<table>
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Biography

Many of the phenomena we want information about are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine:

$$\sin (x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \qquad \text{Sine angle sum identity}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right)$$

$$= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 \qquad \text{Example 5(a) and}$$
$$\qquad\qquad\qquad\qquad\qquad \text{Theorem 7, Section 2.4}$$

$$= \cos x.$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx} (\sin x) = \cos x.$$

EXAMPLE 1   Derivatives Involving the Sine

You Try It

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx} (\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x.$$

Video

(b) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx} (\sin x) + 2x \sin x \qquad \text{Product Rule}$$

$$= x^2 \cos x + 2x \sin x.$$

(c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \dfrac{d}{dx} (\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}. \qquad\qquad\qquad \blacksquare$$

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine,
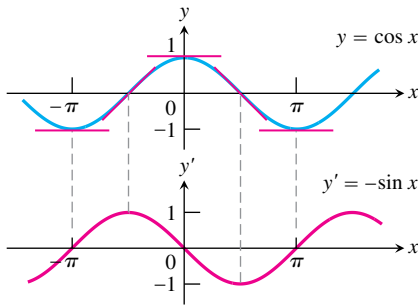
$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we have

FIGURE 3.23   The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \qquad \begin{array}{l}\text{Cosine angle sum} \\ \text{identity}\end{array}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 \qquad \begin{array}{l}\text{Example 5(a) and} \\ \text{Theorem 7, Section 2.4}\end{array}$$

$$= -\sin x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Figure 3.23 shows a way to visualize this result.

EXAMPLE 2   Derivatives Involving the Cosine

(a) $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$

$$= 5 - \sin x.$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x.$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\dfrac{d}{dx}(\cos x) - \cos x \dfrac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$
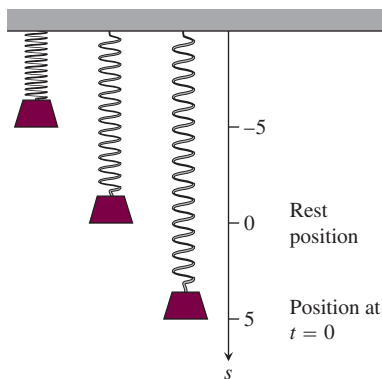
$$= \frac{1}{1 - \sin x}.$$

FIGURE 3.24 A body hanging from a vertical spring and then displaced oscillates above and below its rest position. Its motion is described by trigonometric functions (Example 3).
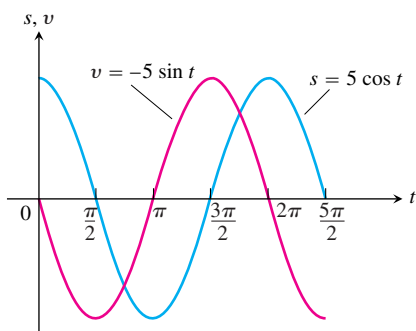

FIGURE 3.25 The graphs of the position and velocity of the body in Example 3.

Simple Harmonic Motion

The motion of a body bobbing freely up and down on the end of a spring or bungee cord is an example of *simple harmonic motion*. The next example describes a case in which there are no opposing forces such as friction or buoyancy to slow the motion down.

EXAMPLE 3   Motion on a Spring

A body hanging from a spring (Figure 3.24) is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t?

Solution   We have

Position: $s = 5 \cos t$

Velocity: $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(5 \cos t) = -5 \sin t$

Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(-5 \sin t) = -5 \cos t.$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π.

2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.25. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion.

3. The acceleration value is always the exact opposite of the position value. When the weight is above the rest position, gravity is pulling it back down; when the weight is below the rest position, the spring is pulling it back up.

4. The acceleration, $a = -5 \cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring offset each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$.   ∎

EXAMPLE 4   Jerk

The jerk of the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t) = 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign.   ∎

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

To show a typical calculation, we derive the derivative of the tangent function. The other derivations are left to Exercise 50.

EXAMPLE 5

Find $d(\tan x)/dx$.

Solution

$$\frac{d}{dx}\left(\tan x\right) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}\left(\sin x\right) - \sin x \frac{d}{dx}\left(\cos x\right)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \qquad \blacksquare$$

EXAMPLE 6

Find y'' if $y = \sec x$.

Solution

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}\left(\tan x\right) + \tan x \frac{d}{dx}\left(\sec x\right) \qquad \text{Product Rule}$$

$$= \sec x\,(\sec^2 x) + \tan x\,(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x \qquad \blacksquare$$

The differentiability of the trigonometric functions throughout their domains gives another proof of their continuity at every point in their domains (Theorem 1, Section 3.1). So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 7   Finding a Trigonometric Limit

$$\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3} \qquad \blacksquare$$

EXERCISES 3.4

Derivatives

In Exercises 1–12, find dy/dx.

1. $y = -10x + 3\cos x$

2. $y = \dfrac{3}{x} + 5\sin x$

3. $y = \csc x - 4\sqrt{x} + 7$

4. $y = x^2 \cot x - \dfrac{1}{x^2}$

5. $y = (\sec x + \tan x)(\sec x - \tan x)$

6. $y = (\sin x + \cos x)\sec x$

7. $y = \dfrac{\cot x}{1 + \cot x}$

8. $y = \dfrac{\cos x}{1 + \sin x}$

9. $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$

10. $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

11. $y = x^2 \sin x + 2x \cos x - 2\sin x$

12. $y = x^2 \cos x - 2x \sin x - 2\cos x$

In Exercises 13–16, find ds/dt.

13. $s = \tan t - t$

14. $s = t^2 - \sec t + 1$

15. $s = \dfrac{1 + \csc t}{1 - \csc t}$

16. $s = \dfrac{\sin t}{1 - \cos t}$

In Exercises 17–20, find $dr/d\theta$.

17. $r = 4 - \theta^2 \sin \theta$

18. $r = \theta \sin \theta + \cos \theta$

19. $r = \sec \theta \csc \theta$

20. $r = (1 + \sec \theta)\sin \theta$

In Exercises 21–24, find dp/dq.

21. $p = 5 + \dfrac{1}{\cot q}$

22. $p = (1 + \csc q)\cos q$

23. $p = \dfrac{\sin q + \cos q}{\cos q}$

24. $p = \dfrac{\tan q}{1 + \tan q}$

25. Find y'' if

a. $y = \csc x$.

b. $y = \sec x$.

26. Find $y^{(4)} = d^4 y/dx^4$ if

a. $y = -2\sin x$.

**b.** $y = 9\cos x$.

Tangent Lines

In Exercises 27–30, graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

27. $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$

$x = -\pi, 0, 3\pi/2$

28. $y = \tan x, \quad -\pi/2 < x < \pi/2$

$x = -\pi/3, 0, \pi/3$

29. $y = \sec x, \quad -\pi/2 < x < \pi/2$

$x = -\pi/3, \pi/4$

30. $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$

$x = -\pi/3, 3\pi/2$

T Do the graphs of the functions in Exercises 31–34 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not? Visualize your findings by graphing the functions with a grapher.

31. $y = x + \sin x$

32. $y = 2x + \sin x$

33. $y = x - \cot x$

34. $y = x + 2\cos x$

35. Find all points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

36. Find all points on the curve $y = \cot x, 0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

In Exercises 37 and 38, find an equation for **(a)** the tangent to the curve at P and **(b)** the horizontal tangent to the curve at Q.

37.

$y = 4 + \cot x - 2\csc x$

38.

$y = 1 + \sqrt{2}\csc x + \cot x$
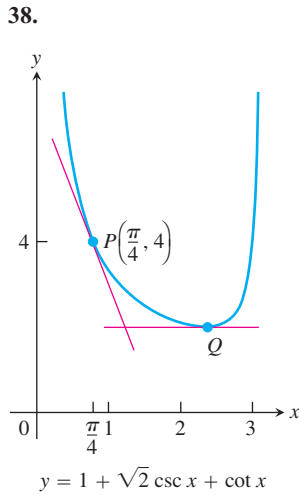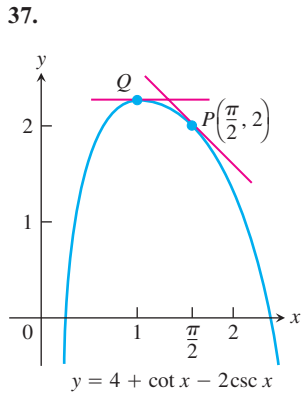
Trigonometric Limits

Find the limits in Exercises 39–44.

39. $\lim\limits_{x \to 2} \sin\left(\dfrac{1}{x} - \dfrac{1}{2}\right)$

40. $\lim\limits_{x \to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

41. $\lim\limits_{x \to 0} \sec\left[\cos x + \pi \tan\left(\dfrac{\pi}{4\sec x}\right) - 1\right]$

42. $\lim\limits_{x \to 0} \sin\left(\dfrac{\pi + \tan x}{\tan x - 2\sec x}\right)$

43. $\lim\limits_{t \to 0} \tan\left(1 - \dfrac{\sin t}{t}\right)$

44. $\lim\limits_{\theta \to 0} \cos\left(\dfrac{\pi\theta}{\sin\theta}\right)$

Simple Harmonic Motion

The equations in Exercises 45 and 46 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

45. $s = 2 - 2\sin t$ **46.** $s = \sin t + \cos t$

Theory and Examples

47. Is there a value of c that will make

$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

48. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

49. Find $d^{999}/dx^{999}(\cos x)$.

50. Derive the formula for the derivative with respect to x of

a. $\sec x$.   **b.** $\csc x$.   **c.** $\cot x$.

T 51. Graph $y = \cos x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\sin(x + h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?

T 52. Graph $y = -\sin x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\cos(x + h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?
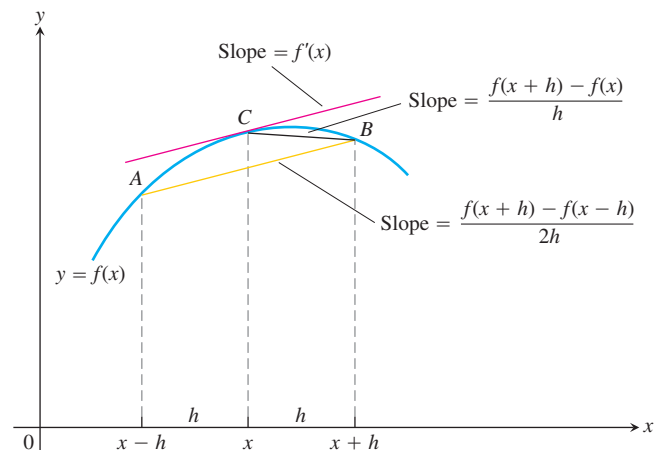
T 53. Centered difference quotients The *centered difference quotient*

$$\frac{f(x + h) - f(x - h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than Fermat's difference quotient

$$\frac{f(x + h) - f(x)}{h}.$$

See the accompanying figure.

a. To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x + h) - \sin(x - h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 51 for the same values of h.

b. To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x + h) - \cos(x - h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 52 for the same values of h.

54. A caution about centered difference quotients   (*Continuation of Exercise 53.*) The quotient

$$\frac{f(x + h) - f(x - h)}{2h}$$

may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \to 0} \frac{|0 + h| - |0 - h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

T 55. Slopes on the graph of the tangent function   Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? a largest slope? Is the slope ever negative? Give reasons for your answers.

T 56. Slopes on the graph of the cotangent function   Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? A largest slope? Is the slope ever positive? Give reasons for your answers.

T 57. Exploring (sin kx)/x   Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the y-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k? Give reasons for your answers.

T 58. Radians versus degrees: degree mode derivatives   What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

a. With your graphing calculator or computer grapher in *degree mode*, graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h \to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

b. With your grapher still in degree mode, estimate

$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

c. Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

d. Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?

e. The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?