Chapter# 2
Limits and Continuity

Sections Covered: 2.3,2.4,2.6

2.3 *Precise Definition of Limit*

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

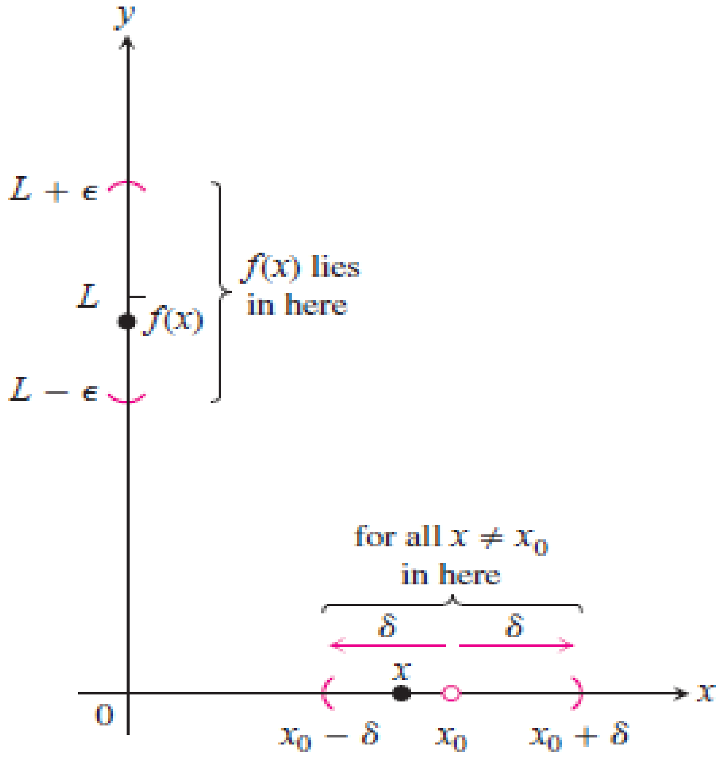


FIGURE 2.14 The relation of δ and ϵ in the definition of limit.

Example

Show that

$$\lim_{x \to 1} (5x-3) = 2$$

by using definition of Limit.

Solution

Here $x_0 = 1$, $f(x) = 5x-3$, $L = 2$.

So definition of limit is

$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$

So

$0 < |x-1| < \delta \implies |f(x) - 2| < \varepsilon$

We find δ by working backward from the ε-inequality.

$$|(5x-3) - 2| = |5x-3-2| < \varepsilon$$

$$|5x-5| < \varepsilon$$

$$5|x-1| < \varepsilon$$

$$|x-1| < \varepsilon/5 \longrightarrow (1)$$

Thus we take $\delta = \varepsilon/5$.

If $0 < |x-1| < \delta = \varepsilon/5$, then

$|5x-3-2| = |5x-5| = 5|x-1| < 5(\varepsilon/3) = \varepsilon$ from (1).

which proves that

$$\lim_{x \to 1} (5x-3) = 2$$

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Example <u>Finding Delta Algebraically.</u>

For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that $\forall x$

$$0 < |x-5| < \delta \implies |\sqrt{x-1} - 2| < 1$$

Solution

① Solve the inequality $|\sqrt{x-1} - 2| < 1$ to find an interval $x_0 = 5$ on ~~the~~ which the inequality holds $\forall$ ~~x~~ $x \neq x_0$

$$|\sqrt{x-1} - 2| < 1$$

$$\Rightarrow -1 < (\sqrt{x-1} - 2) < 1$$

$$\Rightarrow -1 + 2 < \sqrt{x-1} < 1 + 2$$

$$\Rightarrow 1 < \sqrt{x-1} < 3$$

$$\Rightarrow (1)^2 < (\sqrt{x-1})^2 < (3)^2$$

$$1 < x-1 < 9$$

$$\Rightarrow \boxed{2 < x < 10} \longrightarrow ①$$

So the inequality holds for open interval $(2, 10)$ except possibly at $x = 5$.

② $|x - 5| < \delta$

$$\Rightarrow -\delta < (x - 5) < \delta$$

$$\Rightarrow \boxed{-\delta + 5 < x < \delta + 5} \longrightarrow ②$$

Comparing ① & ②

$-\delta + 5 = 2$ & $+\delta + 5 = 10$

$\Rightarrow \delta = 3$ & $\delta = 5$

So the distance from 5 to the nearer endpoint of $(2,10)$ is 3. If we take $\delta = 3$, then the inequality $0 < |x-5| < \delta$ will automatically place x between 2 & 10 to make $|\sqrt{x-1} - 2| < 1$

So

$$0 < |x-5| < 3 \Rightarrow |\sqrt{x-1} - 2| < 1.$$

Exercise

Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, x_0 and $\epsilon > 0$. In each case, find an open interval about x_0 on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \epsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \epsilon = 0.01$
16. $f(x) = 2x - 2, \quad L = -6, \quad x_0 = -2, \quad \epsilon = 0.02$
17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad x_0 = 0, \quad \epsilon = 0.1$
18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad x_0 = 1/4, \quad \epsilon = 0.1$
19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad x_0 = 10, \quad \epsilon = 1$
20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad x_0 = 23, \quad \epsilon = 1$
21. $f(x) = 1/x, \quad L = 1/4, \quad x_0 = 4, \quad \epsilon = 0.05$
22. $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \epsilon = 0.1$
23. $f(x) = x^2, \quad L = 4, \quad x_0 = -2, \quad \epsilon = 0.5$
24. $f(x) = 1/x, \quad L = -1, \quad x_0 = -1, \quad \epsilon = 0.1$
25. $f(x) = x^2 - 5, \quad L = 11, \quad x_0 = 4, \quad \epsilon = 1$

2.4: *Two sided Limits and Limits at Infinity*

One-Sided Limits

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

Right Hand Limit

Intuitively, if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x \to c^+} f(x) = L.$$

The symbol "$x \to c^+$" means that we consider only values of x greater than c.

Left Hand Limit

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider only x values less than c.

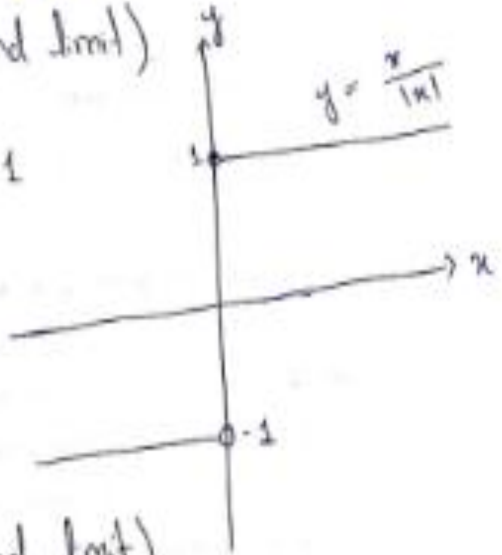
$$f(x) = \frac{x}{|x|}$$

where $|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$

So for $x < 0$ (left hand limit)

$$f(x) = \frac{-x}{|-x|} = \frac{-x}{x} = -1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-1) = -1$$



For $x > 0$ (Right hand limit)

$$f(x) = \frac{x}{|x|} = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (1) = 1.$$

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

EXAMPLE 5 Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Show that **(a)** $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h} \\
&= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \sin \theta \qquad \text{Let } \theta = h/2 \\
&= -(1)(0) = 0.
\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} && \text{Now, Eq. (1) applies with } \theta = 2x.\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

■

EXAMPLE 7 Using Theorem 8

(a) $$\begin{aligned}\lim_{x\to\infty}\left(5+\frac{1}{x}\right) &= \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} && \text{Sum Rule}\\ &= 5 + 0 = 5 && \text{Known limits}\end{aligned}$$

(b) $$\begin{aligned}\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} &= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}\\ &= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} && \text{Product rule}\\ &= \pi\sqrt{3}\cdot 0\cdot 0 = 0 && \text{Known limits}\end{aligned}$$

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator. What happens then depends on the degrees of the polynomials involved.

EXAMPLE 8 Numerator and Denominator of Same Degree

$$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$

Divide numerator and denominator by x^2.

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$

See Fig. 2.33. ■

EXAMPLE 9 Degree of Numerator Less Than Degree of Denominator

$$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$

Divide numerator and denominator by x^3.

$$= \frac{0 + 0}{2 - 0} = 0$$

See Fig. 2.34. ■

Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 21–36.

21. $\lim_{\theta \to 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\lim_{t \to 0} \frac{\sin kt}{t}$ (k constant)

23. $\lim_{y \to 0} \frac{\sin 3y}{4y}$

24. $\lim_{h \to 0^-} \frac{h}{\sin 3h}$

25. $\lim_{x \to 0} \frac{\tan 2x}{x}$

26. $\lim_{t \to 0} \frac{2t}{\tan t}$

27. $\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x}$

28. $\lim_{x \to 0} 6x^2(\cot x)(\csc 2x)$

29. $\lim_{x \to 0} \frac{x + x \cos x}{\sin x \cos x}$

30. $\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x}$

31. $\lim_{t \to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

32. $\lim_{h \to 0} \frac{\sin(\sin h)}{\sin h}$

33. $\lim_{\theta \to 0} \frac{\sin \theta}{\sin 2\theta}$

34. $\lim_{x \to 0} \frac{\sin 5x}{\sin 4x}$

35. $\lim_{x \to 0} \frac{\tan 3x}{\sin 8x}$

36. $\lim_{y \to 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

Calculating Limits as $x \to \pm\infty$

In Exercises 37–42, find the limit of each function (a) as $x \to \infty$ and (b) as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

37. $f(x) = \frac{2}{x} - 3$

38. $f(x) = \pi - \frac{2}{x^2}$

39. $g(x) = \frac{1}{2 + (1/x)}$

40. $g(x) = \frac{1}{8 - (5/x^2)}$

41. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$

42. $h(x) = \frac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 43–46.

43. $\lim_{x \to \infty} \frac{\sin 2x}{x}$

44. $\lim_{\theta \to -\infty} \frac{\cos \theta}{3\theta}$

45. $\lim_{t \to -\infty} \frac{2 - t + \sin t}{t + \cos t}$

46. $\lim_{r \to \infty} \frac{r + \sin r}{2r + 7 - 5 \sin r}$

Limits of Rational Functions

In Exercises 47–56, find the limit of each rational function (a) as $x \to \infty$ and (b) as $x \to -\infty$.

47. $f(x) = \frac{2x + 3}{5x + 7}$

48. $f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7}$

49. $f(x) = \frac{x + 1}{x^2 + 3}$

50. $f(x) = \frac{3x + 7}{x^2 - 2}$

51. $h(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$

52. $g(x) = \frac{1}{x^3 - 4x + 1}$

2.6 Continuity

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \to c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.51).

EXAMPLE 2 A Function Continuous Throughout Its Domain

The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$ (Figure 2.52), including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ■

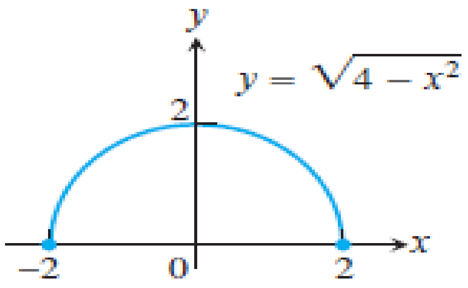


FIGURE 2.52 A function that is continuous at every domain point (Example 2).

EXAMPLE 3 The Unit Step Function Has a Jump Discontinuity

The unit step function $U(x)$, graphed in Figure 2.53, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

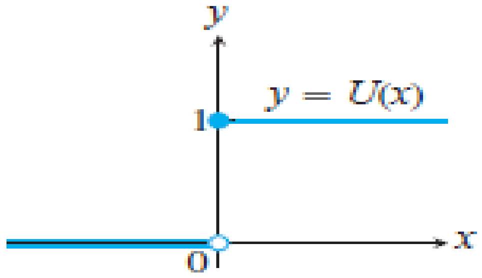


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f)
2.	$\lim_{x\to c} f(x)$ exists	(f has a limit as $x \to c$)
3.	$\lim_{x\to c} f(x) = f(c)$	(the limit equals the function value)

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. For example, the semicircle function graphed in Figure 2.52 is continuous on the interval $[-2, 2]$, which is its domain. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$ (Figure 2.56), but it is continuous over its domain $(-\infty, 0) \cup (0, \infty)$.

EXAMPLE 5 Identifying Continuous Functions

(a) The function $y = 1/x$ (Figure 2.56) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

EXAMPLE 6 Polynomial and Rational Functions Are Continuous

(a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \to c} P(x) = P(c)$ by Theorem 2, Section 2.2.

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by the Quotient Rule in Theorem 9.

EXAMPLE 7 Continuity of the Absolute Value Function

The function $f(x) = |x|$ is continuous at every value of x. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$. ■

Composites

All composites of continuous functions are continuous. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.57). In this case, the limit as $x \to c$ is $g(f(c))$.

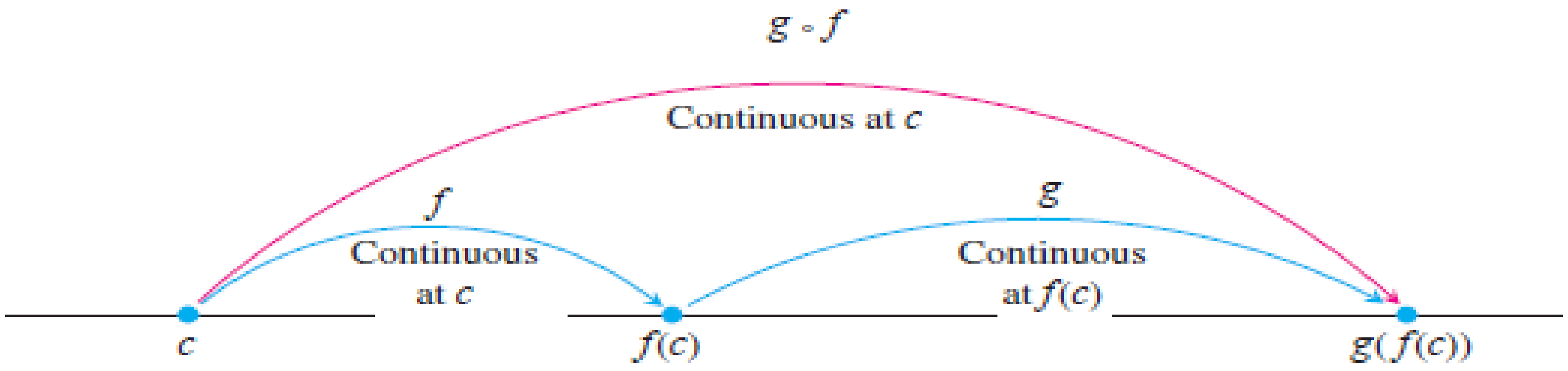


FIGURE 2.57 Composites of continuous functions are continuous.

THEOREM 10 Composite of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

EXAMPLE 8 Applying Theorems 9 and 10

Show that the following functions are continuous everywhere on their respective domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ **(b)** $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ **(d)** $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a rational power of the continuous identity function $f(x) = x$ (Part 6, Theorem 9). The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$.

(b) The numerator is a rational power of the identity function; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

(d) Because the sine function is everywhere-continuous (Exercise 62), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function (Figure 2.58). ■

EXERCISE 2.6

At what points are the functions in Exercises 13–28 continuous?

13. $y = \frac{1}{x-2} - 3x$

14. $y = \frac{1}{(x+2)^2} + 4$

15. $y = \frac{x+1}{x^2 - 4x + 3}$

16. $y = \frac{x+3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \frac{1}{|x| + 1} - \frac{x^2}{2}$

19. $y = \frac{\cos x}{x}$

20. $y = \frac{x+2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \frac{\pi x}{2}$

23. $y = \frac{x \tan x}{x^2 + 1}$

24. $y = \frac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

Composite Functions

Find the limits in Exercises 29–34. Are the functions continuous at the point being approached?

29. $\lim_{x \to \pi} \sin(x - \sin x)$

30. $\lim_{t \to 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right)$

31. $\lim_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

32. $\lim_{x \to 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right)$

33. $\lim_{t \to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right)$

34. $\lim_{x \to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$