

5.2 Euler's Method

Euler's method is the most elementary approximation technique for solving initial-value problems. Although it is seldom used in practice, the simplicity of its derivation can be used to illustrate the techniques involved in the construction of some of the more advanced techniques, without the cumbersome algebra that accompanies these constructions.

The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \quad (5.6)$$

A continuous approximation to the solution $y(t)$ will not be obtained; instead, approximations to y will be generated at various values, called **mesh points**, in the interval $[a, b]$. Once the approximate solution is obtained at the points, the approximate solution at other points in the interval can be found by interpolation.

We first make the stipulation that the mesh points are equally distributed throughout the interval $[a, b]$. This condition is ensured by choosing a positive integer N and selecting the mesh points

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \dots, N.$$

The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$ is called the **step size**.

We will use Taylor's Theorem to derive Euler's method. Suppose that $y(t)$, the unique solution to (5.6), has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \dots, N - 1$,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i),$$

for some number ξ_i in (t_i, t_{i+1}) . Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i),$$

and, because $y(t)$ satisfies the differential equation (5.6),

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i). \quad (5.7)$$

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \dots, N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \dots, N - 1. \end{aligned} \quad (5.8)$$

Illustration In Example 1 we will use an algorithm for Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5,$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

The use of elementary difference methods to approximate the solution to differential equations was one of the numerous mathematical topics that was first presented to the mathematical public by the most prolific of mathematicians, Leonhard Euler (1707–1783).

For this problem $f(t, y) = y - t^2 + 1$, so

$$w_0 = y(0) = 0.5;$$

$$w_1 = w_0 + 0.5(w_0 - (0.0)^2 + 1) = 0.5 + 0.5(1.5) = 1.25;$$

$$w_2 = w_1 + 0.5(w_1 - (0.5)^2 + 1) = 1.25 + 0.5(2.0) = 2.25;$$

$$w_3 = w_2 + 0.5(w_2 - (1.0)^2 + 1) = 2.25 + 0.5(2.25) = 3.375;$$

and

$$y(2) \approx w_4 = w_3 + 0.5(w_3 - (1.5)^2 + 1) = 3.375 + 0.5(2.125) = 4.4375. \quad \square$$

Equation (5.8) is called the **difference equation** associated with Euler's method. As we will see later in this chapter, the theory and solution of difference equations parallel, in many ways, the theory and solution of differential equations. Algorithm 5.1 implements Euler's method.



Euler's

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b ; integer N ; initial condition α .

OUTPUT approximation w to y at the $(N + 1)$ values of t .

Step 1 Set $h = (b - a)/N$;

$$t = a;$$

$$w = \alpha;$$

OUTPUT (t, w) .

Step 2 For $i = 1, 2, \dots, N$ do Steps 3, 4.

Step 3 Set $w = w + hf(t, w)$; (Compute w_i .)

$$t = a + ih. \quad (\text{Compute } t_i.)$$

Step 4 **OUTPUT** (t, w) .

Step 5 **STOP**. ■

To interpret Euler's method geometrically, note that when w_i is a close approximation to $y(t_i)$, the assumption that the problem is well-posed implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i)).$$

The graph of the function highlighting $y(t_i)$ is shown in Figure 5.2. One step in Euler's method appears in Figure 5.3, and a series of steps appears in Figure 5.4.

Figure 5.2

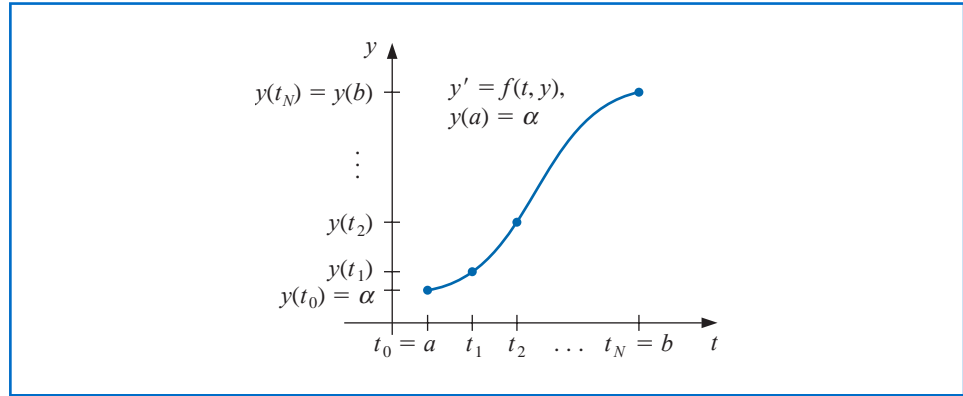


Figure 5.3

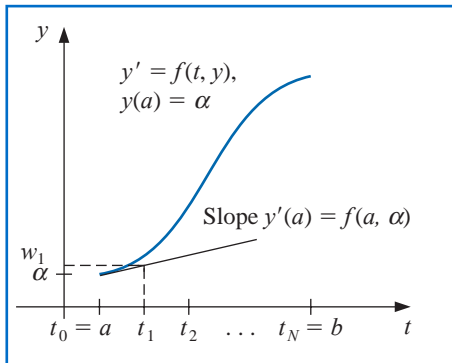
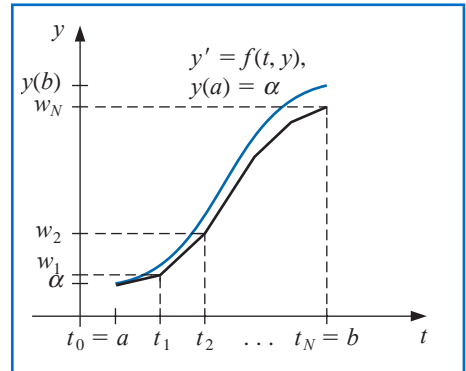


Figure 5.4



Example 1 Euler's method was used in the first illustration with $h = 0.5$ to approximate the solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Use Algorithm 5.1 with $N = 10$ to determine approximations, and compare these with the exact values given by $y(t) = (t + 1)^2 - 0.5e^t$.

Solution With $N = 10$ we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, and

$$w_{i+1} = w_i + h(w_i - t_i^2 + 1) = w_i + 0.2[w_i - 0.04i^2 + 1] = 1.2w_i - 0.008i^2 + 0.2,$$

for $i = 0, 1, \dots, 9$. So

$$w_1 = 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8; \quad w_2 = 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152;$$

and so on. Table 5.1 shows the comparison between the approximate values at t_i and the actual values. ■

Table 5.1

t_i	w_i	$y_i = y(t_i)$	$ y_i - w_i $
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

Note that the error grows slightly as the value of t increases. This controlled error growth is a consequence of the stability of Euler's method, which implies that the error is expected to grow in no worse than a linear manner.

Maple has implemented Euler's method as an option with the command *InitialValueProblem* within the *NumericalAnalysis* subpackage of the *Student* package. To use it for the problem in Example 1 first load the package and the differential equation.

```
with(Student[NumericalAnalysis]): deq := diff(y(t), t) = y(t) - t^2 + 1
```

Then issue the command

```
C := InitialValueProblem(deq, y(0) = 0.5, t = 2, method = euler, numsteps = 10,
output = information, digits = 8)
```

Maple produces

```
[ 1 .. 12 x 1 .. 4 Array
  Data Type: anything
  Storage: rectangular
  Order: Fortran_order ]
```

Double clicking on the output brings up a table that gives the values of t_i , actual solution values $y(t_i)$, the Euler approximations w_i , and the absolute errors $|y(t_i) - w_i|$. These agree with the values in Table 5.1.

To print the Maple table we can issue the commands

```
for k from 1 to 12 do
print(C[k, 1], C[k, 2], C[k, 3], C[k, 4])
end do
```

The options within the *InitialValueProblem* command are the specification of the first order differential equation to be solved, the initial condition, the final value of the independent variable, the choice of method, the number of steps used to determine that $h = (2 - 0) / (\text{numsteps})$, the specification of form of the output, and the number of digits of rounding to be used in the computations. Other output options can specify a particular value of t or a plot of the solution.

Error Bounds for Euler's Method

Although Euler's method is not accurate enough to warrant its use in practice, it is sufficiently elementary to analyze the error that is produced from its application. The error analysis for

the more accurate methods that we consider in subsequent sections follows the same pattern but is more complicated.

To derive an error bound for Euler’s method, we need two computational lemmas.

Lemma 5.7 For all $x \geq -1$ and any positive m , we have $0 \leq (1+x)^m \leq e^{mx}$. ■

Proof Applying Taylor’s Theorem with $f(x) = e^x$, $x_0 = 0$, and $n = 1$ gives

$$e^x = 1 + x + \frac{1}{2}x^2e^\xi,$$

where ξ is between x and zero. Thus

$$0 \leq 1 + x \leq 1 + x + \frac{1}{2}x^2e^\xi = e^x,$$

and, because $1 + x \geq 0$, we have

$$0 \leq (1+x)^m \leq (e^x)^m = e^{mx}. \quad \blacksquare \blacksquare \blacksquare$$

Lemma 5.8 If s and t are positive real numbers, $\{a_i\}_{i=0}^k$ is a sequence satisfying $a_0 \geq -t/s$, and

$$a_{i+1} \leq (1+s)a_i + t, \quad \text{for each } i = 0, 1, 2, \dots, k-1, \quad (5.9)$$

then

$$a_{i+1} \leq e^{(i+1)s} \left(a_0 + \frac{t}{s} \right) - \frac{t}{s}. \quad \blacksquare$$

Proof For a fixed integer i , Inequality (5.9) implies that

$$\begin{aligned} a_{i+1} &\leq (1+s)a_i + t \\ &\leq (1+s)[(1+s)a_{i-1} + t] + t = (1+s)^2a_{i-1} + [1 + (1+s)]t \\ &\leq (1+s)^3a_{i-2} + [1 + (1+s) + (1+s)^2]t \\ &\vdots \\ &\leq (1+s)^{i+1}a_0 + [1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i]t. \end{aligned}$$

But

$$1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i = \sum_{j=0}^i (1+s)^j$$

is a geometric series with ratio $(1+s)$ that sums to

$$\frac{1 - (1+s)^{i+1}}{1 - (1+s)} = \frac{1}{s}[(1+s)^{i+1} - 1].$$

Thus

$$a_{i+1} \leq (1+s)^{i+1}a_0 + \frac{(1+s)^{i+1} - 1}{s}t = (1+s)^{i+1} \left(a_0 + \frac{t}{s} \right) - \frac{t}{s},$$

and using Lemma 5.7 with $x = 1+s$ gives

$$a_{i+1} \leq e^{(i+1)s} \left(a_0 + \frac{t}{s} \right) - \frac{t}{s}. \quad \blacksquare \blacksquare \blacksquare$$

Theorem 5.9 Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b],$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

Let w_0, w_1, \dots, w_N be the approximations generated by Euler's method for some positive integer N . Then, for each $i = 0, 1, 2, \dots, N$,

$$|y(t_i) - w_i| \leq \frac{hM}{2L} [e^{L(t_i-a)} - 1]. \quad (5.10)$$

Proof When $i = 0$ the result is clearly true, since $y(t_0) = w_0 = \alpha$.

From Eq. (5.7), we have

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i),$$

for $i = 0, 1, \dots, N-1$, and from the equations in (5.8),

$$w_{i+1} = w_i + hf(t_i, w_i).$$

Using the notation $y_i = y(t_i)$ and $y_{i+1} = y(t_{i+1})$, we subtract these two equations to obtain

$$y_{i+1} - w_{i+1} = y_i - w_i + h[f(t_i, y_i) - f(t_i, w_i)] + \frac{h^2}{2}y''(\xi_i)$$

Hence

$$|y_{i+1} - w_{i+1}| \leq |y_i - w_i| + h|f(t_i, y_i) - f(t_i, w_i)| + \frac{h^2}{2}|y''(\xi_i)|.$$

Now f satisfies a Lipschitz condition in the second variable with constant L , and $|y''(t)| \leq M$, so

$$|y_{i+1} - w_{i+1}| \leq (1 + hL)|y_i - w_i| + \frac{h^2M}{2}.$$

Referring to Lemma 5.8 and letting $s = hL$, $t = h^2M/2$, and $a_j = |y_j - w_j|$, for each $j = 0, 1, \dots, N$, we see that

$$|y_{i+1} - w_{i+1}| \leq e^{(i+1)hL} \left(|y_0 - w_0| + \frac{h^2M}{2hL} \right) - \frac{h^2M}{2hL}.$$

Because $|y_0 - w_0| = 0$ and $(i+1)h = t_{i+1} - t_0 = t_{i+1} - a$, this implies that

$$|y_{i+1} - w_{i+1}| \leq \frac{hM}{2L} (e^{(t_{i+1}-a)L} - 1),$$

for each $i = 0, 1, \dots, N-1$. ■ ■ ■

The weakness of Theorem 5.9 lies in the requirement that a bound be known for the second derivative of the solution. Although this condition often prohibits us from obtaining a realistic error bound, it should be noted that if $\partial f/\partial t$ and $\partial f/\partial y$ both exist, the chain rule for partial differentiation implies that

$$y''(t) = \frac{dy'}{dt}(t) = \frac{df}{dt}(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t)).$$

So it is at times possible to obtain an error bound for $y''(t)$ without explicitly knowing $y(t)$.

Example 2 The solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5,$$

was approximated in Example 1 using Euler’s method with $h = 0.2$. Use the inequality in Theorem 5.9 to find a bounds for the approximation errors and compare these to the actual errors.

Solution Because $f(t, y) = y - t^2 + 1$, we have $\partial f(t, y)/\partial y = 1$ for all y , so $L = 1$. For this problem, the exact solution is $y(t) = (t + 1)^2 - 0.5e^t$, so $y''(t) = 2 - 0.5e^t$ and

$$|y''(t)| \leq 0.5e^2 - 2, \quad \text{for all } t \in [0, 2].$$

Using the inequality in the error bound for Euler’s method with $h = 0.2$, $L = 1$, and $M = 0.5e^2 - 2$ gives

$$|y_i - w_i| \leq 0.1(0.5e^2 - 2)(e^{t_i} - 1).$$

Hence

$$|y(0.2) - w_1| \leq 0.1(0.5e^2 - 2)(e^{0.2} - 1) = 0.03752;$$

$$|y(0.4) - w_2| \leq 0.1(0.5e^2 - 2)(e^{0.4} - 1) = 0.08334;$$

and so on. Table 5.2 lists the actual error found in Example 1, together with this error bound. Note that even though the true bound for the second derivative of the solution was used, the error bound is considerably larger than the actual error, especially for increasing values of t . ■

Table 5.2

t_i	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Actual Error	0.02930	0.06209	0.09854	0.13875	0.18268	0.23013	0.28063	0.33336	0.38702	0.43969
Error Bound	0.03752	0.08334	0.13931	0.20767	0.29117	0.39315	0.51771	0.66985	0.85568	1.08264

The principal importance of the error-bound formula given in Theorem 5.9 is that the bound depends linearly on the step size h . Consequently, diminishing the step size should give correspondingly greater accuracy to the approximations.

Neglected in the result of Theorem 5.9 is the effect that round-off error plays in the choice of step size. As h becomes smaller, more calculations are necessary and more round-off error is expected. In actuality then, the difference-equation form

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \dots, N - 1,$$

is not used to calculate the approximation to the solution y_i at a mesh point t_i . We use instead an equation of the form

$$\begin{aligned} u_0 &= \alpha + \delta_0, \\ u_{i+1} &= u_i + hf(t_i, u_i) + \delta_{i+1}, \quad \text{for each } i = 0, 1, \dots, N-1, \end{aligned} \quad (5.11)$$

where δ_i denotes the round-off error associated with u_i . Using methods similar to those in the proof of Theorem 5.9, we can produce an error bound for the finite-digit approximations to y_i given by Euler's method.

Theorem 5.10 Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \quad (5.12)$$

and u_0, u_1, \dots, u_N be the approximations obtained using (5.11). If $|\delta_i| < \delta$ for each $i = 0, 1, \dots, N$ and the hypotheses of Theorem 5.9 hold for (5.12), then

$$|y(t_i) - u_i| \leq \frac{1}{L} \left(\frac{hM}{2} + \frac{\delta}{h} \right) [e^{L(t_i-a)} - 1] + |\delta_0|e^{L(t_i-a)}, \quad (5.13)$$

for each $i = 0, 1, \dots, N$. ■

The error bound (5.13) is no longer linear in h . In fact, since

$$\lim_{h \rightarrow 0} \left(\frac{hM}{2} + \frac{\delta}{h} \right) = \infty,$$

the error would be expected to become large for sufficiently small values of h . Calculus can be used to determine a lower bound for the step size h . Letting $E(h) = (hM/2) + (\delta/h)$ implies that $E'(h) = (M/2) - (\delta/h^2)$.

If $h < \sqrt{2\delta/M}$, then $E'(h) < 0$ and $E(h)$ is decreasing.

If $h > \sqrt{2\delta/M}$, then $E'(h) > 0$ and $E(h)$ is increasing.

The minimal value of $E(h)$ occurs when

$$h = \sqrt{\frac{2\delta}{M}}. \quad (5.14)$$

Decreasing h beyond this value tends to increase the total error in the approximation. Normally, however, the value of δ is sufficiently small that this lower bound for h does not affect the operation of Euler's method.

EXERCISE SET 5.2

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.
 - a. $y' = te^{3t} - 2y$, $0 \leq t \leq 1$, $y(0) = 0$, with $h = 0.5$
 - b. $y' = 1 + (t - y)^2$, $2 \leq t \leq 3$, $y(2) = 1$, with $h = 0.5$
 - c. $y' = 1 + y/t$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.25$
 - d. $y' = \cos 2t + \sin 3t$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.25$

2. Use Euler's method to approximate the solutions for each of the following initial-value problems.
- $y' = e^{t-y}$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.5$
 - $y' = \frac{1+t}{1+y}$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.5$
 - $y' = -y + ty^{1/2}$, $2 \leq t \leq 3$, $y(2) = 2$, with $h = 0.25$
 - $y' = t^{-2}(\sin 2t - 2ty)$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.25$
3. The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the error bound.
- $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$
 - $y(t) = t + \frac{1}{1-t}$
 - $y(t) = t \ln t + 2t$
 - $y(t) = \frac{1}{2} \sin 2t - \frac{1}{3} \cos 3t + \frac{4}{3}$
4. The actual solutions to the initial-value problems in Exercise 2 are given here. Compute the actual error and compare this to the error bound if Theorem 5.9 can be applied.
- $y(t) = \ln(e^t + e - 1)$
 - $y(t) = \sqrt{t^2 + 2t + 6} - 1$
 - $y(t) = \left(t - 2 + \sqrt{2e}e^{-t/2}\right)^2$
 - $y(t) = \frac{4 + \cos 2 - \cos 2t}{2t^2}$
5. Use Euler's method to approximate the solutions for each of the following initial-value problems.
- $y' = y/t - (y/t)^2$, $1 \leq t \leq 2$, $y(1) = 1$, with $h = 0.1$
 - $y' = 1 + y/t + (y/t)^2$, $1 \leq t \leq 3$, $y(1) = 0$, with $h = 0.2$
 - $y' = -(y+1)(y+3)$, $0 \leq t \leq 2$, $y(0) = -2$, with $h = 0.2$
 - $y' = -5y + 5t^2 + 2t$, $0 \leq t \leq 1$, $y(0) = \frac{1}{3}$, with $h = 0.1$
6. Use Euler's method to approximate the solutions for each of the following initial-value problems.
- $y' = \frac{2-2ty}{t^2+1}$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.1$
 - $y' = \frac{y^2}{1+t}$, $1 \leq t \leq 2$, $y(1) = -(\ln 2)^{-1}$, with $h = 0.1$
 - $y' = (y^2 + y)/t$, $1 \leq t \leq 3$, $y(1) = -2$, with $h = 0.2$
 - $y' = -ty + 4ty^{-1}$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.1$
7. The actual solutions to the initial-value problems in Exercise 5 are given here. Compute the actual error in the approximations of Exercise 5.
- $y(t) = \frac{t}{1 + \ln t}$
 - $y(t) = t \tan(\ln t)$
 - $y(t) = -3 + \frac{2}{1 + e^{-2t}}$
 - $y(t) = t^2 + \frac{1}{3}e^{-5t}$
8. The actual solutions to the initial-value problems in Exercise 6 are given here. Compute the actual error in the approximations of Exercise 6.
- $y(t) = \frac{2t+1}{t^2+1}$
 - $y(t) = \frac{-1}{\ln(t+1)}$
 - $y(t) = \frac{2t}{1-2t}$
 - $y(t) = \sqrt{4-3e^{-t^2}}$
9. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \leq t \leq 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

- Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y .

- b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y , and compare them to the actual values.
- i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$
- c. Compute the value of h necessary for $|y(t_i) - w_i| \leq 0.1$, using Eq. (5.10).
10. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \leq t \leq 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

- a. Use Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of y .
- b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y , and compare them to the actual values.
- i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$
- c. Compute the value of h necessary for $|y(t_i) - w_i| \leq 0.05$ using Eq. (5.10).
11. Given the initial-value problem

$$y' = -y + t + 1, \quad 0 \leq t \leq 5, \quad y(0) = 1,$$

with exact solution $y(t) = e^{-t} + t$:

- a. Approximate $y(5)$ using Euler's method with $h = 0.2$, $h = 0.1$, and $h = 0.05$.
- b. Determine the optimal value of h to use in computing $y(5)$, assuming $\delta = 10^{-6}$ and that Eq. (5.14) is valid.
12. Consider the initial-value problem

$$y' = -10y, \quad 0 \leq t \leq 2, \quad y(0) = 1,$$

which has solution $y(t) = e^{-10t}$. What happens when Euler's method is applied to this problem with $h = 0.1$? Does this behavior violate Theorem 5.9?

13. Use the results of Exercise 5 and linear interpolation to approximate the following values of $y(t)$. Compare the approximations obtained to the actual values obtained using the functions given in Exercise 7.
- a. $y(1.25)$ and $y(1.93)$ b. $y(2.1)$ and $y(2.75)$
 c. $y(1.3)$ and $y(1.93)$ d. $y(0.54)$ and $y(0.94)$
14. Use the results of Exercise 6 and linear interpolation to approximate the following values of $y(t)$. Compare the approximations obtained to the actual values obtained using the functions given in Exercise 8.
- a. $y(0.25)$ and $y(0.93)$ b. $y(1.25)$ and $y(1.93)$
 c. $y(2.10)$ and $y(2.75)$ d. $y(0.54)$ and $y(0.94)$

15. Let $E(h) = \frac{hM}{2} + \frac{\delta}{h}$.

- a. For the initial-value problem

$$y' = -y + 1, \quad 0 \leq t \leq 1, \quad y(0) = 0,$$

compute the value of h to minimize $E(h)$. Assume $\delta = 5 \times 10^{-(n+1)}$ if you will be using n -digit arithmetic in part (c).

- b. For the optimal h computed in part (a), use Eq. (5.13) to compute the minimal error obtainable.
- c. Compare the actual error obtained using $h = 0.1$ and $h = 0.01$ to the minimal error in part (b). Can you explain the results?
16. In a circuit with impressed voltage \mathcal{E} having resistance R , inductance L , and capacitance C in parallel, the current i satisfies the differential equation

$$\frac{di}{dt} = C \frac{d^2\mathcal{E}}{dt^2} + \frac{1}{R} \frac{d\mathcal{E}}{dt} + \frac{1}{L} \mathcal{E}.$$

Suppose $C = 0.3$ farads, $R = 1.4$ ohms, $L = 1.7$ henries, and the voltage is given by

$$\mathcal{E}(t) = e^{-0.06\pi t} \sin(2t - \pi).$$

If $i(0) = 0$, find the current i for the values $t = 0.1j$, where $j = 0, 1, \dots, 100$.

17. In a book entitled *Looking at History Through Mathematics*, Rashevsky [Ra], pp. 103–110, considers a model for a problem involving the production of nonconformists in society. Suppose that a society has a population of $x(t)$ individuals at time t , in years, and that all nonconformists who mate with other nonconformists have offspring who are also nonconformists, while a fixed proportion r of all other offspring are also nonconformist. If the birth and death rates for all individuals are assumed to be the constants b and d , respectively, and if conformists and nonconformists mate at random, the problem can be expressed by the differential equations

$$\frac{dx(t)}{dt} = (b - d)x(t) \quad \text{and} \quad \frac{dx_n(t)}{dt} = (b - d)x_n(t) + rb(x(t) - x_n(t)),$$

where $x_n(t)$ denotes the number of nonconformists in the population at time t .

- a. Suppose the variable $p(t) = x_n(t)/x(t)$ is introduced to represent the proportion of nonconformists in the society at time t . Show that these equations can be combined and simplified to the single differential equation

$$\frac{dp(t)}{dt} = rb(1 - p(t)).$$

- b. Assuming that $p(0) = 0.01$, $b = 0.02$, $d = 0.015$, and $r = 0.1$, approximate the solution $p(t)$ from $t = 0$ to $t = 50$ when the step size is $h = 1$ year.
- c. Solve the differential equation for $p(t)$ exactly, and compare your result in part (b) when $t = 50$ with the exact value at that time.

5.3 Higher-Order Taylor Methods

Since the object of a numerical techniques is to determine accurate approximations with minimal effort, we need a means for comparing the efficiency of various approximation methods. The first device we consider is called the *local truncation error* of the method.

The local truncation error at a specified step measures the amount by which the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation at that step. This might seem like an unlikely way to compare the error of various methods. We really want to know how well the approximations generated by the methods satisfy the differential equation, not the other way around. However, we don't know the exact solution so we cannot generally determine this, and the local truncation will serve quite well to determine not only the local error of a method but the actual approximation error.

Consider the initial value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

Definition 5.11 The difference method

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \dots, N - 1,$$

has **local truncation error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \dots, N - 1$, where y_i and y_{i+1} denote the solution at t_i and t_{i+1} , respectively. ■

For example, Euler's method has local truncation error at the i th step

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad \text{for each } i = 0, 1, \dots, N - 1.$$

This error is a *local error* because it measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step. As such, it depends on the differential equation, the step size, and the particular step in the approximation.

By considering Eq. (5.7) in the previous section, we see that Euler's method has

$$\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i), \quad \text{for some } \xi_i \text{ in } (t_i, t_{i+1}).$$

When $y''(t)$ is known to be bounded by a constant M on $[a, b]$, this implies

$$|\tau_{i+1}(h)| \leq \frac{h}{2} M,$$

so the local truncation error in Euler's method is $O(h)$.

One way to select difference-equation methods for solving ordinary differential equations is in such a manner that their local truncation errors are $O(h^p)$ for as large a value of p as possible, while keeping the number and complexity of calculations of the methods within a reasonable bound.

Since Euler's method was derived by using Taylor's Theorem with $n = 1$ to approximate the solution of the differential equation, our first attempt to find methods for improving the convergence properties of difference methods is to extend this technique of derivation to larger values of n .

Suppose the solution $y(t)$ to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has $(n + 1)$ continuous derivatives. If we expand the solution, $y(t)$, in terms of its n th Taylor polynomial about t_i and evaluate at t_{i+1} , we obtain

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i), \quad (5.15)$$

for some ξ_i in (t_i, t_{i+1}) .

Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), \quad y''(t) = f'(t, y(t)), \quad \text{and, generally, } y^{(k)}(t) = f^{(k-1)}(t, y(t)).$$

Substituting these results into Eq. (5.15) gives

$$\begin{aligned} y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \\ + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)). \end{aligned} \quad (5.16)$$

The difference-equation method corresponding to Eq. (5.16) is obtained by deleting the remainder term involving ξ_i .

The methods in this section use Taylor polynomials and the knowledge of the derivative at a node to approximate the value of the function at a new node.

Taylor method of order n

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \dots, N - 1, \quad (5.17)$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \dots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i, w_i).$$

Euler’s method is Taylor’s method of order one.

Example 1 Apply Taylor’s method of orders (a) two and (b) four with $N = 10$ to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Solution (a) For the method of order two we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable t . Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t,$$

so

$$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) = w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i)$$

$$= \left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i$$

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \dots, 10$. Thus the second-order method becomes

$$w_0 = 0.5,$$

$$w_{i+1} = w_i + h \left[\left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i \right]$$

$$= w_i + 0.2 \left[\left(1 + \frac{0.2}{2}\right)(w_i - 0.04i^2 + 1) - 0.04i \right]$$

$$= 1.22w_i - 0.0088i^2 - 0.008i + 0.22.$$

Table 5.3

t_i	Taylor Order 2 w_i	Error $ y(t_i) - w_i $
0.0	0.500000	0
0.2	0.830000	0.000701
0.4	1.215800	0.001712
0.6	1.652076	0.003135
0.8	2.132333	0.005103
1.0	2.648646	0.007787
1.2	3.191348	0.011407
1.4	3.748645	0.016245
1.6	4.306146	0.022663
1.8	4.846299	0.031122
2.0	5.347684	0.042212

The first two steps give the approximations

$$y(0.2) \approx w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.22 = 0.83;$$

$$y(0.4) \approx w_2 = 1.22(0.83) - 0.0088(0.2)^2 - 0.008(0.2) + 0.22 = 1.2158$$

All the approximations and their errors are shown in Table 5.3

(b) For Taylor’s method of order four we need the first three derivatives of $f(t, y(t))$ with respect to t . Again using $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = y - t^2 + 1 - 2t,$$

$$f''(t, y(t)) = \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2$$

$$= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1,$$

and

$$f'''(t, y(t)) = \frac{d}{dt}(y - t^2 - 2t - 1) = y' - 2t - 2 = y - t^2 - 2t - 1,$$

so

$$\begin{aligned} T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \frac{h^2}{6}f''(t_i, w_i) + \frac{h^3}{24}f'''(t_i, w_i) \\ &= w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i) + \frac{h^2}{6}(w_i - t_i^2 - 2t_i - 1) \\ &\quad + \frac{h^3}{24}(w_i - t_i^2 - 2t_i - 1) \\ &= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) \\ &\quad + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}. \end{aligned}$$

Hence Taylor's method of order four is

$$\begin{aligned} w_0 &= 0.5, \\ w_{i+1} &= w_i + h \left[\left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)ht_i \right. \\ &\quad \left. + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24} \right], \end{aligned}$$

for $i = 0, 1, \dots, N - 1$.

Because $N = 10$ and $h = 0.2$ the method becomes

$$\begin{aligned} w_{i+1} &= w_i + 0.2 \left[\left(1 + \frac{0.2}{2} + \frac{0.04}{6} + \frac{0.008}{24}\right)(w_i - 0.04i^2) \right. \\ &\quad \left. - \left(1 + \frac{0.2}{3} + \frac{0.04}{12}\right)(0.04i) + 1 + \frac{0.2}{2} - \frac{0.04}{6} - \frac{0.008}{24} \right] \\ &= 1.2214w_i - 0.008856i^2 - 0.00856i + 0.2186, \end{aligned}$$

for each $i = 0, 1, \dots, 9$. The first two steps give the approximations

$$y(0.2) \approx w_1 = 1.2214(0.5) - 0.008856(0)^2 - 0.00856(0) + 0.2186 = 0.8293;$$

$$y(0.4) \approx w_2 = 1.2214(0.8293) - 0.008856(0.2)^2 - 0.00856(0.2) + 0.2186 = 1.214091$$

All the approximations and their errors are shown in Table 5.4.

Compare these results with those of Taylor's method of order 2 in Table 5.4 and you will see that the fourth-order results are vastly superior.

The results from Table 5.4 indicate the Taylor's method of order 4 results are quite accurate at the nodes 0.2, 0.4, etc. But suppose we need to determine an approximation to an intermediate point in the table, for example, at $t = 1.25$. If we use linear interpolation on the Taylor method of order four approximations at $t = 1.2$ and $t = 1.4$, we have

$$y(1.25) \approx \left(\frac{1.25 - 1.4}{1.2 - 1.4}\right) 3.1799640 + \left(\frac{1.25 - 1.2}{1.4 - 1.2}\right) 3.7324321 = 3.3180810.$$

Table 5.4

t_i	Taylor Order 4 w_i	Error $ y(t_i) - w_i $
0.0	0.500000	0
0.2	0.829300	0.000001
0.4	1.214091	0.000003
0.6	1.648947	0.000006
0.8	2.127240	0.000010
1.0	2.640874	0.000015
1.2	3.179964	0.000023
1.4	3.732432	0.000032
1.6	4.283529	0.000045
1.8	4.815238	0.000062
2.0	5.305555	0.000083

Hermite interpolation requires both the value of the function and its derivative at each node. This makes it a natural interpolation method for approximating differential equations since these data are all available.

The true value is $y(1.25) = 3.3173285$, so this approximation has an error of 0.0007525, which is nearly 30 times the average of the approximation errors at 1.2 and 1.4.

We can significantly improve the approximation by using cubic Hermite interpolation. To determine this approximation for $y(1.25)$ requires approximations to $y'(1.2)$ and $y'(1.4)$ as well as approximations to $y(1.2)$ and $y(1.4)$. However, the approximations for $y(1.2)$ and $y(1.4)$ are in the table, and the derivative approximations are available from the differential equation, because $y'(t) = f(t, y(t))$. In our example $y'(t) = y(t) - t^2 + 1$, so

$$y'(1.2) = y(1.2) - (1.2)^2 + 1 \approx 3.1799640 - 1.44 + 1 = 2.7399640$$

and

$$y'(1.4) = y(1.4) - (1.4)^2 + 1 \approx 3.7324327 - 1.96 + 1 = 2.7724321.$$

The divided-difference procedure in Section 3.4 gives the information in Table 5.5. The underlined entries come from the data, and the other entries use the divided-difference formulas.

Table 5.5

1.2	<u>3.1799640</u>			
		<u>2.7399640</u>		
1.2	<u>3.1799640</u>		0.1118825	
		2.7623405		-0.3071225
1.4	<u>3.7324321</u>		0.0504580	
		<u>2.7724321</u>		
1.4	<u>3.7324321</u>			

The cubic Hermite polynomial is

$$y(t) \approx 3.1799640 + (t - 1.2)2.7399640 + (t - 1.2)^2 0.1118825 + (t - 1.2)^2 (t - 1.4)(-0.3071225),$$

so

$$y(1.25) \approx 3.1799640 + 0.1369982 + 0.0002797 + 0.0001152 = 3.3173571,$$

a result that is accurate to within 0.0000286. This is about the average of the errors at 1.2 and at 1.4, and only 4% of the error obtained using linear interpolation. This improvement in accuracy certainly justifies the added computation required for the Hermite method. ■

Theorem 5.12 If Taylor’s method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size h and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$. ■

Proof Note that Eq. (5.16) on page 277 can be rewritten

$$y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \dots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i) = \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)),$$

for some ξ_i in (t_i, t_{i+1}) . So the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)),$$

for each $i = 0, 1, \dots, N-1$. Since $y \in C^{n+1}[a, b]$, we have $y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a, b]$ and $\tau_i(h) = O(h^n)$, for each $i = 1, 2, \dots, N$. ■ ■ ■

Taylor's methods are options within the Maple command *InitialValueProblem*. The form and output for Taylor's methods are the same as available under Euler's method, as discussed in Section 5.1. To obtain Taylor's method of order 2 for the problem in Example 1, first load the package and the differential equation.

```
with(Student[NumericalAnalysis]) : deq := diff(y(t), t) = y(t) - t^2 + 1
```

Then issue

```
C := InitialValueProblem(deq, y(0) = 0.5, t = 2, method = taylor, order = 2,
numsteps = 10, output = information, digits = 8)
```

Maple responds with an array of data similar to that produced with Euler's method. Double clicking on the output will bring up a table that gives the values of t_i , actual solution values $y(t_i)$, the Taylor approximations w_i , and the absolute errors $|y(t_i) - w_i|$. These agree with the values in Table 5.3.

To print the table issue the commands

```
for k from 1 to 12 do
print(C[k, 1], C[k, 2], C[k, 3], C[k, 4])
end do
```

EXERCISE SET 5.3

- Use Taylor's method of order two to approximate the solutions for each of the following initial-value problems.
 - $y' = te^{3t} - 2y$, $0 \leq t \leq 1$, $y(0) = 0$, with $h = 0.5$
 - $y' = 1 + (t - y)^2$, $2 \leq t \leq 3$, $y(2) = 1$, with $h = 0.5$
 - $y' = 1 + y/t$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.25$
 - $y' = \cos 2t + \sin 3t$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.25$
- Use Taylor's method of order two to approximate the solutions for each of the following initial-value problems.
 - $y' = e^{t-y}$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.5$
 - $y' = \frac{1+t}{1+y}$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.5$
 - $y' = -y + ty^{1/2}$, $2 \leq t \leq 3$, $y(2) = 2$, with $h = 0.25$
 - $y' = t^{-2}(\sin 2t - 2ty)$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.25$
- Repeat Exercise 1 using Taylor's method of order four.
- Repeat Exercise 2 using Taylor's method of order four.
- Use Taylor's method of order two to approximate the solution for each of the following initial-value problems.
 - $y' = y/t - (y/t)^2$, $1 \leq t \leq 1.2$, $y(1) = 1$, with $h = 0.1$
 - $y' = \sin t + e^{-t}$, $0 \leq t \leq 1$, $y(0) = 0$, with $h = 0.5$
 - $y' = (y^2 + y)/t$, $1 \leq t \leq 3$, $y(1) = -2$, with $h = 0.5$
 - $y' = -ty + 4ty^{-1}$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.25$
- Use Taylor's method of order two to approximate the solution for each of the following initial-value problems.
 - $y' = \frac{2-2ty}{t^2+1}$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.1$
 - $y' = \frac{y^2}{1+t}$, $1 \leq t \leq 2$, $y(1) = -(\ln 2)^{-1}$, with $h = 0.1$

- c. $y' = (y^2 + y)/t$, $1 \leq t \leq 3$, $y(1) = -2$, with $h = 0.2$
 d. $y' = -ty + 4t/y$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.1$
7. Repeat Exercise 5 using Taylor's method of order four.
 8. Repeat Exercise 6 using Taylor's method of order four.
 9. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \leq t \leq 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

- a. Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of y .
 b. Use the answers generated in part (a) and linear interpolation to approximate y at the following values, and compare them to the actual values of y .
 i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$
 c. Use Taylor's method of order four with $h = 0.1$ to approximate the solution, and compare it with the actual values of y .
 d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate y at the following values, and compare them to the actual values of y .
 i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$
10. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \leq t \leq 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

- a. Use Taylor's method of order two with $h = 0.05$ to approximate the solution, and compare it with the actual values of y .
 b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y , and compare them to the actual values.
 i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$
 c. Use Taylor's method of order four with $h = 0.05$ to approximate the solution, and compare it with the actual values of y .
 d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate the following values of y , and compare them to the actual values.
 i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$
11. A projectile of mass $m = 0.11$ kg shot vertically upward with initial velocity $v(0) = 8$ m/s is slowed due to the force of gravity, $F_g = -mg$, and due to air resistance, $F_r = -kv|v|$, where $g = 9.8$ m/s² and $k = 0.002$ kg/m. The differential equation for the velocity v is given by

$$mv' = -mg - kv|v|.$$

- a. Find the velocity after 0.1, 0.2, ..., 1.0 s.
 b. To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.
12. Use the Taylor method of order two with $h = 0.1$ to approximate the solution to

$$y' = 1 + t \sin(ty), \quad 0 \leq t \leq 2, \quad y(0) = 0.$$

5.4 Runge-Kutta Methods

The Taylor methods outlined in the previous section have the desirable property of high-order local truncation error, but the disadvantage of requiring the computation and evaluation of the derivatives of $f(t, y)$. This is a complicated and time-consuming procedure for most problems, so the Taylor methods are seldom used in practice.