CHAPTER

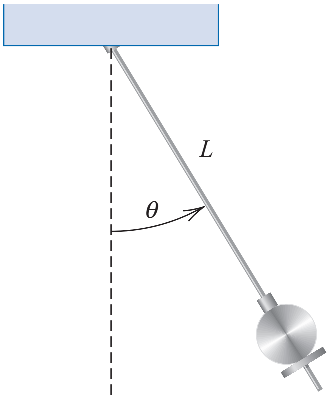
5

Initial-Value Problems for Ordinary Differential Equations

Introduction

The motion of a swinging pendulum under certain simplifying assumptions is described by the second-order differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0,$$



where L is the length of the pendulum, $g \approx 32.17$ ft/s^2 is the gravitational constant of the earth, and θ is the angle the pendulum makes with the vertical. If, in addition, we specify the position of the pendulum when the motion begins, $\theta(t_0) = \theta_0$, and its velocity at that point, $\theta'(t_0) = \theta'_0$, we have what is called an *initial-value problem*.

For small values of θ, the approximation $\theta \approx \sin\theta$ can be used to simplify this problem to the linear initial-value problem

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \quad \theta(t_0) = \theta_0, \quad \theta'(t_0) = \theta'_0.$$

This problem can be solved by a standard differential-equation technique. For larger values of θ, the assumption that $\theta = \sin\theta$ is not reasonable so approximation methods must be used. A problem of this type is considered in Exercise 8 of Section 5.9.

Any textbook on ordinary differential equations details a number of methods for explicitly finding solutions to first-order initial-value problems. In practice, however, few of the problems originating from the study of physical phenomena can be solved exactly.

The first part of this chapter is concerned with approximating the solution $y(t)$ to a problem of the form

$$\frac{dy}{dt} = f(t, y), \quad \text{for } a \le t \le b,$$

subject to an initial condition $y(a) = \alpha$. Later in the chapter we deal with the extension of these methods to a system of first-order differential equations in the form

$$\begin{aligned}
\frac{dy_1}{dt} &= f_1(t, y_1, y_2, \dots, y_n), \\
\frac{dy_2}{dt} &= f_2(t, y_1, y_2, \dots, y_n), \\
&\vdots \\
\frac{dy_n}{dt} &= f_n(t, y_1, y_2, \dots, y_n),
\end{aligned}$$

for $a \le t \le b$, subject to the initial conditions

$$y_1(a) = \alpha_1, \quad y_2(a) = \alpha_2, \quad \dots, \quad y_n(a) = \alpha_n.$$

We also examine the relationship of a system of this type to the general nth-order initial-value problem of the form

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}),$$

for $a \le t \le b$, subject to the initial conditions

$$y(a) = \alpha_1, \quad y'(a) = \alpha_2, \quad \dots, \quad y^{n-1}(a) = \alpha_n.$$

5.1 The Elementary Theory of Initial-Value Problems

Differential equations are used to model problems in science and engineering that involve the change of some variable with respect to another. Most of these problems require the solution of an *initial-value problem*, that is, the solution to a differential equation that satisfies a given initial condition.

In common real-life situations, the differential equation that models the problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to modify the problem by simplifying the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original problem. The other approach, which we will examine in this chapter, uses methods for approximating the solution of the original problem. This is the approach that is most commonly taken because the approximation methods give more accurate results and realistic error information.

The methods that we consider in this chapter do not produce a continuous approximation to the solution of the initial-value problem. Rather, approximations are found at certain specified, and often equally spaced, points. Some method of interpolation, commonly Hermite, is used if intermediate values are needed.

We need some definitions and results from the theory of ordinary differential equations before considering methods for approximating the solutions to initial-value problems.

Definition 5.1 A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2,)| \leq L|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a **Lipschitz constant** for f. ■

Example 1 Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) \mid 1 \leq t \leq 2 \text{ and } -3 \leq y \leq 4\}$.

Solution For each pair of points (t, y_1) and (t, y_2) in D we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|||y_1| - |y_2|| \leq 2|y_1 - y_2|.$$

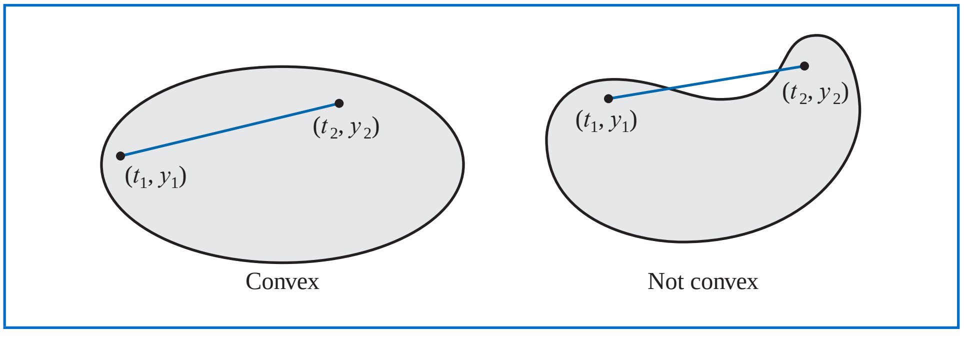
Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2. The smallest value possible for the Lipschitz constant for this problem is $L = 2$, because, for example,

$$|f(2, 1) - f(2, 0)| = |2 - 0| = 2|1 - 0|.$$ ■

Definition 5.2 A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever (t_1, y_1) and (t_2, y_2) belong to D, then $((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$ also belongs to D for every λ in $[0, 1]$. ■

In geometric terms, Definition 5.2 states that a set is convex provided that whenever two points belong to the set, the entire straight-line segment between the points also belongs to the set. (See Figure 5.1.) The sets we consider in this chapter are generally of the form $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$ for some constants a and b. It is easy to verify (see Exercise 7) that these sets are convex.

Figure 5.1



Theorem 5.3 Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \quad \text{for all } (t, y) \in D, \tag{5.1}$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L. ■

Rudolf Lipschitz (1832–1903) worked in many branches of mathematics, including number theory, Fourier series, differential equations, analytical mechanics, and potential theory. He is best known for this generalization of the work of Augustin-Louis Cauchy (1789–1857) and Guiseppe Peano (1856–1932).

The proof of Theorem 5.3 is discussed in Exercise 6; it is similar to the proof of the corresponding result for functions of one variable discussed in Exercise 27 of Section 1.1.

As the next theorem will show, it is often of significant interest to determine whether the function involved in an initial-value problem satisfies a Lipschitz condition in its second variable, and condition (5.1) is generally easier to apply than the definition. We should note, however, that Theorem 5.3 gives only sufficient conditions for a Lipschitz condition to hold. The function in Example 1, for instance, satisfies a Lipschitz condition, but the partial derivative with respect to y does not exist when $y = 0$.

The following theorem is a version of the fundamental existence and uniqueness theorem for first-order ordinary differential equations. Although the theorem can be proved with the hypothesis reduced somewhat, this form of the theorem is sufficient for our purposes. (The proof of the theorem, in approximately this form, can be found in [BiR], pp. 142–155.)

Theorem 5.4 Suppose that $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$. ■

Example 2 Use Theorem 5.4 to show that there is a unique solution to the initial-value problem

$$y' = 1 + t \sin(ty), \quad 0 \leq t \leq 2, \quad y(0) = 0.$$

Solution Holding t constant and applying the Mean Value Theorem to the function

$$f(t, y) = 1 + t \sin(ty),$$

we find that when $y_1 < y_2$, a number ξ in (y_1, y_2) exists with

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t, \xi) = t^2 \cos(\xi t).$$

Thus

$$|f(t, y_2) - f(t, y_1)| = |y_2 - y_1||t^2 \cos(\xi t)| \leq 4|y_2 - y_1|,$$

and f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$. Additionally, $f(t, y)$ is continuous when $0 \leq t \leq 2$ and $-\infty < y < \infty$, so Theorem 5.4 implies that a unique solution exists to this initial-value problem.

If you have completed a course in differential equations you might try to find the exact solution to this problem. ■

Well-Posed Problems

Now that we have, to some extent, taken care of the question of when initial-value problems have unique solutions, we can move to the second important consideration when approximating the solution to an initial-value problem. Initial-value problems obtained by observing physical phenomena generally only approximate the true situation, so we need to know whether small changes in the statement of the problem introduce correspondingly small changes in the solution. This is also important because of the introduction of round-off error when numerical methods are used. That is,

- Question: How do we determine whether a particular problem has the property that small changes, or perturbations, in the statement of the problem introduce correspondingly small changes in the solution?

As usual, we first need to give a workable definition to express this concept.

Definition 5.5 The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha, \tag{5.2}$$

is said to be a **well-posed problem** if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0, \tag{5.3}$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$ ■

The problem specified by (5.3) is called a **perturbed problem** associated with the original problem (5.2). It assumes the possibility of an error being introduced in the statement of the differential equation, as well as an error δ_0 being present in the initial condition.

Numerical methods will always be concerned with solving a perturbed problem because any round-off error introduced in the representation perturbs the original problem. Unless the original problem is well-posed, there is little reason to expect that the numerical solution to a perturbed problem will accurately approximate the solution to the original problem.

The following theorem specifies conditions that ensure that an initial-value problem is well-posed. The proof of this theorem can be found in [BiR], pp. 142–147.

Theorem 5.6 Suppose $D = \{(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is well-posed. ■

Example 3 Show that the initial-value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5. \tag{5.4}$$

is well posed on $D = \{(t, y) \mid 0 \le t \le 2 \text{ and } -\infty < y < \infty\}$.

Solution Because

$$\left|\frac{\partial(y - t^2 + 1)}{\partial y}\right| = |1| = 1,$$

Theorem 5.3 implies that $f(t, y) = y - t^2 + 1$ satisfies a Lipschitz condition in y on D with Lipschitz constant 1. Since f is continuous on D, Theorem 5.6 implies that the problem is well-posed.

As an illustration, consider the solution to the perturbed problem

$$\frac{dz}{dt} = z - t^2 + 1 + \delta, \quad 0 \le t \le 2, \quad z(0) = 0.5 + \delta_0, \tag{5.5}$$

where δ and δ_0 are constants. The solutions to Eqs. (5.4) and (5.5) are

$$y(t) = (t+1)^2 - 0.5e^t \quad \text{and} \quad z(t) = (t+1)^2 + (\delta + \delta_0 - 0.5)e^t - \delta,$$

respectively.

Suppose that ε is a positive number. If $|\delta| < \varepsilon$ and $|\delta_0| < \varepsilon$, then

$$|y(t) - z(t)| = |(\delta + \delta_0)e^t - \delta| \leq |\delta + \delta_0|e^2 + |\delta| \leq (2e^2 + 1)\varepsilon,$$

for all t. This implies that problem (5.4) is well-posed with $k(\varepsilon) = 2e^2 + 1$ for all $\varepsilon > 0$. ■

Maple can be used to solve many initial-value problems. Consider the problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Maple reserves the letter *D* to represent differentiation.

To define the differential equation and initial condition, enter

$deq := D(y)(t) = y(t) - t^2 + 1; init := y(0) = 0.5$

The names *deq* and *init* have been chosen by the user. The command to solve the initial-value problems is

$deqsol := dsolve(\{deq, init\}, y(t))$

and Maple responds with

$$y(t) = 1 + t^2 + 2t - \frac{1}{2}e^t$$

To use the solution to obtain a specific value, such as $y(1.5)$, we enter

$q := rhs(deqsol) : \ evalf(subs(t = 1.5, q))$

which gives

4.009155465

The function *rhs* (for *r*ight *h*and *s*ide) is used to assign the solution of the initial-value problem to the function q, which we then evaluate at $t = 1.5$.

The function *dsolve* can fail if an explicit solution to the initial-value problem cannot be found. For example, for the initial-value problem given in Example 2, the command

$deqsol2 := dsolve(\{D(y)(t) = 1 + t \cdot \sin(t \cdot y(t)), y(0) = 0\}, y(t))$

does not succeed because an explicit solution cannot be found. In this case a numerical method must be used.

EXERCISE SET 5.1

1. Use Theorem 5.4 to show that each of the following initial-value problems has a unique solution, and find the solution.

a. $y' = y\cos t, \quad 0 \leq t \leq 1, \quad y(0) = 1.$

b. $y' = \frac{2}{t}y + t^2e^t, \quad 1 \leq t \leq 2, \quad y(1) = 0.$

c. $y' = -\frac{2}{t}y + t^2e^t, \quad 1 \leq t \leq 2, \quad y(1) = \sqrt{2}e.$

d. $y' = \frac{4t^3y}{1+t^4}, \quad 0 \leq t \leq 1, \quad y(0) = 1.$

2. Show that each of the following initial-value problems has a unique solution and find the solution. Can Theorem 5.4 be applied in each case?

a. $y' = e^{t-y}, \quad 0 \le t \le 1, \quad y(0) = 1.$

b. $y' = t^{-2}(\sin 2t - 2ty), \quad 1 \le t \le 2, \quad y(1) = 2.$

c. $y' = -y + ty^{1/2}, \quad 2 \le t \le 3, \quad y(2) = 2.$

d. $y' = \dfrac{ty + y}{ty + t}, \quad 2 \le t \le 4, \quad y(2) = 4.$

3. For each choice of $f(t, y)$ given in parts (a)–(d):

i. Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$?

ii. Can Theorem 5.6 be used to show that the initial-value problem

$$y' = f(t, y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

is well-posed?

a. $f(t, y) = t^2 y + 1$ **b.** $f(t, y) = ty$ **c.** $f(t, y) = 1 - y$ **d.** $f(t, y) = -ty + \dfrac{4t}{y}$

4. For each choice of $f(t, y)$ given in parts (a)–(d):

i. Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$?

ii. Can Theorem 5.6 be used to show that the initial-value problem

$$y' = f(t, y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

is well-posed?

a. $f(t, y) = e^{t-y}$ **b.** $f(t, y) = \dfrac{1 + y}{1 + t}$ **c.** $f(t, y) = \cos(yt)$ **d.** $f(t, y) = \dfrac{y^2}{1 + t}$

5. For the following initial-value problems, show that the given equation implicitly defines a solution. Approximate $y(2)$ using Newton's method.

a. $y' = -\dfrac{y^3 + y}{(3y^2 + 1)t}, \quad 1 \le t \le 2, \quad y(1) = 1; \quad y^3 t + yt = 2$

b. $y' = -\dfrac{y \cos t + 2te^y}{\sin t + t^2 e^y + 2}, \quad 1 \le t \le 2, \quad y(1) = 0; \quad y \sin t + t^2 e^y + 2y = 1$

6. Prove Theorem 5.3 by applying the Mean Value Theorem 1,8 to $f(t, y)$, holding t fixed.

7. Show that, for any constants a and b, the set $D = \{(t, y) \mid a \le t \le b, -\infty < y < \infty\}$ is convex.

8. Suppose the perturbation $\delta(t)$ is proportional to t, that is, $\delta(t) = \delta t$ for some constant δ. Show directly that the following initial-value problems are well-posed.

a. $y' = 1 - y, \quad 0 \le t \le 2, \quad y(0) = 0$

b. $y' = t + y, \quad 0 \le t \le 2, \quad y(0) = -1$

c. $y' = \dfrac{2}{t} y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0$

d. $y' = -\dfrac{2}{t} y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = \sqrt{2}e$

9. *Picard's method* for solving the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is described as follows: Let $y_0(t) = \alpha$ for each t in $[a, b]$. Define a sequence $\{y_k(t)\}$ of functions by

$$y_k(t) = \alpha + \int_a^t f(\tau, y_{k-1}(\tau))\, d\tau, \quad k = 1, 2, \ldots.$$

a. Integrate $y' = f(t, y(t))$, and use the initial condition to derive Picard's method.

b. Generate $y_0(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$ for the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1.$$

c. Compare the result in part (b) to the Maclaurin series of the actual solution $y(t) = t + e^{-t}$.

5.2 Euler's Method

Euler's method is the most elementary approximation technique for solving initial-value problems. Although it is seldom used in practice, the simplicity of its derivation can be used to illustrate the techniques involved in the construction of some of the more advanced techniques, without the cumbersome algebra that accompanies these constructions.

The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha. \tag{5.6}$$

A continuous approximation to the solution $y(t)$ will not be obtained; instead, approximations to y will be generated at various values, called **mesh points**, in the interval $[a, b]$. Once the approximate solution is obtained at the points, the approximate solution at other points in the interval can be found by interpolation.

We first make the stipulation that the mesh points are equally distributed throughout the interval $[a, b]$. This condition is ensured by choosing a positive integer N and selecting the mesh points

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N.$$

The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$ is called the **step size**.

The use of elementary difference methods to approximate the solution to differential equations was one of the numerous mathematical topics that was first presented to the mathematical public by the most prolific of mathematicians, Leonhard Euler (1707–1783).

We will use Taylor's Theorem to derive Euler's method. Suppose that $y(t)$, the unique solution to (5.6), has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N - 1$,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i),$$

for some number ξ_i in (t_i, t_{i+1}). Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i),$$

and, because $y(t)$ satisfies the differential equation (5.6),

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i). \tag{5.7}$$

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1. \end{aligned} \tag{5.8}$$

Illustration In Example 1 we will use an algorithm for Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.