C'. && \text{Taking } C' = C - \ln 2
\end{aligned}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

EXAMPLE 2 Using the Substitution $x = a\sin\theta$

You Try It

Evaluate

$$\int \frac{x^2\, dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

$$\begin{aligned}
\int \frac{x^2\, dx}{\sqrt{9-x^2}} &= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|} \\
&= 9\int \sin^2\theta\, d\theta && \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= 9\int \frac{1-\cos 2\theta}{2}\, d\theta \\
&= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C && \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C && \text{Fig. 8.5} \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.
\end{aligned}$$

■

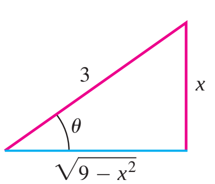


FIGURE 8.5 Reference triangle for $x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

EXAMPLE 3 Using the Substitution $x = a\sec\theta$

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)} = 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for $0 < \theta < \pi/2$

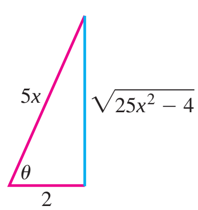


FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 3).

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\,d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$

Fig. 8.6 ■

A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.

EXAMPLE 4 Finding the Volume of a Solid of Revolution

Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2 + 4)$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution We sketch the region (Figure 8.7) and use the disk method:

$$V = \int_0^2 \pi[R(x)]^2\,dx = 16\pi\int_0^2 \frac{dx}{(x^2 + 4)^2}.$$

$R(x) = \frac{4}{x^2 + 4}$

To evaluate the integral, we set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\,d\theta, \qquad \theta = \tan^{-1}\frac{x}{2},$$

$$x^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$$

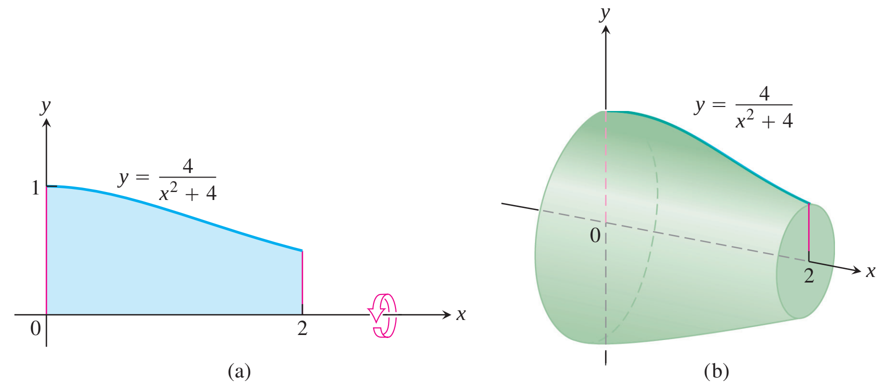


FIGURE 8.7 The region (a) and solid (b) in Example 4.

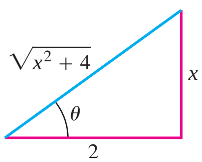


FIGURE 8.8 Reference triangle for $x = 2 \tan \theta$ (Example 4).

(Figure 8.8). With these substitutions,

$$V = 16\pi \int_0^2 \frac{dx}{(x^2 + 4)^2}$$

$$= 16\pi \int_0^{\pi/4} \frac{2 \sec^2 \theta \, d\theta}{(4 \sec^2 \theta)^2}$$

$\theta = 0$ when $x = 0$; $\theta = \pi/4$ when $x = 2$

$$= 16\pi \int_0^{\pi/4} \frac{2 \sec^2 \theta \, d\theta}{16 \sec^4 \theta} = \pi \int_0^{\pi/4} 2 \cos^2 \theta \, d\theta$$

$$= \pi \int_0^{\pi/4} (1 + \cos 2\theta) \, d\theta = \pi \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$2 \cos^2 \theta = 1 + \cos 2\theta$

$$= \pi \left[\frac{\pi}{4} + \frac{1}{2} \right] \approx 4.04.$$

■

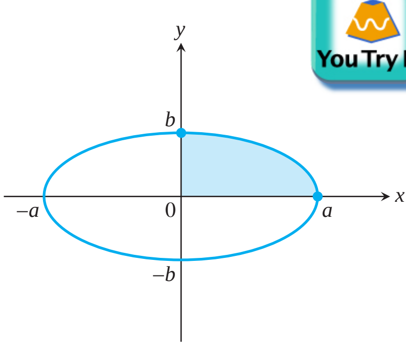


FIGURE 8.9 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in Example 5.

EXAMPLE 5 Finding the Area of an Ellipse

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (Figure 8.9). Solving the equation of the ellipse for $y \geq 0$, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2},$$

or

$$y = \frac{b}{a} \sqrt{a^2 - x^2} \qquad 0 \leq x \leq a$$

The area of the ellipse is

$$\begin{aligned}
A &= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx \\
&= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta \\
&= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= 2ab\left[\frac{\pi}{2} + 0 - 0\right] = \pi ab.
\end{aligned}$$

$x = a\sin\theta$, $dx = a\cos\theta\,d\theta$,
$\theta = 0$ when $x = 0$;
$\theta = \pi/2$ when $x = a$

If $a = b = r$ we get that the area of a circle with radius r is πr^2. ■

EXERCISES 8.5

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int \frac{dy}{\sqrt{9+y^2}}$

2. $\displaystyle\int \frac{3\,dy}{\sqrt{1+9y^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

15. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

17. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

18. $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

19. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

20. $\displaystyle\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

21. $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

22. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

23. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

24. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\displaystyle\int \frac{8\,dx}{(4x^2+1)^2}$

26. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\displaystyle\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

28. $\displaystyle\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

30. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

32. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

34. $\displaystyle\int \frac{dx}{1+x^2}$

35. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

36. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $\displaystyle x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

38. $\displaystyle \sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $\displaystyle (x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$

40. $\displaystyle (x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2/3}$.

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1 + x^2)$, and the line $x = 1$.

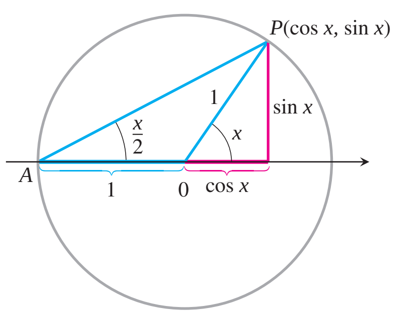
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{1}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\begin{aligned}
\cos x &= 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1 \\
&= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1
\end{aligned}$$

$$\cos x = \frac{1 - z^2}{1 + z^2}, \tag{2}$$

and

$$\begin{aligned}
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)} \cdot \cos^2\left(\frac{x}{2}\right) \\
&= 2\tan\frac{x}{2} \cdot \frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}
\end{aligned}$$

$$\sin x = \frac{2z}{1 + z^2}. \tag{3}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{4}$$

Examples

a.
$$\begin{aligned}
\int \frac{1}{1 + \cos x}\,dx &= \int \frac{1 + z^2}{2}\,\frac{2\,dz}{1 + z^2} \\
&= \int dz = z + C \\
&= \tan\left(\frac{x}{2}\right) + C
\end{aligned}$$

b.
$$\begin{aligned}
\int \frac{1}{2 + \sin x}\,dx &= \int \frac{1 + z^2}{2 + 2z + 2z^2}\,\frac{2\,dz}{1 + z^2} \\
&= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4} \\
&= \int \frac{du}{u^2 + a^2} \\
&= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \\
&= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z + 1}{\sqrt{3}} + C \\
&= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1 + 2\tan(x/2)}{\sqrt{3}} + C
\end{aligned}$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 43–50. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

43. $\displaystyle\int \frac{dx}{1 - \sin x}$

44. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

46. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$

47. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}$

48. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta + \sin\theta}$

49. $\displaystyle\int \frac{dt}{\sin t - \cos t}$

50. $\displaystyle\int \frac{\cos t\,dt}{1 - \cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \sec\theta\,d\theta$

52. $\displaystyle\int \csc\theta\,d\theta$