7. Verify the entries for the values of $n = 2$ and 3 in Table 4.12 on page 232 by finding the roots of the respective Legendre polynomials, and use the equations preceding this table to find the coefficients associated with the values.

8. Show that the formula $Q(P) = \sum_{i=1}^{n} c_i P(x_i)$ cannot have degree of precision greater than $2n - 1$, regardless of the choice of $c_1, \ldots, c_n$ and $x_1, \ldots, x_n$. [*Hint:* Construct a polynomial that has a double root at each of the x_i's.]

9. Apply Maple's Composite Gaussian Quadrature routine to approximate $\int_{-1}^{1} x^2 e^x\, dx$ in the following manner.

a. Use Gaussian Quadrature with $n = 8$ on the single interval $[-1, 1]$.

b. Use Gaussian Quadrature with $n = 4$ on the intervals $[-1, 0]$ and $[0, 1]$.

c. Use Gaussian Quadrature with $n = 2$ on the intervals $[-1, -0.5]$, $[-0.5, 0]$, $[0, 0.5]$ and $[0.5, 1]$.

d. Give an explanation for the accuracy of the results.

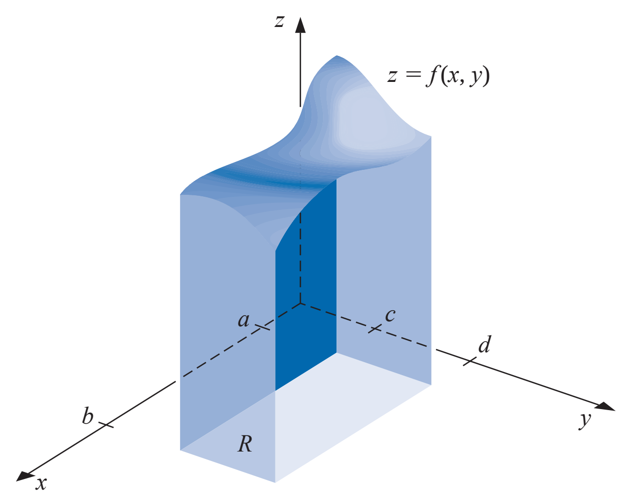
4.8 Multiple Integrals

The techniques discussed in the previous sections can be modified for use in the approximation of multiple integrals. Consider the double integral

$$\iint_R f(x, y)\, dA,$$

where $R = \{\, (x, y) \mid a \le x \le b, c \le y \le d \,\}$, for some constants a, b, c, and d, is a rectangular region in the plane. (See Figure 4.18.)

Figure 4.18



The following illustration shows how the Composite Trapezoidal rule using two subintervals in each coordinate direction would be applied to this integral.

Illustration Writing the double integral as an iterated integral gives

$$\iint_R f(x, y)\, dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

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To simplify notation, let $k = (d-c)/2$ and $h = (b-a)/2$. Apply the Composite Trapezoidal rule to the interior integral to obtain

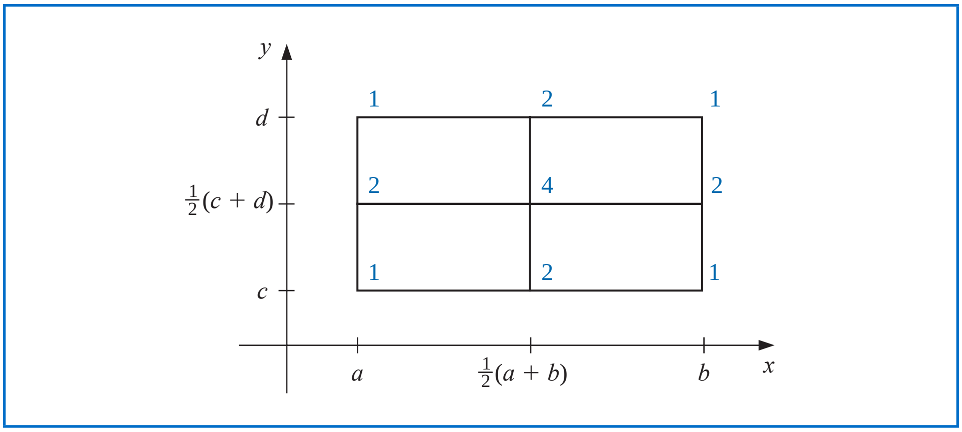
$$\int_c^d f(x, y)\, dy \approx \frac{k}{2}\left[f(x, c) + f(x, d) + 2f\left(x, \frac{c+d}{2}\right)\right].$$

This approximation is of order $O\left((d-c)^3\right)$. Then apply the Composite Trapezoidal rule again to approximate the integral of this function of x:

$$\begin{aligned}
\int_a^b \left(\int_c^d f(x, y)\, dy\right) dx &\approx \int_a^b \left(\frac{d-c}{4}\right)\left[f(x, c) + 2f\left(x, \frac{c+d}{2}\right) + f(d)\right] dx \\
&= \frac{b-a}{4}\left(\frac{d-c}{4}\right)\left[f(a, c) + 2f\left(a, \frac{c+d}{2}\right) + f(a, d)\right] \\
&\quad + \frac{b-a}{4}\left(2\left(\frac{d-c}{4}\right)\left[f\left(\frac{a+b}{2}, c\right)\right.\right. \\
&\quad \left.\left. + 2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \left(\frac{a+b}{2}, d\right)\right]\right) \\
&\quad + \frac{b-a}{4}\left(\frac{d-c}{4}\right)\left[f(b, c) + 2f\left(b, \frac{c+d}{2}\right) + f(b, d)\right] \\
&= \frac{(b-a)(d-c)}{16}\left[f(a, c) + f(a, d) + f(b, c) + f(b, d)\right. \\
&\quad + 2\left(f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + f\left(a, \frac{c+d}{2}\right)\right. \\
&\quad \left.\left. + f\left(b, \frac{c+d}{2}\right)\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]
\end{aligned}$$

This approximation is of order $O\left((b-a)(d-c)\left[(b-a)^2 + (d-c)^2\right]\right)$. Figure 4.19 shows a grid with the number of functional evaluations at each of the nodes used in the approximation. □

Figure 4.19



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As the illustration shows, the procedure is quite straightforward. But the number of function evaluations grows with the square of the number required for a single integral. In a practical situation we would not expect to use a method as elementary as the Composite Trapezoidal rule. Instead we will employ the Composite Simpson's rule to illustrate the general approximation technique, although any other composite formula could be used in its place.

To apply the Composite Simpson's rule, we divide the region R by partitioning both $[a, b]$ and $[c, d]$ into an even number of subintervals. To simplify the notation, we choose even integers n and m and partition $[a, b]$ and $[c, d]$ with the evenly spaced mesh points $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_m$, respectively. These subdivisions determine step sizes $h = (b-a)/n$ and $k = (d-c)/m$. Writing the double integral as the iterated integral

$$\iint_R f(x, y)\, dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx,$$

we first use the Composite Simpson's rule to approximate

$$\int_c^d f(x, y)\, dy,$$

treating x as a constant.

Let $y_j = c + jk$, for each $j = 0, 1, \ldots, m$. Then

$$\int_c^d f(x, y)\, dy = \frac{k}{3}\left[f(x, y_0) + 2\sum_{j=1}^{(m/2)-1} f(x, y_{2j}) + 4\sum_{j=1}^{m/2} f(x, y_{2j-1}) + f(x, y_m) \right] - \frac{(d-c)k^4}{180}\frac{\partial^4 f}{\partial y^4}(x, \mu),$$

for some μ in (c, d). Thus

$$\begin{aligned}\int_a^b \int_c^d f(x, y)\, dy\, dx = \frac{k}{3}\Bigg[&\int_a^b f(x, y_0)\, dx + 2\sum_{j=1}^{(m/2)-1} \int_a^b f(x, y_{2j})\, dx \\ &+ 4\sum_{j=1}^{m/2} \int_a^b f(x, y_{2j-1})\, dx + \int_a^b f(x, y_m)\, dx \Bigg] \\ &- \frac{(d-c)k^4}{180} \int_a^b \frac{\partial^4 f}{\partial y^4}(x, \mu)\, dx.\end{aligned}$$

Composite Simpson's rule is now employed on the integrals in this equation. Let $x_i = a + ih$, for each $i = 0, 1, \ldots, n$. Then for each $j = 0, 1, \ldots, m$, we have

$$\int_a^b f(x, y_j)\, dx = \frac{h}{3}\left[f(x_0, y_j) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i}, y_j) + 4\sum_{i=1}^{n/2} f(x_{2i-1}, y_j) + f(x_n, y_j) \right] - \frac{(b-a)h^4}{180}\frac{\partial^4 f}{\partial x^4}(\xi_j, y_j),$$

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for some ξ_j in (a, b). The resulting approximation has the form

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \approx \frac{hk}{9}\Bigg\{\Bigg[f(x_0,y_0) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i},y_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1},y_0) + f(x_n,y_0)\Bigg]$$

$$+ 2\Bigg[\sum_{j=1}^{(m/2)-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{(m/2)-1}\sum_{i=1}^{(n/2)-1} f(x_{2i},y_{2j}) + 4\sum_{j=1}^{(m/2)-1}\sum_{i=1}^{n/2} f(x_{2i-1},y_{2j}) + \sum_{j=1}^{(m/2)-1} f(x_n,y_{2j})\Bigg]$$

$$+ 4\Bigg[\sum_{j=1}^{m/2} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m/2}\sum_{i=1}^{(n/2)-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{m/2}\sum_{i=1}^{n/2} f(x_{2i-1},y_{2j-1}) + \sum_{j=1}^{m/2} f(x_n,y_{2j-1})\Bigg]$$

$$+ \Bigg[f(x_0,y_m) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i},y_m) + 4\sum_{i=1}^{n/2} f(x_{2i-1},y_m) + f(x_n,y_m)\Bigg]\Bigg\}.$$

The error term E is given by

$$E = \frac{-k(b-a)h^4}{540}\Bigg[\frac{\partial^4 f}{\partial x^4}(\xi_0,y_0) + 2\sum_{j=1}^{(m/2)-1}\frac{\partial^4 f}{\partial x^4}(\xi_{2j},y_{2j}) + 4\sum_{j=1}^{m/2}\frac{\partial^4 f}{\partial x^4}(\xi_{2j-1},y_{2j-1})$$

$$+ \frac{\partial^4 f}{\partial x^4}(\xi_m,y_m)\Bigg] - \frac{(d-c)k^4}{180}\int_a^b \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx.$$

If $\partial^4 f/\partial x^4$ is continuous, the Intermediate Value Theorem 1.11 can be repeatedly applied to show that the evaluation of the partial derivatives with respect to x can be replaced by a common value and that

$$E = \frac{-k(b-a)h^4}{540}\left[3m\frac{\partial^4 f}{\partial x^4}(\overline{\eta},\overline{\mu})\right] - \frac{(d-c)k^4}{180}\int_a^b \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx,$$

for some $(\overline{\eta},\overline{\mu})$ in R. If $\partial^4 f/\partial y^4$ is also continuous, the Weighted Mean Value Theorem for Integrals 1.13 implies that

$$\int_a^b \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx = (b-a)\frac{\partial^4 f}{\partial y^4}(\hat{\eta},\hat{\mu}),$$

for some $(\hat{\eta},\hat{\mu})$ in R. Because $m = (d-c)/k$, the error term has the form

$$E = \frac{-k(b-a)h^4}{540}\left[3m\frac{\partial^4 f}{\partial x^4}(\overline{\eta},\overline{\mu})\right] - \frac{(d-c)(b-a)}{180}k^4\frac{\partial^4 f}{\partial y^4}(\hat{\eta},\hat{\mu})$$

which simplifies to

$$E = -\frac{(d-c)(b-a)}{180}\left[h^4\frac{\partial^4 f}{\partial x^4}(\overline{\eta},\overline{\mu}) + k^4\frac{\partial^4 f}{\partial y^4}(\hat{\eta},\hat{\mu})\right],$$

for some $(\overline{\eta},\overline{\mu})$ and $(\hat{\eta},\hat{\mu})$ in R.

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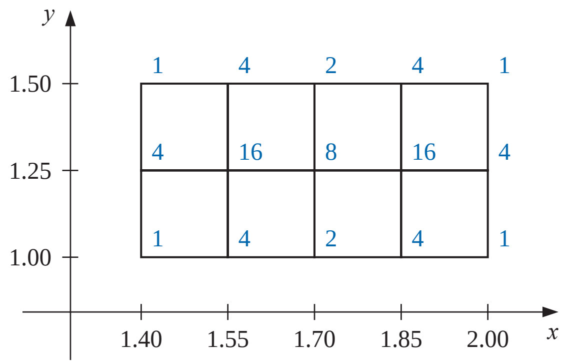
Example 1 Use Composite Simpson's rule with $n = 4$ and $m = 2$ to approximate

$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx,$$

Solution The step sizes for this application are $h = (2.0 - 1.4)/4 = 0.15$ and $k = (1.5 - 1.0)/2 = 0.25$. The region of integration R is shown in Figure 4.20, together with the nodes (x_i, y_j), where $i = 0, 1, 2, 3, 4$ and $j = 0, 1, 2$. It also shows the coefficients $w_{i,j}$ of $f(x_i, y_i) = \ln(x_i + 2y_i)$ in the sum that gives the Composite Simpson's rule approximation to the integral.

Figure 4.20



The approximation is

$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx \approx \frac{(0.15)(0.25)}{9} \sum_{i=0}^{4} \sum_{j=0}^{2} w_{i,j} \ln(x_i + 2y_j)$$

$$= 0.4295524387.$$

We have

$$\frac{\partial^4 f}{\partial x^4}(x, y) = \frac{-6}{(x + 2y)^4} \quad \text{and} \quad \frac{\partial^4 f}{\partial y^4}(x, y) = \frac{-96}{(x + 2y)^4},$$

and the maximum values of the absolute values of these partial derivatives occur on R when $x = 1.4$ and $y = 1.0$. So the error is bounded by

$$|E| \le \frac{(0.5)(0.6)}{180} \left[(0.15)^4 \max_{(x,y) \text{in} R} \frac{6}{(x + 2y)^4} + (0.25)^4 \max_{(x,y) \text{in} R} \frac{96}{(x + 2y)^4} \right] \le 4.72 \times 10^{-6}.$$

The actual value of the integral to ten decimal places is

$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx = 0.4295545265,$$

so the approximation is accurate to within 2.1×10^{-6}. ■

The same techniques can be applied for the approximation of triple integrals as well as higher integrals for functions of more than three variables. The number of functional evaluations required for the approximation is the product of the number of functional evaluations required when the method is applied to each variable.

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Gaussian Quadrature for Double Integral Approximation

To reduce the number of functional evaluations, more efficient methods such as Gaussian quadrature, Romberg integration, or Adaptive quadrature can be incorporated in place of the Newton-Cotes formulas. The following example illustrates the use of Gaussian quadrature for the integral considered in Example 1.

Example 2 Use Gaussian quadrature with $n = 3$ in both dimensions to approximate the integral

$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx.$$

Solution Before employing Gaussian quadrature to approximate this integral, we need to transform the region of integration

$$R = \{ (x, y) \mid 1.4 \le x \le 2.0, 1.0 \le y \le 1.5 \}$$

into

$$\hat{R} = \{ (u, v) \mid -1 \le u \le 1, -1 \le v \le 1 \}.$$

The linear transformations that accomplish this are

$$u = \frac{1}{2.0 - 1.4}(2x - 1.4 - 2.0) \quad \text{and} \quad v = \frac{1}{1.5 - 1.0}(2y - 1.0 - 1.5),$$

or, equivalently, $x = 0.3u + 1.7$ and $y = 0.25v + 1.25$. Employing this change of variables gives an integral on which Gaussian quadrature can be applied:

$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx = 0.075 \int_{-1}^{1} \int_{-1}^{1} \ln(0.3u + 0.5v + 4.2)\, dv\, du.$$

The Gaussian quadrature formula for $n = 3$ in both u and v requires that we use the nodes

$$u_1 = v_1 = r_{3,2} = 0, \quad u_0 = v_0 = r_{3,1} = -0.7745966692,$$

and

$$u_2 = v_2 = r_{3,3} = 0.7745966692.$$

The associated weights are $c_{3,2} = 0.\overline{8}$ and $c_{3,1} = c_{3,3} = 0.\overline{5}$. (These are given in Table 4.12 on page 232.) The resulting approximation is

$$\begin{aligned}\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx &\approx 0.075 \sum_{i=1}^{3} \sum_{j=1}^{3} c_{3,i} c_{3,j} \ln(0.3 r_{3,i} + 0.5 r_{3,j} + 4.2) \\ &= 0.4295545313.\end{aligned}$$

Although this result requires only 9 functional evaluations compared to 15 for the Composite Simpson's rule considered in Example 1, it is accurate to within 4.8×10^{-9}, compared to 2.1×10^{-6} accuracy in Example 1. ■

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Non-Rectangular Regions

The use of approximation methods for double integrals is not limited to integrals with rectangular regions of integration. The techniques previously discussed can be modified to approximate double integrals of the form

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx \tag{4.42}$$

or

$$\int_c^d \int_{a(y)}^{b(y)} f(x, y)\, dx\, dy. \tag{4.43}$$

In fact, integrals on regions not of this type can also be approximated by performing appropriate partitions of the region. (See Exercise 10.)

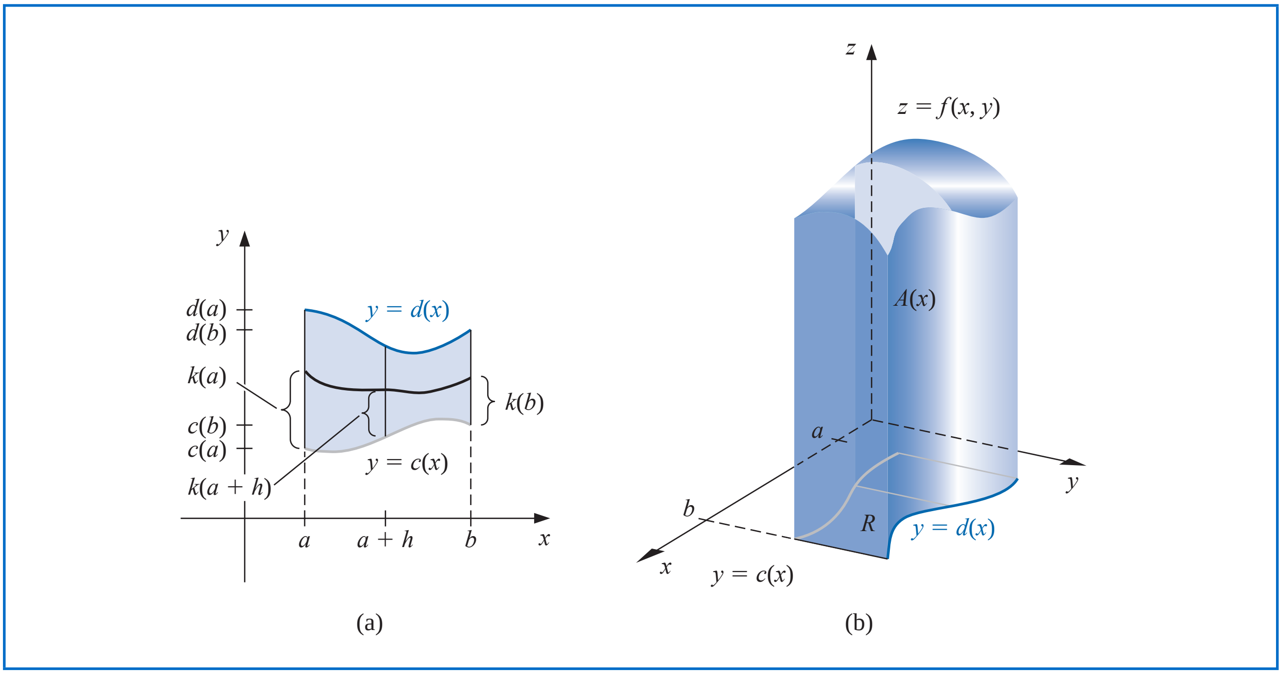
To describe the technique involved with approximating an integral in the form

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx,$$

we will use the basic Simpson's rule to integrate with respect to both variables. The step size for the variable x is $h = (b - a)/2$, but the step size for y varies with x (see Figure 4.21) and is written

$$k(x) = \frac{d(x) - c(x)}{2}.$$

Figure 4.21



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This gives

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \approx \int_a^b \frac{k(x)}{3}[f(x,c(x)) + 4f(x,c(x)+k(x)) + f(x,d(x))]\,dx$$

$$\approx \frac{h}{3}\left\{\frac{k(a)}{3}[f(a,c(a)) + 4f(a,c(a)+k(a)) + f(a,d(a))]\right.$$

$$+ \frac{4k(a+h)}{3}[f(a+h,c(a+h)) + 4f(a+h,c(a+h)$$

$$+ k(a+h)) + f(a+h,d(a+h))]$$

$$\left. + \frac{k(b)}{3}[f(b,c(b)) + 4f(b,c(b)+k(b)) + f(b,d(b))]\right\}.$$

Algorithm 4.4 applies the Composite Simpson's rule to an integral in the form (4.42). Integrals in the form (4.43) can, of course, be handled similarly.

ALGORITHM 4.4

Simpson's Double Integral

To approximate the integral

$$I = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx :$$

INPUT endpoints a, b: even positive integers m, n.

OUTPUT approximation J to I.

Step 1 Set $h = (b-a)/n$;
$J_1 = 0$; (*End terms.*)
$J_2 = 0$; (*Even terms.*)
$J_3 = 0$. (*Odd terms.*)

Step 2 For $i = 0, 1, \ldots, n$ do Steps 3–8.

Step 3 Set $x = a + ih$; (*Composite Simpson's method for* x.)
$HX = (d(x) - c(x))/m$;
$K_1 = f(x, c(x)) + f(x, d(x))$; (*End terms.*)
$K_2 = 0$; (*Even terms.*)
$K_3 = 0$. (*Odd terms.*)

Step 4 For $j = 1, 2, \ldots, m-1$ do Step 5 and 6.

Step 5 Set $y = c(x) + jHX$;
$Q = f(x, y)$.

Step 6 If j is even then set $K_2 = K_2 + Q$
else set $K_3 = K_3 + Q$.

Step 7 Set $L = (K_1 + 2K_2 + 4K_3)HX/3$.

$$\left(L \approx \int_{c(x_i)}^{d(x_i)} f(x_i, y)\,dy \quad \textit{by the Composite Simpson's method.}\right)$$

Step 8 If $i = 0$ or $i = n$ then set $J_1 = J_1 + L$
else if i is even then set $J_2 = J_2 + L$
else set $J_3 = J_3 + L$.

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Step 9 Set $J = h(J_1 + 2J_2 + 4J_3)/3$.

Step 10 OUTPUT (J);
STOP.

To apply Gaussian quadrature to the double integral

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx,$$

first requires transforming, for each x in $[a, b]$, the variable y in the interval $[c(x), d(x)]$ into the variable t in the interval $[-1, 1]$. This linear transformation gives

$$f(x, y) = f\left(x, \frac{(d(x) - c(x))t + d(x) + c(x)}{2}\right) \quad \text{and} \quad dy = \frac{d(x) - c(x)}{2}\, dt.$$

The reduced calculation makes it generally worthwhile to apply Gaussian quadrature rather than a Simpson's technique when approximating double integrals.

Then, for each x in $[a, b]$, we apply Gaussian quadrature to the resulting integral

$$\int_{c(x)}^{d(x)} f(x, y)\, dy = \int_{-1}^{1} f\left(x, \frac{(d(x) - c(x))t + d(x) + c(x)}{2}\right) dt$$

to produce

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx \approx \int_a^b \frac{d(x)-c(x)}{2} \sum_{j=1}^{n} c_{n,j} f\left(x, \frac{(d(x) - c(x))r_{n,j} + d(x) + c(x)}{2}\right) dx,$$

where, as before, the roots $r_{n,j}$ and coefficients $c_{n,j}$ come from Table 4.12 on page 232. Now the interval $[a, b]$ is transformed to $[-1, 1]$, and Gaussian quadrature is applied to approximate the integral on the right side of this equation. The details are given in Algorithm 4.5.

ALGORITHM
4.5

Gaussian Double Integral

To approximate the integral

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx :$$

INPUT endpoints a, b; positive integers m, n.
(The roots $r_{i,j}$ and coefficients $c_{i,j}$ need to be available for $i = \max\{m, n\}$ and for $1 \le j \le i$.)

OUTPUT approximation J to I.

Step 1 Set $h_1 = (b - a)/2$;
$h_2 = (b + a)/2$;
$J = 0$.

Step 2 For $i = 1, 2, \ldots, m$ do Steps 3–5.

Step 3 Set $JX = 0$;
$x = h_1 r_{m,i} + h_2$;
$d_1 = d(x)$;
$c_1 = c(x)$;
$k_1 = (d_1 - c_1)/2$;
$k_2 = (d_1 + c_1)/2$.

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Step 4 For $j = 1, 2, \ldots, n$ do
set $y = k_1 r_{n,j} + k_2$;
$Q = f(x, y)$;
$JX = JX + c_{n,j} Q$.

Step 5 Set $J = J + c_{m,i} k_1 JX$.

Step 6 Set $J = h_1 J$.

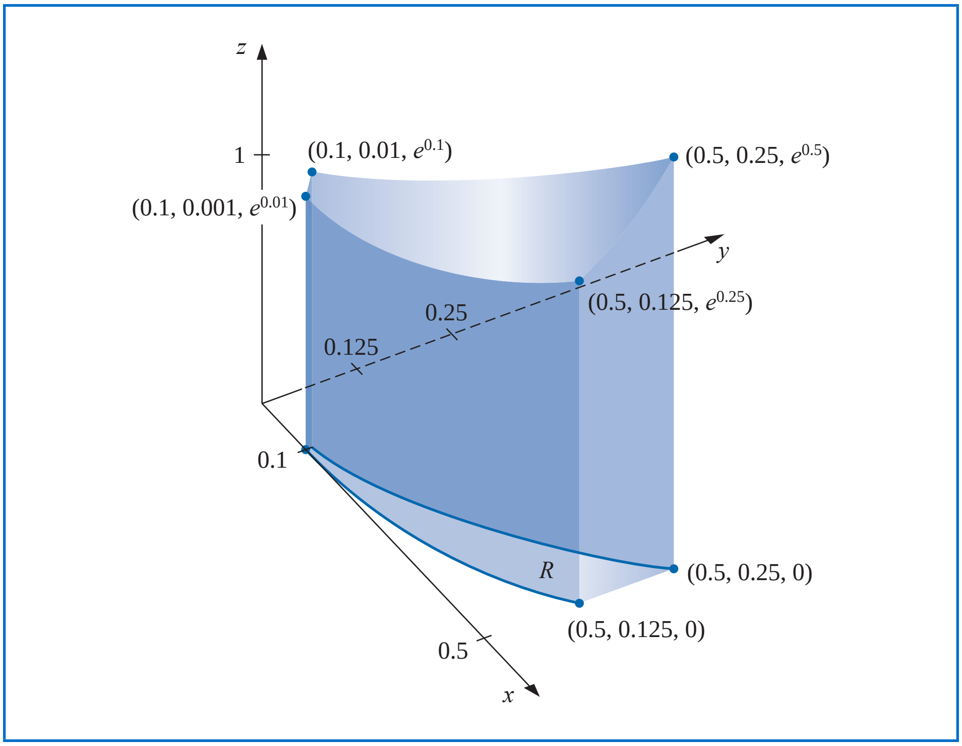
Step 7 OUTPUT (J);
STOP. ■

Illustration The volume of the solid in Figure 4.22 is approximated by applying Simpson's Double Integral Algorithm with $n = m = 10$ to

$$\int_{0.1}^{0.5} \int_{x^3}^{x^2} e^{y/x}\, dy\, dx.$$

This requires 121 evaluations of the function $f(x, y) = e^{y/x}$ and produces the value 0.0333054, which approximates the volume of the solid shown in Figure 4.22 to nearly seven decimal places. Applying the Gaussian Quadrature Algorithm with $n = m = 5$ requires only 25 function evaluations and gives the approximation 0.03330556611, which is accurate to 11 decimal places. □

Figure 4.22



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Triple Integral Approximation

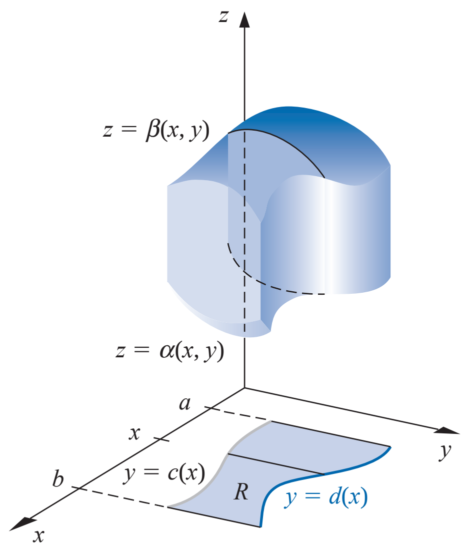
Triple integrals of the form

$$\int_a^b \int_{c(x)}^{d(x)} \int_{\alpha(x,y)}^{\beta(x,y)} f(x, y, z)\, dz\, dy\, dx$$

The reduced calculation makes it almost always worthwhile to apply Gaussian quadrature rather than a Simpson's technique when approximating triple or higher integrals.

(see Figure 4.23) are approximated in a similar manner. Because of the number of calculations involved, Gaussian quadrature is the method of choice. Algorithm 4.6 implements this procedure.

Figure 4.23



ALGORITHM 4.6

Gaussian Triple Integral

To approximate the integral

$$\int_a^b \int_{c(x)}^{d(x)} \int_{\alpha(x,y)}^{\beta(x,y)} f(x, y, z)\, dz\, dy\, dx :$$

INPUT endpoints a, b; positive integers m, n, p.
(The roots $r_{i,j}$ and coefficients $c_{i,j}$ need to be available for $i = \max\{n, m, p\}$ and for $1 \le j \le i$.)

OUTPUT approximation J to I.

Step 1 Set $h_1 = (b - a)/2$;
$h_2 = (b + a)/2$;
$J = 0$.

Step 2 For $i = 1, 2, \ldots, m$ do Steps 3–8.

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Step 3 Set $JX = 0$;
$x = h_1 r_{m,i} + h_2$;
$d_1 = d(x)$;
$c_1 = c(x)$;
$k_1 = (d_1 - c_1)/2$;
$k_2 = (d_1 + c_1)/2$.

Step 4 For $j = 1, 2, \ldots, n$ do Steps 5–7.

Step 5 Set $JY = 0$;
$y = k_1 r_{n,j} + k_2$;
$\beta_1 = \beta(x, y)$;
$\alpha_1 = \alpha(x, y)$;
$l_1 = (\beta_1 - \alpha_1)/2$;
$l_2 = (\beta_1 + \alpha_1)/2$.

Step 6 For $k = 1, 2, \ldots, p$ do
set $z = l_1 r_{p,k} + l_2$;
$Q = f(x, y, z)$;
$JY = JY + c_{p,k} Q$.

Step 7 Set $JX = JX + c_{n,j} l_1 JY$.

Step 8 Set $J = J + c_{m,i} k_1 JX$.

Step 9 Set $J = h_1 J$.

Step 10 OUTPUT (J);
STOP.

The following example requires the evaluation of four triple integrals.

Illustration The center of a mass of a solid region D with density function σ occurs at

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right),$$

where

$$M_{yz} = \iiint_D x\sigma(x, y, z)\, dV, \quad M_{xz} = \iiint_D y\sigma(x, y, z)\, dV$$

and

$$M_{xy} = \iiint_D z\sigma(x, y, z)\, dV$$

are the moments about the coordinate planes and the mass of D is

$$M = \iiint_D \sigma(x, y, z)\, dV.$$

The solid shown in Figure 4.24 is bounded by the upper nappe of the cone $z^2 = x^2 + y^2$ and the plane $z = 2$. Suppose that this solid has density function given by

$$\sigma(x, y, z) = \sqrt{x^2 + y^2}.$$

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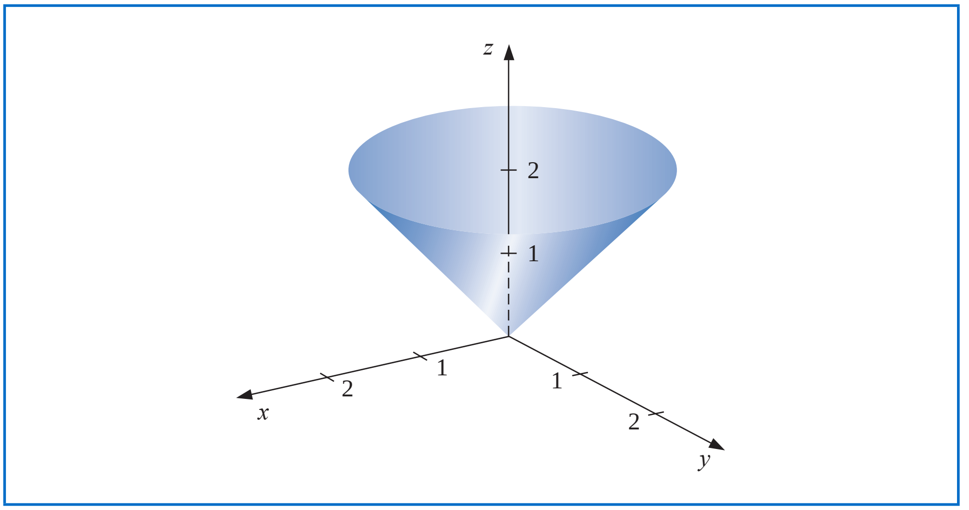
Figure 4.24



Applying the Gaussian Triple Integral Algorithm 4.6 with $n = m = p = 5$ requires 125 function evaluations per integral and gives the following approximations:

$$M = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} \sqrt{x^2 + y^2}\, dz\, dy\, dx$$

$$= 4 \int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} \sqrt{x^2 + y^2}\, dz\, dy\, dx \approx 8.37504476,$$

$$M_{yz} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} x\sqrt{x^2 + y^2}\, dz\, dy\, dx \approx -5.55111512 \times 10^{-17},$$

$$M_{xz} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} y\sqrt{x^2 + y^2}\, dz\, dy\, dx \approx -8.01513675 \times 10^{-17},$$

$$M_{xy} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} z\sqrt{x^2 + y^2}\, dz\, dy\, dx \approx 13.40038156.$$

This implies that the approximate location of the center of mass is

$$(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 1.60003701).$$

These integrals are quite easy to evaluate directly. If you do this, you will find that the exact center of mass occurs at $(0, 0, 1.6)$. □

Multiple integrals can be evaluated in Maple using the *MultInt* command in the *MultivariateCalculus* subpackage of the *Student* package. For example, to evaluate the multiple integral

$$\int_{2}^{4} \int_{x-1}^{x+6} \int_{-2}^{4+y^2} x^2 + y^2 + z\, dz\, dy\, dx$$

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we first load the package and define the function with

$with(Student[MultivariateCalculus])$: $f := (x, y, z) \to x^2 + y^2 + z$

Then issue the command

$MultiInt(f(x, y, z), z = -2..4 + y^2, y = x - 1..x + 6, x = 2..4)$

which produces the result

$$1.995885970$$

EXERCISE SET 4.8

1. Use Algorithm 4.4 with $n = m = 4$ to approximate the following double integrals, and compare the results to the exact answers.

a. $\int_{2.1}^{2.5} \int_{1.2}^{1.4} xy^2\, dy\, dx$

b. $\int_{0}^{0.5} \int_{0}^{0.5} e^{y-x}\, dy\, dx$

c. $\int_{2}^{2.2} \int_{x}^{2x} (x^2 + y^3)\, dy\, dx$

d. $\int_{1}^{1.5} \int_{0}^{x} (x^2 + \sqrt{y})\, dy\, dx$

2. Find the smallest values for $n = m$ so that Algorithm 4.4 can be used to approximate the integrals in Exercise 1 to within 10^{-6} of the actual value.

3. Use Algorithm 4.4 with (i) $n = 4$, $m = 8$, (ii) $n = 8$, $m = 4$, and (iii) $n = m = 6$ to approximate the following double integrals, and compare the results to the exact answers.

a. $\int_{0}^{\pi/4} \int_{\sin x}^{\cos x} (2y \sin x + \cos^2 x)\, dy\, dx$

b. $\int_{1}^{e} \int_{1}^{x} \ln xy\, dy\, dx$

c. $\int_{0}^{1} \int_{x}^{2x} (x^2 + y^3)\, dy\, dx$

d. $\int_{0}^{1} \int_{x}^{2x} (y^2 + x^3)\, dy\, dx$

e. $\int_{0}^{\pi} \int_{0}^{x} \cos x\, dy\, dx$

f. $\int_{0}^{\pi} \int_{0}^{x} \cos y\, dy\, dx$

g. $\int_{0}^{\pi/4} \int_{0}^{\sin x} \frac{1}{\sqrt{1 - y^2}}\, dy\, dx$

h. $\int_{-\pi}^{3\pi/2} \int_{0}^{2\pi} (y \sin x + x \cos y)\, dy\, dx$

4. Find the smallest values for $n = m$ so that Algorithm 4.4 can be used to approximate the integrals in Exercise 3 to within 10^{-6} of the actual value.

5. Use Algorithm 4.5 with $n = m = 2$ to approximate the integrals in Exercise 1, and compare the results to those obtained in Exercise 1.

6. Find the smallest values of $n = m$ so that Algorithm 4.5 can be used to approximate the integrals in Exercise 1 to within 10^{-6}. Do not continue beyond $n = m = 5$. Compare the number of functional evaluations required to the number required in Exercise 2.

7. Use Algorithm 4.5 with (i) $n = m = 3$, (ii) $n = 3$, $m = 4$, (iii) $n = 4$, $m = 3$, and (iv) $n = m = 4$ to approximate the integrals in Exercise 3.

8. Use Algorithm 4.5 with $n = m = 5$ to approximate the integrals in Exercise 3. Compare the number of functional evaluations required to the number required in Exercise 4.

9. Use Algorithm 4.4 with $n = m = 14$ and Algorithm 4.5 with $n = m = 4$ to approximate

$$\iint_R e^{-(x+y)}\, dA,$$

for the region R in the plane bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

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10. Use Algorithm 4.4 to approximate

$$\iint_R \sqrt{xy+y^2}\, dA,$$

where R is the region in the plane bounded by the lines $x+y=6$, $3y-x=2$, and $3x-y=2$. First partition R into two regions R_1 and R_2 on which Algorithm 4.4 can be applied. Use $n=m=6$ on both R_1 and R_2.

11. A plane lamina is a thin sheet of continuously distributed mass. If σ is a function describing the density of a lamina having the shape of a region R in the xy-plane, then the center of the mass of the lamina $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{\iint_R x\sigma(x,y)\, dA}{\iint_R \sigma(x,y)\, dA}, \quad \bar{y} = \frac{\iint_R y\sigma(x,y)\, dA}{\iint_R \sigma(x,y)\, dA}.$$

Use Algorithm 4.4 with $n=m=14$ to find the center of mass of the lamina described by $R=\{(x,y) \mid 0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}\}$ with the density function $\sigma(x,y)=e^{-(x^2+y^2)}$. Compare the approximation to the exact result.

12. Repeat Exercise 11 using Algorithm 4.5 with $n=m=5$.

13. The area of the surface described by $z=f(x,y)$ for (x,y) in R is given by

$$\iint_R \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\, dA.$$

Use Algorithm 4.4 with $n=m=8$ to find an approximation to the area of the surface on the hemisphere $x^2+y^2+z^2=9$, $z \ge 0$ that lies above the region in the plane described by $R=\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

14. Repeat Exercise 13 using Algorithm 4.5 with $n=m=4$.

15. Use Algorithm 4.6 with $n=m=p=2$ to approximate the following triple integrals, and compare the results to the exact answers.

a. $\displaystyle\int_0^1 \int_1^2 \int_0^{0.5} e^{x+y+z}\, dz\, dy\, dx$

b. $\displaystyle\int_0^1 \int_x^1 \int_0^{y} y^2 z\, dz\, dy\, dx$

c. $\displaystyle\int_0^1 \int_{x^2}^x \int_{x-y}^{x+y} y\, dz\, dy\, dx$

d. $\displaystyle\int_0^1 \int_{x^2}^x \int_{x-y}^{x+y} z\, dz\, dy\, dx$

e. $\displaystyle\int_0^\pi \int_0^x \int_0^{xy} \frac{1}{y} \sin\frac{z}{y}\, dz\, dy\, dx$

f. $\displaystyle\int_0^1 \int_0^1 \int_{-xy}^{xy} e^{x^2+y^2}\, dz\, dy\, dx$

16. Repeat Exercise 15 using $n=m=p=3$.

17. Repeat Exercise 15 using $n=m=p=4$ and $n=m=p=5$.

18. Use Algorithm 4.6 with $n=m=p=4$ to approximate

$$\iiint_S xy \sin(yz)\, dV,$$

where S is the solid bounded by the coordinate planes and the planes $x=\pi$, $y=\pi/2$, $z=\pi/3$. Compare this approximation to the exact result.

19. Use Algorithm 4.6 with $n=m=p=5$ to approximate

$$\iiint_S \sqrt{xyz}\, dV,$$

where S is the region in the first octant bounded by the cylinder $x^2+y^2=4$, the sphere $x^2+y^2+z^2=4$, and the plane $x+y+z=8$. How many functional evaluations are required for the approximation?

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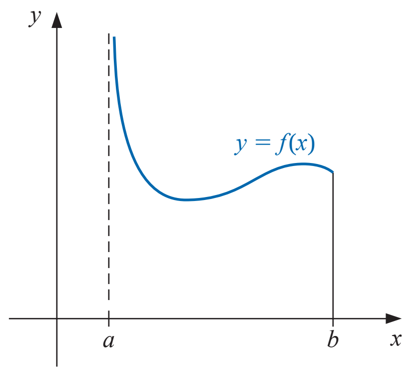
4.9 Improper Integrals

Improper integrals result when the notion of integration is extended either to an interval of integration on which the function is unbounded or to an interval with one or more infinite endpoints. In either circumstance, the normal rules of integral approximation must be modified.

Left Endpoint Singularity

We will first consider the situation when the integrand is unbounded at the left endpoint of the interval of integration, as shown in Figure 4.25. In this case we say that f has a **singularity** at the endpoint a. We will then show how other improper integrals can be reduced to problems of this form.

Figure 4.25



It is shown in calculus that the improper integral with a singularity at the left endpoint,

$$\int_a^b \frac{dx}{(x-a)^p},$$

converges if and only if $0 < p < 1$, and in this case, we define

$$\int_a^b \frac{1}{(x-a)^p}\, dx = \lim_{M\to a^+} \left.\frac{(x-a)^{1-p}}{1-p}\right|_{x=M}^{x=b} = \frac{(b-a)^{1-p}}{1-p}.$$

Example 1 Show that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges but $\int_0^1 \frac{1}{x^2}\, dx$ diverges.

Solution For the first integral we have

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{M\to 0^+} \int_M^1 x^{-1/2}\, dx = \lim_{M\to 0^+} 2x^{1/2}\Big|_{x=M}^{x=1} = 2 - 0 = 2,$$

but the second integral

$$\int_0^1 \frac{1}{x^2}\, dx = \lim_{M\to 0^+} \int_M^1 x^{-2}\, dx = \lim_{M\to 0^+} -x^{-1}\Big|_{x=M}^{x=1}$$

is unbounded.

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If f is a function that can be written in the form

$$f(x) = \frac{g(x)}{(x-a)^p},$$

where $0 < p < 1$ and g is continuous on $[a, b]$, then the improper integral

$$\int_a^b f(x)\,dx$$

also exists. We will approximate this integral using the Composite Simpson's rule, provided that $g \in C^5[a, b]$. In that case, we can construct the fourth Taylor polynomial, $P_4(x)$, for g about a,

$$P_4(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \frac{g'''(a)}{3!}(x-a)^3 + \frac{g^{(4)}(a)}{4!}(x-a)^4,$$

and write

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x) - P_4(x)}{(x-a)^p}\,dx + \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx. \tag{4.44}$$

Because $P(x)$ is a polynomial, we can exactly determine the value of

$$\int_a^b \frac{P_4(x)}{(x-a)^p}\,dx = \sum_{k=0}^{4} \int_a^b \frac{g^{(k)}(a)}{k!}(x-a)^{k-p}\,dx = \sum_{k=0}^{4} \frac{g^{(k)}(a)}{k!(k+1-p)}(b-a)^{k+1-p}. \tag{4.45}$$

This is generally the dominant portion of the approximation, especially when the Taylor polynomial $P_4(x)$ agrees closely with $g(x)$ throughout the interval $[a, b]$.

To approximate the integral of f, we must add to this value the approximation of

$$\int_a^b \frac{g(x) - P_4(x)}{(x-a)^p}\,dx.$$

To determine this, we first define

$$G(x) = \begin{cases} \frac{g(x)-P_4(x)}{(x-a)^p}, & \text{if } a < x \le b, \\ 0, & \text{if } x = a. \end{cases}$$

■

This gives us a continuous function on $[a, b]$. In fact, $0 < p < 1$ and $P_4^{(k)}(a)$ agrees with $g^{(k)}(a)$ for each $k = 0, 1, 2, 3, 4$, so we have $G \in C^4[a, b]$. This implies that the Composite Simpson's rule can be applied to approximate the integral of G on $[a, b]$. Adding this approximation to the value in Eq. (4.45) gives an approximation to the improper integral of f on $[a, b]$, within the accuracy of the Composite Simpson's rule approximation.

Example 2 Use Composite Simpson's rule with $h = 0.25$ to approximate the value of the improper integral

$$\int_0^1 \frac{e^x}{\sqrt{x}}\,dx.$$

Solution The fourth Taylor polynomial for e^x about $x = 0$ is

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24},$$

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so the dominant portion of the approximation to $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ is

$$\begin{aligned}\int_0^1 \frac{P_4(x)}{\sqrt{x}}\,dx &= \int_0^1 \left(x^{-1/2} + x^{1/2} + \frac{1}{2}x^{3/2} + \frac{1}{6}x^{5/2} + \frac{1}{24}x^{7/2}\right) dx\\ &= \lim_{M\to 0^+}\left[2x^{1/2} + \frac{2}{3}x^{3/2} + \frac{1}{5}x^{5/2} + \frac{1}{21}x^{7/2} + \frac{1}{108}x^{9/2}\right]_M^1\\ &= 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{108} \approx 2.9235450.\end{aligned}$$

For the second portion of the approximation to $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ we need to approximate $\int_0^1 G(x)\,dx$, where

$$G(x) = \begin{cases} \frac{1}{\sqrt{x}}(e^x - P_4(x)), & \text{if } 0 < x \le 1,\\ 0, & \text{if } x = 0.\end{cases}$$

Table 4.13

x	$G(x)$
0.00	0
0.25	0.0000170
0.50	0.0004013
0.75	0.0026026
1.00	0.0099485

Table 4.13 lists the values needed for the Composite Simpson's rule for this approximation.

Using these data and the Composite Simpson's rule gives

$$\begin{aligned}\int_0^1 G(x)\,dx &\approx \frac{0.25}{3}[0 + 4(0.0000170) + 2(0.0004013) + 4(0.0026026) + 0.0099485]\\ &= 0.0017691.\end{aligned}$$

Hence

$$\int_0^1 \frac{e^x}{\sqrt{x}}\,dx \approx 2.9235450 + 0.0017691 = 2.9253141.$$

This result is accurate to within the accuracy of the Composite Simpson's rule approximation for the function G. Because $|G^{(4)}(x)| < 1$ on $[0, 1]$, the error is bounded by

$$\frac{1-0}{180}(0.25)^4 = 0.0000217.$$

■

Right Endpoint Singularity

To approximate the improper integral with a singularity at the right endpoint, we could develop a similar technique but expand in terms of the right endpoint b instead of the left endpoint a. Alternatively, we can make the substitution

$$z = -x, \qquad dz = -\,dx$$

to change the improper integral into one of the form

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-z)\,dz, \tag{4.46}$$

which has its singularity at the left endpoint. Then we can apply the left endpoint singularity technique we have already developed. (See Figure 4.26.)

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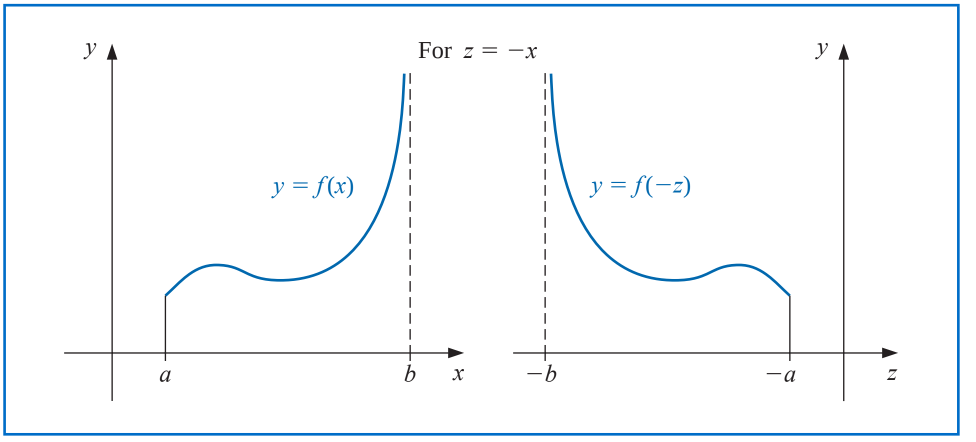
Figure 4.26



An improper integral with a singularity at c, where $a < c < b$, is treated as the sum of improper integrals with endpoint singularities since

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Infinite Singularity

The other type of improper integral involves infinite limits of integration. The basic integral of this type has the form

$$\int_a^\infty \frac{1}{x^p}\,dx,$$

for $p > 1$. This is converted to an integral with left endpoint singularity at 0 by making the integration substitution

$$t = x^{-1}, \quad dt = -x^{-2}\,dx, \quad \text{so} \quad dx = -x^2\,dt = -t^{-2}\,dt.$$

Then

$$\int_a^\infty \frac{1}{x^p}\,dx = \int_{1/a}^0 -\frac{t^p}{t^2}\,dt = \int_0^{1/a} \frac{1}{t^{2-p}}\,dt.$$

In a similar manner, the variable change $t = x^{-1}$ converts the improper integral $\int_a^\infty f(x)\,dx$ into one that has a left endpoint singularity at zero:

$$\int_a^\infty f(x)\,dx = \int_0^{1/a} t^{-2} f\left(\frac{1}{t}\right)\,dt. \tag{4.47}$$

It can now be approximated using a quadrature formula of the type described earlier.

Example 3 Approximate the value of the improper integral

$$I = \int_1^\infty x^{-3/2} \sin\frac{1}{x}\,dx.$$

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Solution We first make the variable change $t = x^{-1}$, which converts the infinite singularity into one with a left endpoint singularity. Then

$$dt = -x^{-2}\,dx, \quad \text{so} \quad dx = -x^2\,dt = -\frac{1}{t^2}\,dt,$$

and

$$I = \int_{x=1}^{x=\infty} x^{-3/2} \sin\frac{1}{x}\,dx = \int_{t=1}^{t=0} \left(\frac{1}{t}\right)^{-3/2} \sin t \left(-\frac{1}{t^2}\,dt\right) = \int_0^1 t^{-1/2} \sin t\,dt.$$

The fourth Taylor polynomial, $P_4(t)$, for $\sin t$ about 0 is

$$P_4(t) = t - \frac{1}{6}t^3,$$

so

$$G(t) = \begin{cases} \dfrac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}}, & \text{if } 0 < t \le 1 \\ 0, & \text{if } t = 0 \end{cases}$$

is in $C^4[0, 1]$, and we have

$$\begin{aligned} I &= \int_0^1 t^{-1/2}\left(t - \frac{1}{6}t^3\right) dt + \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}}\,dt \\ &= \left[\frac{2}{3}t^{3/2} - \frac{1}{21}t^{7/2}\right]_0^1 + \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}}\,dt \\ &= 0.61904761 + \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}}\,dt. \end{aligned}$$

The result from the Composite Simpson's rule with $n = 16$ for the remaining integral is 0.0014890097. This gives a final approximation of

$$I = 0.0014890097 + 0.61904761 = 0.62053661,$$

which is accurate to within 4.0×10^{-8}. ■

EXERCISE SET 4.9

1. Use Simpson's Composite rule and the given values of n to approximate the following improper integrals.

a. $\displaystyle\int_0^1 x^{-1/4} \sin x\,dx, \quad n = 4$

b. $\displaystyle\int_0^1 \frac{e^{2x}}{\sqrt[5]{x^2}}\,dx, \quad n = 6$

c. $\displaystyle\int_1^2 \frac{\ln x}{(x-1)^{1/5}}\,dx, \quad n = 8$

d. $\displaystyle\int_0^1 \frac{\cos 2x}{x^{1/3}}\,dx, \quad n = 6$

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2. Use the Composite Simpson's rule and the given values of n to approximate the following improper integrals.

a. $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{1-x}}\,dx, \quad n=6$ **b.** $\displaystyle\int_0^2 \frac{xe^x}{\sqrt[3]{(x-1)^2}}\,dx, \quad n=8$

3. Use the transformation $t = x^{-1}$ and then the Composite Simpson's rule and the given values of n to approximate the following improper integrals.

a. $\displaystyle\int_1^\infty \frac{1}{x^2+9}\,dx, \quad n=4$ **b.** $\displaystyle\int_1^\infty \frac{1}{1+x^4}\,dx, \quad n=4$

c. $\displaystyle\int_1^\infty \frac{\cos x}{x^3}\,dx, \quad n=6$ **d.** $\displaystyle\int_1^\infty x^{-4}\sin x\,dx, \quad n=6$

4. The improper integral $\int_0^\infty f(x)\,dx$ cannot be converted into an integral with finite limits using the substitution $t = 1/x$ because the limit at zero becomes infinite. The problem is resolved by first writing $\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$. Apply this technique to approximate the following improper integrals to within 10^{-6}.

a. $\displaystyle\int_0^\infty \frac{1}{1+x^4}\,dx$ **b.** $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^3}\,dx$

5. Suppose a body of mass m is traveling vertically upward starting at the surface of the earth. If all resistance except gravity is neglected, the escape velocity v is given by

$$v^2 = 2gR\int_1^\infty z^{-2}\,dz, \quad \text{where } z = \frac{x}{R},$$

$R = 3960$ miles is the radius of the earth, and $g = 0.00609$ mi/s^2 is the force of gravity at the earth's surface. Approximate the escape velocity v.

6. The Laguerre polynomials $\{L_0(x), L_1(x)\ldots\}$ form an orthogonal set on $[0,\infty)$ and satisfy $\int_0^\infty e^{-x}L_i(x)L_j(x)\,dx = 0$, for $i \neq j$. (See Section 8.2.) The polynomial $L_n(x)$ has n distinct zeros $x_1, x_2, \ldots, x_n$ in $[0,\infty)$. Let

$$c_{n,i} = \int_0^\infty e^{-x}\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}\,dx.$$

Show that the quadrature formula

$$\int_0^\infty f(x)e^{-x}\,dx = \sum_{i=1}^{n} c_{n,i}f(x_i)$$

has degree of precision $2n-1$. (Hint: Follow the steps in the proof of Theorem 4.7.)

7. The Laguerre polynomials $L_0(x) = 1$, $L_1(x) = 1-x$, $L_2(x) = x^2-4x+2$, and $L_3(x) = -x^3 + 9x^2 - 18x + 6$ are derived in Exercise 11 of Section 8.2. As shown in Exercise 6, these polynomials are useful in approximating integrals of the form

$$\int_0^\infty e^{-x}f(x)\,dx = 0.$$

a. Derive the quadrature formula using $n = 2$ and the zeros of $L_2(x)$.

b. Derive the quadrature formula using $n = 3$ and the zeros of $L_3(x)$.

8. Use the quadrature formulas derived in Exercise 7 to approximate the integral

$$\int_0^\infty \sqrt{x}e^{-x}\,dx.$$

9. Use the quadrature formulas derived in Exercise 7 to approximate the integral

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$$

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