EXAMPLE 5

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$

$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$ ■

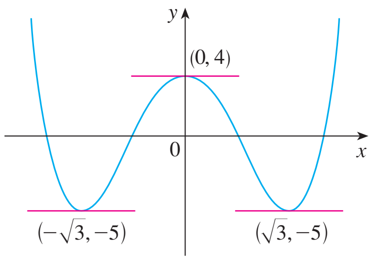

FIGURE 5
The curve $y = x^4 - 6x^2 + 4$ and its horizontal tangents

EXAMPLE 6 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$$

$$= 4x^3 - 12x + 0 = 4x(x^2 - 3)$$

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0$, $\sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 5.) ■

EXAMPLE 7 The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

SOLUTION The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 14 \text{ cm/s}^2$. ■

Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = b^x$ using the definition of a derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h\to 0}\frac{b^x b^h - b^x}{h} = \lim_{h\to 0}\frac{b^x(b^h - 1)}{h}$$

The factor b^x doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = b^x \lim_{h\to 0}\frac{b^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \to 0} \frac{b^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = b^x$ is differentiable at 0, then it is differentiable everywhere and

4 $$f'(x) = f'(0)b^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself*. (The slope is proportional to the height.)

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $b = 2$ and $b = 3$. (Values are stated correct to four decimal places.) It appears that the limits exist and

$$\text{for } b = 2, \qquad f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$$

$$\text{for } b = 3, \qquad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$$

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

$$\frac{d}{dx}(2^x)\bigg|_{x=0} \approx 0.693147 \qquad \frac{d}{dx}(3^x)\bigg|_{x=0} \approx 1.098612$$

Thus, from Equation 4, we have

5 $$\frac{d}{dx}(2^x) \approx (0.69)2^x \qquad \frac{d}{dx}(3^x) \approx (1.10)3^x$$

Of all possible choices for the base b in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$. In view of the estimates of $f'(0)$ for $b = 2$ and $b = 3$, it seems reasonable that there is a number b between 2 and 3 for which $f'(0) = 1$. It is traditional to denote this value by the letter e. (In fact, that is how we introduced e in Section 1.4.) Thus we have the following definition.

In Exercise 1 we will see that e lies between 2.7 and 2.8. Later we will be able to show that, correct to five decimal places,

$$e \approx 2.71828$$

Definition of the Number *e*

$$e \text{ is the number such that } \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Geometrically, this means that of all the possible exponential functions $y = b^x$, the function $f(x) = e^x$ is the one whose tangent line at (0, 1) has a slope $f'(0)$ that is exactly 1. (See Figures 6 and 7.)

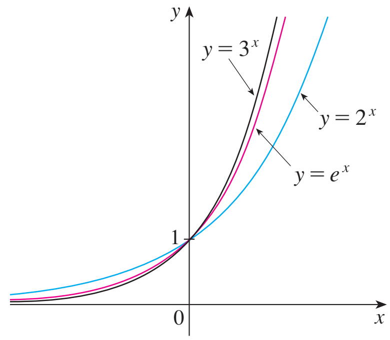

FIGURE 6

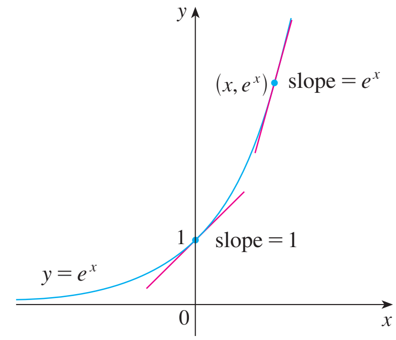

FIGURE 7

If we put $b = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

TEC Visual 3.1 uses the slope-a-scope to illustrate this formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y-coordinate of the point (see Figure 7).

EXAMPLE 8 If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

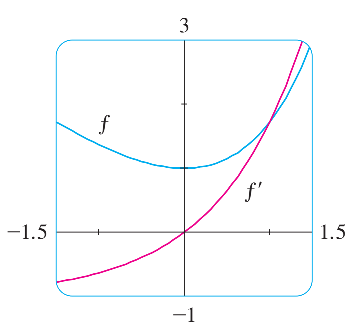

FIGURE 8

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ■

EXAMPLE 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x-coordinate of the point in question be a. Then the slope of the tangent line at that point is e^a. This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \qquad a = \ln 2$$

Therefore the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.) ■

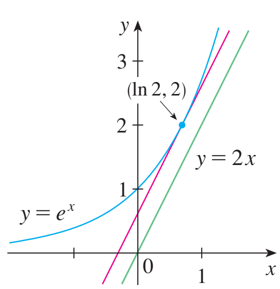

FIGURE 9

3.1 EXERCISES

1. (a) How is the number e defined?
(b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
(b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
(c) Which of the two functions in part (b) grows more rapidly when x is large?

3–32 Differentiate the function.

3. $f(x) = 2^{40}$

4. $f(x) = e^5$

5. $f(x) = 5.2x + 2.3$

6. $g(x) = \frac{7}{4}x^2 - 3x + 12$

7. $f(t) = 2t^3 - 3t^2 - 4t$

8. $f(t) = 1.4t^5 - 2.5t^2 + 6.7$

9. $g(x) = x^2(1 - 2x)$

10. $H(u) = (3u - 1)(u + 2)$

11. $g(t) = 2t^{-3/4}$

12. $B(y) = cy^{-6}$

13. $F(r) = \dfrac{5}{r^3}$

14. $y = x^{5/3} - x^{2/3}$

15. $R(a) = (3a + 1)^2$

16. $h(t) = \sqrt[4]{t} - 4e^t$

17. $S(p) = \sqrt{p} - p$

18. $y = \sqrt[3]{x}(2 + x)$

19. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$

20. $S(R) = 4\pi R^2$

21. $h(u) = Au^3 + Bu^2 + Cu$

22. $y = \dfrac{\sqrt{x} + x}{x^2}$

23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

24. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t}$

25. $j(x) = x^{2.4} + e^{2.4}$

26. $k(r) = e^r + r^e$

27. $G(q) = (1 + q^{-1})^2$

28. $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$

29. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v}$

30. $D(t) = \dfrac{1 + 16t^2}{(4t)^3}$

31. $z = \dfrac{A}{y^{10}} + Be^y$

32. $y = e^{x+1} + 1$

33–36 Find an equation of the tangent line to the curve at the given point.

33. $y = 2x^3 - x^2 + 2, \quad (1, 3)$

34. $y = 2e^x + x, \quad (0, 2)$

35. $y = x + \dfrac{2}{x}, \quad (2, 3)$

36. $y = \sqrt[4]{x} - x, \quad (1, 0)$

37–38 Find equations of the tangent line and normal line to the curve at the given point.

37. $y = x^4 + 2e^x, \quad (0, 2)$

38. $y^2 = x^3, \quad (1, 1)$

39–40 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

39. $y = 3x^2 - x^3, \quad (1, 2)$

40. $y = x - \sqrt{x}, \quad (1, 0)$

41–42 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

41. $f(x) = x^4 - 2x^3 + x^2$

42. $f(x) = x^5 - 2x^3 + x - 1$

43. (a) Graph the function

$$f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$$

in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 2.8.1.)
(c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

44. (a) Graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 2.8.1.)
(c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g'. Compare with your sketch in part (b).

45–46 Find the first and second derivatives of the function.

45. $f(x) = 0.001x^5 - 0.02x^3$

46. $G(r) = \sqrt{r} + \sqrt[3]{r}$

47–48 Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of f, f', and f''.

47. $f(x) = 2x - 5x^{3/4}$

48. $f(x) = e^x - x^3$