Chapter # 3
Differentiation

3.1: Derivative as a Function

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

We use the notation $f(x)$ rather than simply f in the definition to emphasize the independent variable x, which we are differentiating with respect to. The domain of f' is the set of points in the domain of f for which the limit exists, and the domain may be the same or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative)** at x. If f' exists at every point in the domain of f, we call f **differentiable**.

EXAMPLE 1 Applying the Definition

Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution Here we have $f(x) = \dfrac{x}{x-1}$

and

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. \end{aligned}$$

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EXAMPLE 2 Derivative of the Square Root Function

(a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

(a) We use the equivalent form to calculate f':

$$\begin{aligned}
f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)} \\
&= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation and are called **differentiation operators**. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio (until we introduce the idea of "differentials" in Section 3.8).

Be careful not to confuse the notation $D(f)$ as meaning the domain of the function f instead of the derivative function f'. The distinction should be clear from the context.

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

For instance, in Example 2b we could write

$$f'(4) = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

To evaluate an expression, we sometimes use the right bracket] in place of the vertical bar | .

EXAMPLE 6 $y = \sqrt{x}$ Is Not Differentiable at $x = 0$

In Example 2 we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h \to 0^+} \frac{\sqrt{0 + h} - \sqrt{0}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(h, \sqrt{h})$ on a graph of $y = \sqrt{x}$ approach ∞, the graph has a *vertical tangent* at the origin. ■

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1 **Differentiability Implies Continuity**
If f has a derivative at $x = c$, then f is continuous at $x = c$.

CAUTION The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, ~~as we saw in Example 5.~~

EXERCISES 3.1

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$
2. $F(x) = (x - 1)^2 + 1; \quad F'(-1), F'(0), F'(2)$
3. $g(t) = \dfrac{1}{t^2}; \quad g'(-1), g'(2), g'\left(\sqrt{3}\right)$
4. $k(z) = \dfrac{1 - z}{2z}; \quad k'(-1), k'(1), k'\left(\sqrt{2}\right)$
5. $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s + 1}; \quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\dfrac{dy}{dx}$ if $y = 2x^3$
8. $\dfrac{dr}{ds}$ if $r = \dfrac{s^3}{2} + 1$

Differentiation Rules

Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

EXAMPLE 1

If f has the constant value $f(x) = 8$, then

$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

Similarly,

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0.$$

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RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if n is a positive integer, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

EXAMPLE 3

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.9).

(b) A useful special case

The derivative of the negative of a differentiable function u is the negative of the function's derivative. Rule 3 with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 4 Derivative of a Sum

$$\begin{aligned} y &= x^4 + 12x \\ \frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \\ &= 4x^3 + 12 \end{aligned}$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule,** which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives.

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

EXAMPLE 5 Derivative of a Polynomial

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0$$

$$= 3x^2 + \frac{8}{3}x - 5$$

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RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation*, $(uv)' = uv' + vu'$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

EXAMPLE 7 Using the Product Rule

Find the derivative of

$$y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right).$$

Solution We apply the Product Rule with $u = 1/x$ and $v = x^2 + (1/x)$:

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{x}\left(x^2 + \frac{1}{x}\right)\right] &= \frac{1}{x}\left(2x - \frac{1}{x^2}\right) + \left(x^2 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\\ &= 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3}\\ &= 1 - \frac{2}{x^3}.\end{aligned}$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ by Example 3, Section 2.7.

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EXAMPLE 9 Differentiating a Product in Two Ways

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[\left(x^2 + 1\right)\left(x^3 + 3\right)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$
$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x.$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$
$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. ■

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 10 Using the Quotient Rule

Find the derivative of

$$y = \frac{t^2 - 1}{t^2 + 1}.$$

Solution

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\begin{aligned} \frac{dy}{dt} &= \frac{(t^2 + 1)\cdot 2t - (t^2 - 1)\cdot 2t}{(t^2 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2 + 1)^2} \\ &= \frac{4t}{(t^2 + 1)^2}. \end{aligned}$$

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Negative Integer Powers of *x*

The Power Rule for negative integers is the same as the rule for positive integers.

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 11

(a) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$ Agrees with Example 3, Section 2.7

(b) $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$ ■

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. Notationally,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(6x^5\right) = 30x^4.$$

Thus $D^2\left(x^6\right) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n.

EXAMPLE 14 Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

$$\begin{aligned}
&\text{First derivative:} && y' = 3x^2 - 6x \\
&\text{Second derivative:} && y'' = 6x - 6 \\
&\text{Third derivative:} && y''' = 6 \\
&\text{Fourth derivative:} && y^{(4)} = 0.
\end{aligned}$$

The function has derivatives of all orders, the fifth and later derivatives all being zero.

Exercise

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \frac{4x^3}{3} - x$
6. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$
7. $w = 3z^{-2} - \frac{1}{z}$
8. $s = -2t^{-1} + \frac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \frac{1}{3s^2} - \frac{5}{2s}$
12. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

In Exercises 13–16, find y' **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (x - 1)(x^2 + x + 1)$
15. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$
16. $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$

Find the derivatives of the functions in Exercises 17–28.

17. $y = \frac{2x + 5}{3x - 2}$

18. $z = \frac{2x + 1}{x^2 - 1}$

19. $g(x) = \frac{x^2 - 4}{x + 0.5}$

20. $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$

21. $v = (1 - t)(1 + t^2)^{-1}$

22. $w = (2x - 7)^{-1}(x + 5)$

23. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$

24. $u = \frac{5x + 1}{2\sqrt{x}}$

25. $v = \frac{1 + x - 4\sqrt{x}}{x}$

26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

27. $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$

28. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the derivatives of all orders of the functions in Exercises 29 and 30.

29. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$

30. $y = \frac{x^5}{120}$

Find the first and second derivatives of the functions in Exercises 31–38.

31. $y = \frac{x^3 + 7}{x}$

32. $s = \frac{t^2 + 5t - 1}{t^2}$

33. $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

34. $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$

35. $w = \left(\frac{1 + 3z}{3z}\right)(3 - z)$

36. $w = (z + 1)(z - 1)(z^2 + 1)$