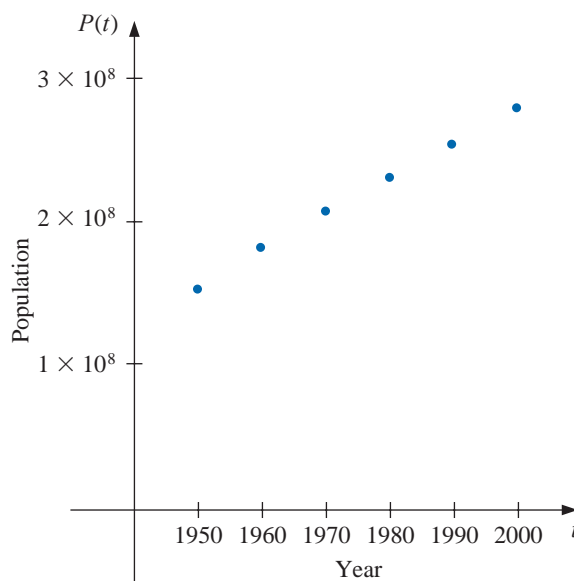


Interpolation and Polynomial Approximation

Introduction

A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000, and the data are also represented in the figure.

Year	1950	1960	1970	1980	1990	2000
Population (in thousands)	151,326	179,323	203,302	226,542	249,633	281,422



In reviewing these data, we might ask whether they could be used to provide a reasonable estimate of the population, say, in 1975 or even in the year 2020. Predictions of this type can be obtained by using a function that fits the given data. This process is called *interpolation* and is the subject of this chapter. This population problem is considered throughout the chapter and in Exercises 18 of Section 3.1, 18 of Section 3.3, and 28 of Section 3.5.

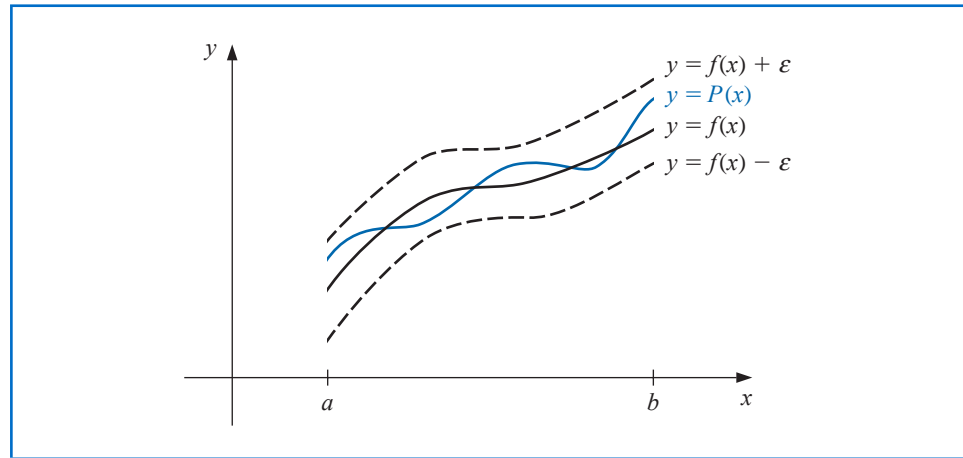
3.1 Interpolation and the Lagrange Polynomial

One of the most useful and well-known classes of functions mapping the set of real numbers into itself is the *algebraic polynomials*, the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and a_0, \dots, a_n are real constants. One reason for their importance is that they uniformly approximate continuous functions. By this we mean that given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as “close” to the given function as desired. This result is expressed precisely in the Weierstrass Approximation Theorem. (See Figure 3.1.)

Figure 3.1



Theorem 3.1 (Weierstrass Approximation Theorem)

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \text{ in } [a, b]. \quad \blacksquare$$

The proof of this theorem can be found in most elementary texts on real analysis (see, for example, [Bart], pp. 165–172).

Another important reason for considering the class of polynomials in the approximation of functions is that the derivative and indefinite integral of a polynomial are easy to determine and are also polynomials. For these reasons, polynomials are often used for approximating continuous functions.

The Taylor polynomials were introduced in Section 1.1, where they were described as one of the fundamental building blocks of numerical analysis. Given this prominence, you might expect that polynomial interpolation would make heavy use of these functions. However this is not the case. The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point. A good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this. For example, suppose we calculate the first six Taylor polynomials about $x_0 = 0$ for $f(x) = e^x$. Since the derivatives of $f(x)$ are all e^x , which evaluated at $x_0 = 0$ gives 1, the Taylor polynomials are

Karl Weierstrass (1815–1897) is often referred to as the father of modern analysis because of his insistence on rigor in the demonstration of mathematical results. He was instrumental in developing tests for convergence of series, and determining ways to rigorously define irrational numbers. He was the first to demonstrate that a function could be everywhere continuous but nowhere differentiable, a result that shocked some of his contemporaries.

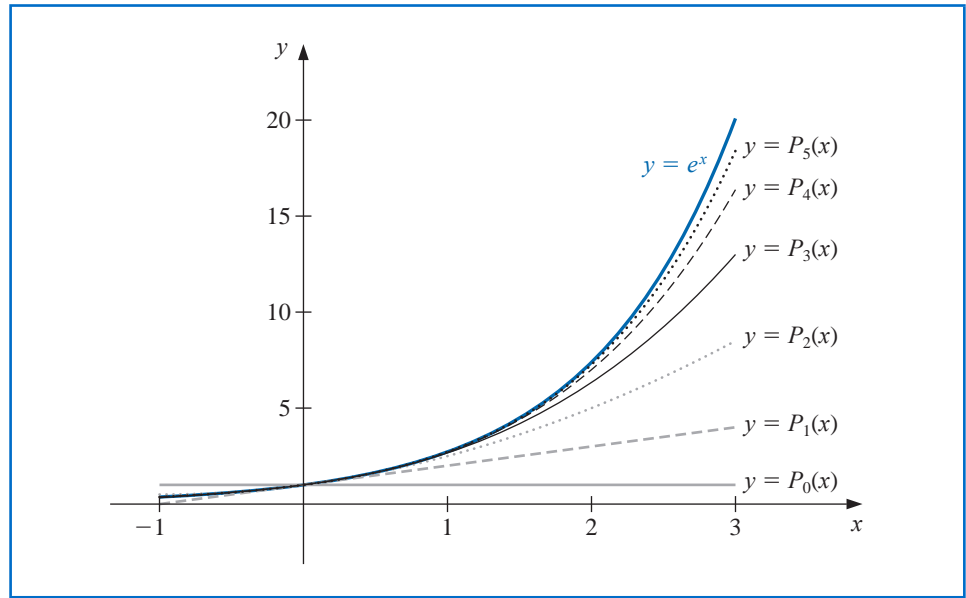
Very little of Weierstrass's work was published during his lifetime, but his lectures, particularly on the theory of functions, had significant influence on an entire generation of students.

$$P_0(x) = 1, \quad P_1(x) = 1 + x, \quad P_2(x) = 1 + x + \frac{x^2}{2}, \quad P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}, \quad \text{and} \quad P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.$$

The graphs of the polynomials are shown in Figure 3.2. (Notice that even for the higher-degree polynomials, the error becomes progressively worse as we move away from zero.)

Figure 3.2



Although better approximations are obtained for $f(x) = e^x$ if higher-degree Taylor polynomials are used, this is not true for all functions. Consider, as an extreme example, using Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$. Since

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = (-1)2 \cdot x^{-3},$$

and, in general,

$$f^{(k)}(x) = (-1)^k k! x^{-k-1},$$

the Taylor polynomials are

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(1)}{k!} (x-1)^k = \sum_{k=0}^n (-1)^k (x-1)^k.$$

To approximate $f(3) = 1/3$ by $P_n(3)$ for increasing values of n , we obtain the values in Table 3.1—rather a dramatic failure! When we approximate $f(3) = 1/3$ by $P_n(3)$ for larger values of n , the approximations become increasingly inaccurate.

Table 3.1

n	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

For the Taylor polynomials all the information used in the approximation is concentrated at the single number x_0 , so these polynomials will generally give inaccurate approximations as we move away from x_0 . This limits Taylor polynomial approximation to the situation in which approximations are needed only at numbers close to x_0 . For ordinary computational purposes it is more efficient to use methods that include information at various points. We consider this in the remainder of the chapter. The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and error estimation.

Lagrange Interpolating Polynomials

The problem of determining a polynomial of degree one that passes through the distinct points (x_0, y_0) and (x_1, y_1) is the same as approximating a function f for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first-degree polynomial **interpolating**, or agreeing with, the values of f at the given points. Using this polynomial for approximation within the interval given by the endpoints is called polynomial **interpolation**.

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

The linear **Lagrange interpolating polynomial** through (x_0, y_0) and (x_1, y_1) is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Note that

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_1(x_0) = 0, \quad \text{and} \quad L_1(x_1) = 1,$$

which implies that

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

and

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1.$$

So P is the unique polynomial of degree at most one that passes through (x_0, y_0) and (x_1, y_1) .

Example 1 Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Solution In this case we have

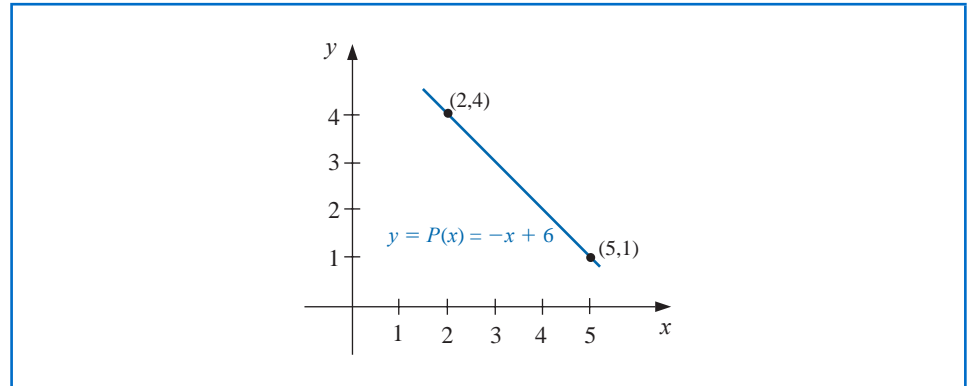
$$L_0(x) = \frac{x - 5}{2 - 5} = -\frac{1}{3}(x - 5) \quad \text{and} \quad L_1(x) = \frac{x - 2}{5 - 2} = \frac{1}{3}(x - 2),$$

so

$$P(x) = -\frac{1}{3}(x - 5) \cdot 4 + \frac{1}{3}(x - 2) \cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x + 6.$$

The graph of $y = P(x)$ is shown in Figure 3.3. ■

Figure 3.3

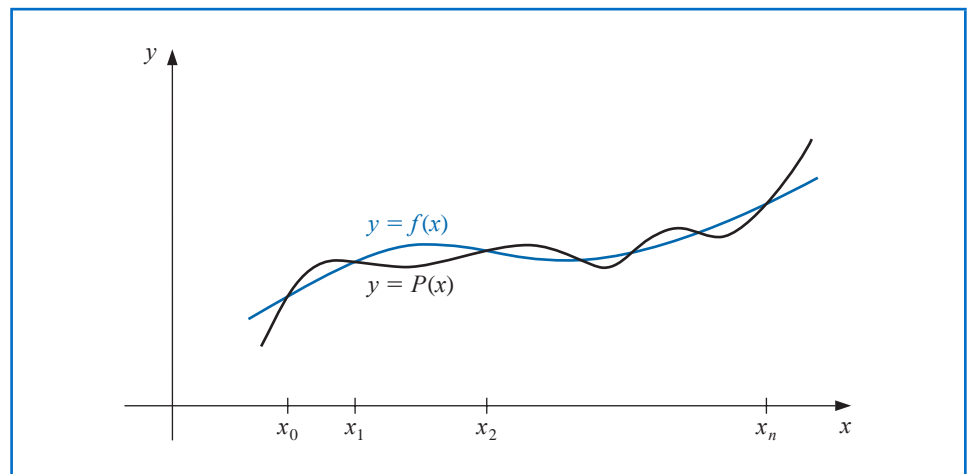


To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through the $n + 1$ points

$$(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n)).$$

(See Figure 3.4.)

Figure 3.4



In this case we first construct, for each $k = 0, 1, \dots, n$, a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ when $i \neq k$ and $L_{n,k}(x_k) = 1$. To satisfy $L_{n,k}(x_i) = 0$ for each $i \neq k$ requires that the numerator of $L_{n,k}(x)$ contain the term

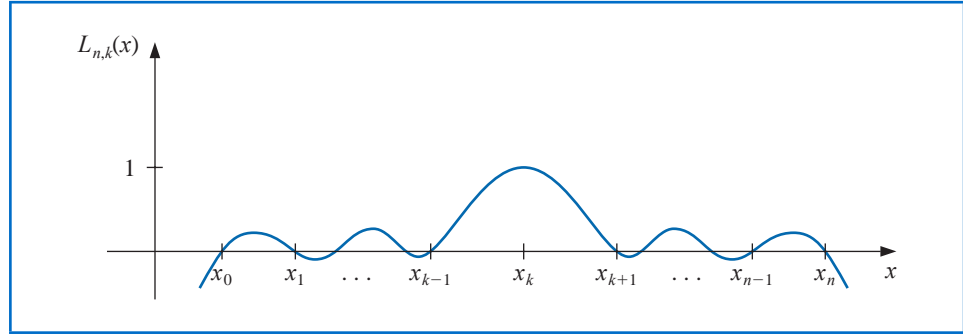
$$(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n).$$

To satisfy $L_{n,k}(x_k) = 1$, the denominator of $L_{n,k}(x)$ must be this same term but evaluated at $x = x_k$. Thus

$$L_{n,k}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}.$$

A sketch of the graph of a typical $L_{n,k}$ (when n is even) is shown in Figure 3.5.

Figure 3.5



The interpolating polynomial is easily described once the form of $L_{n,k}$ is known. This polynomial, called the **n th Lagrange interpolating polynomial**, is defined in the following theorem.

Theorem 3.2

If x_0, x_1, \dots, x_n are $n + 1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k), \quad \text{for each } k = 0, 1, \dots, n.$$

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \dots + f(x_n)L_{n,n}(x) = \sum_{k=0}^n f(x_k)L_{n,k}(x), \quad (3.1)$$

where, for each $k = 0, 1, \dots, n$,

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \quad (3.2)$$

$$= \prod_{\substack{i=0 \\ i \neq k}}^n \frac{(x - x_i)}{(x_k - x_i)}.$$

We will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree.

Example 2

- (a) Use the numbers (called *nodes*) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.
- (b) Use this polynomial to approximate $f(3) = 1/3$.

Solution (a) We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$. In nested form they are

$$L_0(x) = \frac{(x - 2.75)(x - 4)}{(2 - 2.75)(2 - 4)} = \frac{2}{3}(x - 2.75)(x - 4),$$

$$L_1(x) = \frac{(x - 2)(x - 4)}{(2.75 - 2)(2.75 - 4)} = -\frac{16}{15}(x - 2)(x - 4),$$

and

$$L_2(x) = \frac{(x - 2)(x - 2.75)}{(4 - 2)(4 - 2.5)} = \frac{2}{5}(x - 2)(x - 2.75).$$

The interpolation formula named for Joseph Louis Lagrange (1736–1813) was likely known by Isaac Newton around 1675, but it appears to first have been published in 1779 by Edward Waring (1736–1798). Lagrange wrote extensively on the subject of interpolation and his work had significant influence on later mathematicians. He published this result in 1795.

The symbol \prod is used to write products compactly and parallels the symbol \sum , which is used for writing sums.

Also, $f(x_0) = f(2) = 1/2$, $f(x_1) = f(2.75) = 4/11$, and $f(x_2) = f(4) = 1/4$, so

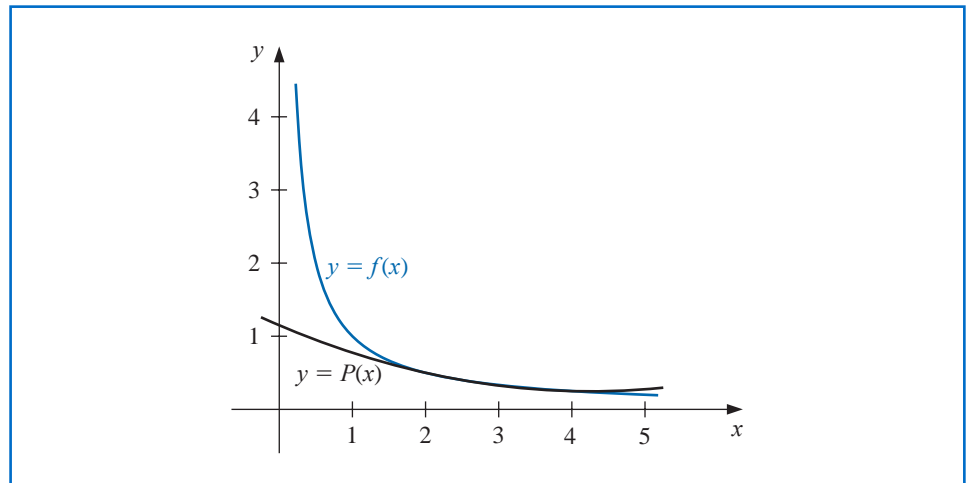
$$\begin{aligned} P(x) &= \sum_{k=0}^2 f(x_k)L_k(x) \\ &= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\ &= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}. \end{aligned}$$

(b) An approximation to $f(3) = 1/3$ (see Figure 3.6) is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

Recall that in the opening section of this chapter (see Table 3.1) we found that no Taylor polynomial expanded about $x_0 = 1$ could be used to reasonably approximate $f(x) = 1/x$ at $x = 3$. ■

Figure 3.6



The interpolating polynomial P of degree less than or equal to 3 is defined in Maple with

$$P := x \rightarrow \text{interp}([2, 11/4, 4], [1/2, 4/11, 1/4], x)$$

$$x \rightarrow \text{interp} \left(\left[2, \frac{11}{4}, 4 \right], \left[\frac{1}{2}, \frac{4}{11}, \frac{1}{4} \right], x \right)$$

To see the polynomial, enter

$$P(x)$$

$$\frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$

Evaluating $P(3)$ as an approximation to $f(3) = 1/3$, is found with

$$\text{evalf}(P(3))$$

$$0.3295454545$$

The interpolating polynomial can also be defined in Maple using the *CurveFitting* package and the call *PolynomialInterpolation*.

The next step is to calculate a remainder term or bound for the error involved in approximating a function by an interpolating polynomial.

Theorem 3.3 Suppose x_0, x_1, \dots, x_n are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between x_0, x_1, \dots, x_n , and hence in (a, b) , exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n), \tag{3.3}$$

where $P(x)$ is the interpolating polynomial given in Eq. (3.1). ■

There are other ways that the error term for the Lagrange polynomial can be expressed, but this is the most useful form and the one that most closely agrees with the standard Taylor polynomial error form.

Proof Note first that if $x = x_k$, for any $k = 0, 1, \dots, n$, then $f(x_k) = P(x_k)$, and choosing $\xi(x_k)$ arbitrarily in (a, b) yields Eq. (3.3).

If $x \neq x_k$, for all $k = 0, 1, \dots, n$, define the function g for t in $[a, b]$ by

$$\begin{aligned} g(t) &= f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)} \\ &= f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^n \frac{(t-x_i)}{(x-x_i)}. \end{aligned}$$

Since $f \in C^{n+1}[a, b]$, and $P \in C^\infty[a, b]$, it follows that $g \in C^{n+1}[a, b]$. For $t = x_k$, we have

$$g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)] \prod_{i=0}^n \frac{(x_k - x_i)}{(x - x_i)} = 0 - [f(x) - P(x)] \cdot 0 = 0.$$

Moreover,

$$g(x) = f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^n \frac{(x - x_i)}{(x - x_i)} = f(x) - P(x) - [f(x) - P(x)] = 0.$$

Thus $g \in C^{n+1}[a, b]$, and g is zero at the $n + 2$ distinct numbers x, x_0, x_1, \dots, x_n . By Generalized Rolle's Theorem 1.10, there exists a number ξ in (a, b) for which $g^{(n+1)}(\xi) = 0$. So

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^n \frac{(t-x_i)}{(x-x_i)} \right]_{t=\xi}. \tag{3.4}$$

However $P(x)$ is a polynomial of degree at most n , so the $(n + 1)$ st derivative, $P^{(n+1)}(x)$, is identically zero. Also, $\prod_{i=0}^n [(t - x_i)/(x - x_i)]$ is a polynomial of degree $(n + 1)$, so

$$\prod_{i=0}^n \frac{(t-x_i)}{(x-x_i)} = \left[\frac{1}{\prod_{i=0}^n (x-x_i)} \right] t^{n+1} + (\text{lower-degree terms in } t),$$

and

$$\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^n \frac{(t-x_i)}{(x-x_i)} = \frac{(n+1)!}{\prod_{i=0}^n (x-x_i)}.$$

Equation (3.4) now becomes

$$0 = f^{(n+1)}(\xi) - 0 - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^n (x - x_i)},$$

and, upon solving for $f(x)$, we have

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i). \quad \blacksquare \quad \blacksquare \quad \blacksquare$$

The error formula in Theorem 3.3 is an important theoretical result because Lagrange polynomials are used extensively for deriving numerical differentiation and integration methods. Error bounds for these techniques are obtained from the Lagrange error formula.

Note that the error form for the Lagrange polynomial is quite similar to that for the Taylor polynomial. The n th Taylor polynomial about x_0 concentrates all the known information at x_0 and has an error term of the form

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x - x_0)^{n+1}.$$

The Lagrange polynomial of degree n uses information at the distinct numbers x_0, x_1, \dots, x_n and, in place of $(x - x_0)^n$, its error formula uses a product of the $n + 1$ terms $(x - x_0), (x - x_1), \dots, (x - x_n)$:

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x - x_0)(x - x_1) \cdots (x - x_n).$$

Example 3 In Example 2 we found the second Lagrange polynomial for $f(x) = 1/x$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

Solution Because $f(x) = x^{-1}$, we have

$$f'(x) = -x^{-2}, \quad f''(x) = 2x^{-3}, \quad \text{and} \quad f'''(x) = -6x^{-4}.$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!} (x - x_0)(x - x_1)(x - x_2) = -(\xi(x))^{-4} (x - 2)(x - 2.75)(x - 4), \quad \text{for } \xi(x) \text{ in } (2, 4).$$

The maximum value of $(\xi(x))^{-4}$ on the interval is $2^{-4} = 1/16$. We now need to determine the maximum value on this interval of the absolute value of the polynomial

$$g(x) = (x - 2)(x - 2.75)(x - 4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22.$$

Because

$$D_x \left(x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22 \right) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x - 7)(2x - 7),$$

the critical points occur at

$$x = \frac{7}{3}, \quad \text{with } g\left(\frac{7}{3}\right) = \frac{25}{108}, \quad \text{and} \quad x = \frac{7}{2}, \quad \text{with } g\left(\frac{7}{2}\right) = -\frac{9}{16}.$$

Hence, the maximum error is

$$\frac{f'''(\xi(x))}{3!} |(x - x_0)(x - x_1)(x - x_2)| \leq \frac{1}{16 \cdot 6} \left| -\frac{9}{16} \right| = \frac{3}{512} \approx 0.00586. \quad \blacksquare$$

The next example illustrates how the error formula can be used to prepare a table of data that will ensure a specified interpolation error within a specified bound.

Example 4 Suppose a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x -values, the step size, is h . What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?

Solution Let x_0, x_1, \dots be the numbers at which f is evaluated, x be in $[0, 1]$, and suppose j satisfies $x_j \leq x \leq x_{j+1}$. Eq. (3.3) implies that the error in linear interpolation is

$$|f(x) - P(x)| = \left| \frac{f^{(2)}(\xi)}{2!} (x - x_j)(x - x_{j+1}) \right| = \frac{|f^{(2)}(\xi)|}{2} |(x - x_j)|(x - x_{j+1}).$$

The step size is h , so $x_j = jh$, $x_{j+1} = (j + 1)h$, and

$$|f(x) - P(x)| \leq \frac{|f^{(2)}(\xi)|}{2!} |(x - jh)(x - (j + 1)h)|.$$

Hence

$$\begin{aligned} |f(x) - P(x)| &\leq \frac{\max_{\xi \in [0,1]} e^\xi}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j + 1)h)| \\ &\leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j + 1)h)|. \end{aligned}$$

Consider the function $g(x) = (x - jh)(x - (j + 1)h)$, for $jh \leq x \leq (j + 1)h$. Because

$$g'(x) = (x - (j + 1)h) + (x - jh) = 2 \left(x - jh - \frac{h}{2} \right),$$

the only critical point for g is at $x = jh + h/2$, with $g(jh + h/2) = (h/2)^2 = h^2/4$.

Since $g(jh) = 0$ and $g((j + 1)h) = 0$, the maximum value of $|g'(x)|$ in $[jh, (j + 1)h]$ must occur at the critical point which implies that

$$|f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}.$$

Consequently, to ensure that the error in linear interpolation is bounded by 10^{-6} , it is sufficient for h to be chosen so that

$$\frac{eh^2}{8} \leq 10^{-6}. \quad \text{This implies that } h < 1.72 \times 10^{-3}.$$

Because $n = (1 - 0)/h$ must be an integer, a reasonable choice for the step size is $h = 0.001$. ■

EXERCISE SET 3.1

1. For the given functions $f(x)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$, and find the absolute error.
 - a. $f(x) = \cos x$
 - b. $f(x) = \sqrt{1 + x}$
 - c. $f(x) = \ln(x + 1)$
 - d. $f(x) = \tan x$

2. For the given functions $f(x)$, let $x_0 = 1$, $x_1 = 1.25$, and $x_2 = 1.6$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(1.4)$, and find the absolute error.
 - a. $f(x) = \sin \pi x$
 - b. $f(x) = \sqrt[3]{x-1}$
 - c. $f(x) = \log_{10}(3x-1)$
 - d. $f(x) = e^{2x} - x$
3. Use Theorem 3.3 to find an error bound for the approximations in Exercise 1.
4. Use Theorem 3.3 to find an error bound for the approximations in Exercise 2.
5. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:
 - a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
 - b. $f(-\frac{1}{3})$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
 - c. $f(0.25)$ if $f(0.1) = 0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$
 - d. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$
6. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:
 - a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
 - b. $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$
 - c. $f(0.18)$ if $f(0.1) = -0.29004986$, $f(0.2) = -0.56079734$, $f(0.3) = -0.81401972$, $f(0.4) = -1.0526302$
 - d. $f(0.25)$ if $f(-1) = 0.86199480$, $f(-0.5) = 0.95802009$, $f(0) = 1.0986123$, $f(0.5) = 1.2943767$
7. The data for Exercise 5 were generated using the following functions. Use the error formula to find a bound for the error, and compare the bound to the actual error for the cases $n = 1$ and $n = 2$.
 - a. $f(x) = x \ln x$
 - b. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$
 - c. $f(x) = x \cos x - 2x^2 + 3x - 1$
 - d. $f(x) = \sin(e^x - 2)$
8. The data for Exercise 6 were generated using the following functions. Use the error formula to find a bound for the error, and compare the bound to the actual error for the cases $n = 1$ and $n = 2$.
 - a. $f(x) = e^{2x}$
 - b. $f(x) = x^4 - x^3 + x^2 - x + 1$
 - c. $f(x) = x^2 \cos x - 3x$
 - d. $f(x) = \ln(e^x + 2)$
9. Let $P_3(x)$ be the interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$, and $(2, 2)$. The coefficient of x^3 in $P_3(x)$ is 6. Find y .
10. Let $f(x) = \sqrt{x-x^2}$ and $P_2(x)$ be the interpolation polynomial on $x_0 = 0$, x_1 and $x_2 = 1$. Find the largest value of x_1 in $(0, 1)$ for which $f(0.5) - P_2(0.5) = -0.25$.
11. Use the following values and four-digit rounding arithmetic to construct a third Lagrange polynomial approximation to $f(1.09)$. The function being approximated is $f(x) = \log_{10}(\tan x)$. Use this knowledge to find a bound for the error in the approximation.

$$f(1.00) = 0.1924 \quad f(1.05) = 0.2414 \quad f(1.10) = 0.2933 \quad f(1.15) = 0.3492$$

12. Use the Lagrange interpolating polynomial of degree three or less and four-digit chopping arithmetic to approximate $\cos 0.750$ using the following values. Find an error bound for the approximation.

$$\cos 0.698 = 0.7661 \quad \cos 0.733 = 0.7432 \quad \cos 0.768 = 0.7193 \quad \cos 0.803 = 0.6946$$

The actual value of $\cos 0.750$ is 0.7317 (to four decimal places). Explain the discrepancy between the actual error and the error bound.

13. Construct the Lagrange interpolating polynomials for the following functions, and find a bound for the absolute error on the interval $[x_0, x_n]$.
 - a. $f(x) = e^{2x} \cos 3x$, $x_0 = 0, x_1 = 0.3, x_2 = 0.6, n = 2$
 - b. $f(x) = \sin(\ln x)$, $x_0 = 2.0, x_1 = 2.4, x_2 = 2.6, n = 2$
 - c. $f(x) = \ln x$, $x_0 = 1, x_1 = 1.1, x_2 = 1.3, x_3 = 1.4, n = 3$
 - d. $f(x) = \cos x + \sin x$, $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 1.0, n = 3$
14. Let $f(x) = e^x$, for $0 \leq x \leq 2$.
 - a. Approximate $f(0.25)$ using linear interpolation with $x_0 = 0$ and $x_1 = 0.5$.
 - b. Approximate $f(0.75)$ using linear interpolation with $x_0 = 0.5$ and $x_1 = 1$.
 - c. Approximate $f(0.25)$ and $f(0.75)$ by using the second interpolating polynomial with $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.
 - d. Which approximations are better and why?
15. Repeat Exercise 11 using Maple with *Digits* set to 10.
16. Repeat Exercise 12 using Maple with *Digits* set to 10.
17. Suppose you need to construct eight-decimal-place tables for the common, or base-10, logarithm function from $x = 1$ to $x = 10$ in such a way that linear interpolation is accurate to within 10^{-6} . Determine a bound for the step size for this table. What choice of step size would you make to ensure that $x = 10$ is included in the table?
18.
 - a. The introduction to this chapter included a table listing the population of the United States from 1950 to 2000. Use Lagrange interpolation to approximate the population in the years 1940, 1975, and 2020.
 - b. The population in 1940 was approximately 132,165,000. How accurate do you think your 1975 and 2020 figures are?
19. It is suspected that the high amounts of tannin in mature oak leaves inhibit the growth of the winter moth (*Operophtera bromata* L., *Geometridae*) larvae that extensively damage these trees in certain years. The following table lists the average weight of two samples of larvae at times in the first 28 days after birth. The first sample was reared on young oak leaves, whereas the second sample was reared on mature leaves from the same tree.
 - a. Use Lagrange interpolation to approximate the average weight curve for each sample.
 - b. Find an approximate maximum average weight for each sample by determining the maximum of the interpolating polynomial.

Day	0	6	10	13	17	20	28
Sample 1 average weight (mg)	6.67	17.33	42.67	37.33	30.10	29.31	28.74
Sample 2 average weight (mg)	6.67	16.11	18.89	15.00	10.56	9.44	8.89

20. In Exercise 26 of Section 1.1 a Maclaurin series was integrated to approximate $\text{erf}(1)$, where $\text{erf}(x)$ is the normal distribution error function defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

- a. Use the Maclaurin series to construct a table for $\text{erf}(x)$ that is accurate to within 10^{-4} for $\text{erf}(x_i)$, where $x_i = 0.2i$, for $i = 0, 1, \dots, 5$.
 - b. Use both linear interpolation and quadratic interpolation to obtain an approximation to $\text{erf}(\frac{1}{3})$. Which approach seems most feasible?
21. Prove Taylor's Theorem 1.14 by following the procedure in the proof of Theorem 3.3. [Hint: Let

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \cdot \frac{(t - x_0)^{n+1}}{(x - x_0)^{n+1}},$$

where P is the n th Taylor polynomial, and use the Generalized Rolle's Theorem 1.10.]

$$P_2(1.5) = \frac{(1.5 - 1.6)(1.5 - 1.9)}{(1.3 - 1.6)(1.3 - 1.9)}(0.6200860) + \frac{(1.5 - 1.3)(1.5 - 1.9)}{(1.6 - 1.3)(1.6 - 1.9)}(0.4554022) \\ + \frac{(1.5 - 1.3)(1.5 - 1.6)}{(1.9 - 1.3)(1.9 - 1.6)}(0.2818186) = 0.5112857,$$

and one with $x_0 = 1.0$, $x_1 = 1.3$, and $x_2 = 1.6$, which gives $\hat{P}_2(1.5) = 0.5124715$.

In the third-degree case, there are also two reasonable choices for the polynomial. One with $x_0 = 1.3$, $x_1 = 1.6$, $x_2 = 1.9$, and $x_3 = 2.2$, which gives $P_3(1.5) = 0.5118302$.

The second third-degree approximation is obtained with $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, and $x_3 = 1.9$, which gives $\hat{P}_3(1.5) = 0.5118127$. The fourth-degree Lagrange polynomial uses all the entries in the table. With $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$, and $x_4 = 2.2$, the approximation is $P_4(1.5) = 0.5118200$.

Because $P_3(1.5)$, $\hat{P}_3(1.5)$, and $P_4(1.5)$ all agree to within 2×10^{-5} units, we expect this degree of accuracy for these approximations. We also expect $P_4(1.5)$ to be the most accurate approximation, since it uses more of the given data.

The function we are approximating is actually the Bessel function of the first kind of order zero, whose value at 1.5 is known to be 0.5118277. Therefore, the true accuracies of the approximations are as follows:

$$|P_1(1.5) - f(1.5)| \approx 1.53 \times 10^{-3},$$

$$|P_2(1.5) - f(1.5)| \approx 5.42 \times 10^{-4},$$

$$|\hat{P}_2(1.5) - f(1.5)| \approx 6.44 \times 10^{-4},$$

$$|P_3(1.5) - f(1.5)| \approx 2.5 \times 10^{-6},$$

$$|\hat{P}_3(1.5) - f(1.5)| \approx 1.50 \times 10^{-5},$$

$$|P_4(1.5) - f(1.5)| \approx 7.7 \times 10^{-6}.$$

Although $P_3(1.5)$ is the most accurate approximation, if we had no knowledge of the actual value of $f(1.5)$, we would accept $P_4(1.5)$ as the best approximation since it includes the most data about the function. The Lagrange error term derived in Theorem 3.3 cannot be applied here because we have no knowledge of the fourth derivative of f . Unfortunately, this is generally the case. \square

Neville's Method

A practical difficulty with Lagrange interpolation is that the error term is difficult to apply, so the degree of the polynomial needed for the desired accuracy is generally not known until computations have been performed. A common practice is to compute the results given from various polynomials until appropriate agreement is obtained, as was done in the previous Illustration. However, the work done in calculating the approximation by the second polynomial does not lessen the work needed to calculate the third approximation; nor is the fourth approximation easier to obtain once the third approximation is known, and so on. We will now derive these approximating polynomials in a manner that uses the previous calculations to greater advantage.

Definition 3.4 Let f be a function defined at $x_0, x_1, x_2, \dots, x_n$, and suppose that m_1, m_2, \dots, m_k are k distinct integers, with $0 \leq m_i \leq n$ for each i . The Lagrange polynomial that agrees with $f(x)$ at the k points $x_{m_1}, x_{m_2}, \dots, x_{m_k}$ is denoted $P_{m_1, m_2, \dots, m_k}(x)$. \blacksquare

Example 1 Suppose that $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 6$, and $f(x) = e^x$. Determine the interpolating polynomial denoted $P_{1,2,4}(x)$, and use this polynomial to approximate $f(5)$.

Solution This is the Lagrange polynomial that agrees with $f(x)$ at $x_1 = 2$, $x_2 = 3$, and $x_4 = 6$. Hence

$$P_{1,2,4}(x) = \frac{(x-3)(x-6)}{(2-3)(2-6)}e^2 + \frac{(x-2)(x-6)}{(3-2)(3-6)}e^3 + \frac{(x-2)(x-3)}{(6-2)(6-3)}e^6.$$

So

$$\begin{aligned} f(5) \approx P(5) &= \frac{(5-3)(5-6)}{(2-3)(2-6)}e^2 + \frac{(5-2)(5-6)}{(3-2)(3-6)}e^3 + \frac{(5-2)(5-3)}{(6-2)(6-3)}e^6 \\ &= -\frac{1}{2}e^2 + e^3 + \frac{1}{2}e^6 \approx 218.105. \end{aligned}$$

The next result describes a method for recursively generating Lagrange polynomial approximations.

Theorem 3.5 Let f be defined at x_0, x_1, \dots, x_k , and let x_j and x_i be two distinct numbers in this set. Then

$$P(x) = \frac{(x-x_j)P_{0,1,\dots,j-1,j+1,\dots,k}(x) - (x-x_i)P_{0,1,\dots,i-1,i+1,\dots,k}(x)}{(x_i-x_j)}$$

is the k th Lagrange polynomial that interpolates f at the $k+1$ points x_0, x_1, \dots, x_k .

Proof For ease of notation, let $Q \equiv P_{0,1,\dots,i-1,i+1,\dots,k}$ and $\hat{Q} \equiv P_{0,1,\dots,j-1,j+1,\dots,k}$. Since $Q(x)$ and $\hat{Q}(x)$ are polynomials of degree $k-1$ or less, $P(x)$ is of degree at most k .

First note that $\hat{Q}(x_i) = f(x_i)$, implies that

$$P(x_i) = \frac{(x_i-x_j)\hat{Q}(x_i) - (x_i-x_i)Q(x_i)}{x_i-x_j} = \frac{(x_i-x_j)}{(x_i-x_j)}f(x_i) = f(x_i).$$

Similarly, since $Q(x_j) = f(x_j)$, we have $P(x_j) = f(x_j)$.

In addition, if $0 \leq r \leq k$ and r is neither i nor j , then $Q(x_r) = \hat{Q}(x_r) = f(x_r)$. So

$$P(x_r) = \frac{(x_r-x_j)\hat{Q}(x_r) - (x_r-x_i)Q(x_r)}{x_i-x_j} = \frac{(x_i-x_j)}{(x_i-x_j)}f(x_r) = f(x_r).$$

But, by definition, $P_{0,1,\dots,k}(x)$ is the unique polynomial of degree at most k that agrees with f at x_0, x_1, \dots, x_k . Thus, $P \equiv P_{0,1,\dots,k}$.

Theorem 3.5 implies that the interpolating polynomials can be generated recursively. For example, we have

$$\begin{aligned} P_{0,1} &= \frac{1}{x_1-x_0}[(x-x_0)P_1 - (x-x_1)P_0], & P_{1,2} &= \frac{1}{x_2-x_1}[(x-x_1)P_2 - (x-x_2)P_1], \\ P_{0,1,2} &= \frac{1}{x_2-x_0}[(x-x_0)P_{1,2} - (x-x_2)P_{0,1}], \end{aligned}$$

and so on. They are generated in the manner shown in Table 3.3, where each row is completed before the succeeding rows are begun.

Table 3.3

x_0	P_0				
x_1	P_1	$P_{0,1}$			
x_2	P_2	$P_{1,2}$	$P_{0,1,2}$		
x_3	P_3	$P_{2,3}$	$P_{1,2,3}$	$P_{0,1,2,3}$	
x_4	P_4	$P_{3,4}$	$P_{2,3,4}$	$P_{1,2,3,4}$	$P_{0,1,2,3,4}$

The procedure that uses the result of Theorem 3.5 to recursively generate interpolating polynomial approximations is called **Neville’s method**. The P notation used in Table 3.3 is cumbersome because of the number of subscripts used to represent the entries. Note, however, that as an array is being constructed, only two subscripts are needed. Proceeding down the table corresponds to using consecutive points x_i with larger i , and proceeding to the right corresponds to increasing the degree of the interpolating polynomial. Since the points appear consecutively in each entry, we need to describe only a starting point and the number of additional points used in constructing the approximation.

To avoid the multiple subscripts, we let $Q_{i,j}(x)$, for $0 \leq j \leq i$, denote the interpolating polynomial of degree j on the $(j + 1)$ numbers $x_{i-j}, x_{i-j+1}, \dots, x_{i-1}, x_i$; that is,

$$Q_{i,j} = P_{i-j,i-j+1,\dots,i-1,i}.$$

Using this notation provides the Q notation array in Table 3.4.

Table 3.4

x_0	$P_0 = Q_{0,0}$				
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$		
x_3	$P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$	
x_4	$P_4 = Q_{4,0}$	$P_{3,4} = Q_{4,1}$	$P_{2,3,4} = Q_{4,2}$	$P_{1,2,3,4} = Q_{4,3}$	$P_{0,1,2,3,4} = Q_{4,4}$

Example 2 Values of various interpolating polynomials at $x = 1.5$ were obtained in the Illustration at the beginning of the Section using the data shown in Table 3.5. Apply Neville’s method to the data by constructing a recursive table of the form shown in Table 3.4.

Table 3.5

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Solution Let $x_0 = 1.0, x_1 = 1.3, x_2 = 1.6, x_3 = 1.9,$ and $x_4 = 2.2,$ then $Q_{0,0} = f(1.0), Q_{1,0} = f(1.3), Q_{2,0} = f(1.6), Q_{3,0} = f(1.9),$ and $Q_{4,0} = f(2.2).$ These are the five polynomials of degree zero (constants) that approximate $f(1.5),$ and are the same as data given in Table 3.5.

Calculating the first-degree approximation $Q_{1,1}(1.5)$ gives

$$\begin{aligned} Q_{1,1}(1.5) &= \frac{(x - x_0)Q_{1,0} - (x - x_1)Q_{0,0}}{x_1 - x_0} \\ &= \frac{(1.5 - 1.0)Q_{1,0} - (1.5 - 1.3)Q_{0,0}}{1.3 - 1.0} \\ &= \frac{0.5(0.6200860) - 0.2(0.7651977)}{0.3} = 0.5233449. \end{aligned}$$

Similarly,

$$\begin{aligned} Q_{2,1}(1.5) &= \frac{(1.5 - 1.3)(0.4554022) - (1.5 - 1.6)(0.6200860)}{1.6 - 1.3} = 0.5102968, \\ Q_{3,1}(1.5) &= 0.5132634, \quad \text{and} \quad Q_{4,1}(1.5) = 0.5104270. \end{aligned}$$

Eric Harold Neville (1889–1961) gave this modification of the Lagrange formula in a paper published in 1932.[N]

The best linear approximation is expected to be $Q_{2,1}$ because 1.5 is between $x_1 = 1.3$ and $x_2 = 1.6$.

In a similar manner, approximations using higher-degree polynomials are given by

$$Q_{2,2}(1.5) = \frac{(1.5 - 1.0)(0.5102968) - (1.5 - 1.6)(0.5233449)}{1.6 - 1.0} = 0.5124715,$$

$$Q_{3,2}(1.5) = 0.5112857, \quad \text{and} \quad Q_{4,2}(1.5) = 0.5137361.$$

The higher-degree approximations are generated in a similar manner and are shown in Table 3.6. ■

Table 3.6

1.0	0.7651977					
1.3	0.6200860	0.5233449				
1.6	0.4554022	0.5102968	0.5124715			
1.9	0.2818186	0.5132634	0.5112857	0.5118127		
2.2	0.1103623	0.5104270	0.5137361	0.5118302	0.5118200	

If the latest approximation, $Q_{4,4}$, was not sufficiently accurate, another node, x_5 , could be selected, and another row added to the table:

$$x_5 \quad Q_{5,0} \quad Q_{5,1} \quad Q_{5,2} \quad Q_{5,3} \quad Q_{5,4} \quad Q_{5,5}.$$

Then $Q_{4,4}$, $Q_{5,4}$, and $Q_{5,5}$ could be compared to determine further accuracy.

The function in Example 2 is the Bessel function of the first kind of order zero, whose value at 2.5 is -0.0483838 , and the next row of approximations to $f(1.5)$ is

$$2.5 \quad -0.0483838 \quad 0.4807699 \quad 0.5301984 \quad 0.5119070 \quad 0.5118430 \quad 0.5118277.$$

The final new entry, 0.5118277, is correct to all seven decimal places.

The *NumericalAnalysis* package in Maple can be used to apply Neville's method for the values of x and $f(x) = y$ in Table 3.6. After loading the package we define the data with

```
xy := [[1.0, 0.7651977], [1.3, 0.6200860], [1.6, 0.4554022], [1.9, 0.2818186]]
```

Neville's method using this data gives the approximation at $x = 1.5$ with the command

```
p3 := PolynomialInterpolation(xy, method = neville, extrapolate = [1.5])
```

The output from Maple for this command is

```
POLYINTERP([[1.0, 0.7651977], [1.3, 0.6200860], [1.6, 0.4554022], [1.9, 0.2818186]],
method = neville, extrapolate = [1.5], INFO)
```

which isn't very informative. To display the information, we enter the command

```
NevilleTable(p3, 1.5)
```

and Maple returns an array with four rows and four columns. The nonzero entries corresponding to the top four rows of Table 3.6 (with the first column deleted), the zero entries are simply used to fill up the array.

To add the additional row to the table using the additional data (2.2, 0.1103623) we use the command

$p3a := \text{AddPoint}(p3, [2.2, 0.1103623])$

and a new array with all the approximation entries in Table 3.6 is obtained with

$\text{NevilleTable}(p3a, 1.5)$

Example 3 Table 3.7 lists the values of $f(x) = \ln x$ accurate to the places given. Use Neville’s method and four-digit rounding arithmetic to approximate $f(2.1) = \ln 2.1$ by completing the Neville table.

Table 3.7

i	x_i	$\ln x_i$
0	2.0	0.6931
1	2.2	0.7885
2	2.3	0.8329

Solution Because $x - x_0 = 0.1$, $x - x_1 = -0.1$, $x - x_2 = -0.2$, and we are given $Q_{0,0} = 0.6931$, $Q_{1,0} = 0.7885$, and $Q_{2,0} = 0.8329$, we have

$$Q_{1,1} = \frac{1}{0.2} [(0.1)0.7885 - (-0.1)0.6931] = \frac{0.1482}{0.2} = 0.7410$$

and

$$Q_{2,1} = \frac{1}{0.1} [(-0.1)0.8329 - (-0.2)0.7885] = \frac{0.07441}{0.1} = 0.7441.$$

The final approximation we can obtain from this data is

$$Q_{2,1} = \frac{1}{0.3} [(0.1)0.7441 - (-0.2)0.7410] = \frac{0.2276}{0.3} = 0.7420.$$

These values are shown in Table 3.8. ■

Table 3.8

i	x_i	$x - x_i$	Q_{i0}	Q_{i1}	Q_{i2}
0	2.0	0.1	0.6931		
1	2.2	-0.1	0.7885	0.7410	
2	2.3	-0.2	0.8329	0.7441	0.7420

In the preceding example we have $f(2.1) = \ln 2.1 = 0.7419$ to four decimal places, so the absolute error is

$$|f(2.1) - P_2(2.1)| = |0.7419 - 0.7420| = 10^{-4}.$$

However, $f'(x) = 1/x$, $f''(x) = -1/x^2$, and $f'''(x) = 2/x^3$, so the Lagrange error formula (3.3) in Theorem 3.3 gives the error bound

$$\begin{aligned} |f(2.1) - P_2(2.1)| &= \left| \frac{f'''(\xi(2.1))}{3!} (x - x_0)(x - x_1)(x - x_2) \right| \\ &= \left| \frac{1}{3(\xi(2.1))^3} (0.1)(-0.1)(-0.2) \right| \leq \frac{0.002}{3(2)^3} = 8.\bar{3} \times 10^{-5}. \end{aligned}$$

Notice that the actual error, 10^{-4} , exceeds the error bound, $8.\bar{3} \times 10^{-5}$. This apparent contradiction is a consequence of finite-digit computations. We used four-digit rounding arithmetic, and the Lagrange error formula (3.3) assumes infinite-digit arithmetic. This caused our actual errors to exceed the theoretical error estimate.

- Remember: You cannot expect more accuracy than the arithmetic provides.

Algorithm 3.1 constructs the entries in Neville’s method by rows.



ALGORITHM
3.1

Neville's Iterated Interpolation

To evaluate the interpolating polynomial P on the $n + 1$ distinct numbers x_0, \dots, x_n at the number x for the function f :

INPUT numbers x, x_0, x_1, \dots, x_n ; values $f(x_0), f(x_1), \dots, f(x_n)$ as the first column $Q_{0,0}, Q_{1,0}, \dots, Q_{n,0}$ of Q .

OUTPUT the table Q with $P(x) = Q_{n,n}$.

Step 1 For $i = 1, 2, \dots, n$
for $j = 1, 2, \dots, i$

$$\text{set } Q_{i,j} = \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}.$$

Step 2 OUTPUT (Q);
STOP.

The algorithm can be modified to allow for the addition of new interpolating nodes. For example, the inequality

$$|Q_{i,i} - Q_{i-1,i-1}| < \varepsilon$$

can be used as a stopping criterion, where ε is a prescribed error tolerance. If the inequality is true, $Q_{i,i}$ is a reasonable approximation to $f(x)$. If the inequality is false, a new interpolation point, x_{i+1} , is added.

EXERCISE SET 3.2

- Use Neville's method to obtain the approximations for Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:
 - $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
 - $f(-\frac{1}{3})$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
 - $f(0.25)$ if $f(0.1) = 0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$
 - $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$
- Use Neville's method to obtain the approximations for Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:
 - $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
 - $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$
 - $f(0.18)$ if $f(0.1) = -0.29004986$, $f(0.2) = -0.56079734$, $f(0.3) = -0.81401972$, $f(0.4) = -1.0526302$
 - $f(0.25)$ if $f(-1) = 0.86199480$, $f(-0.5) = 0.95802009$, $f(0) = 1.0986123$, $f(0.5) = 1.2943767$
- Use Neville's method to approximate $\sqrt{3}$ with the following functions and values.
 - $f(x) = 3^x$ and the values $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 2$.
 - $f(x) = \sqrt{x}$ and the values $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 5$.
 - Compare the accuracy of the approximation in parts (a) and (b).
- Let $P_3(x)$ be the interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$, and $(2, 2)$. Use Neville's method to find y if $P_3(1.5) = 0$.

5. Neville's method is used to approximate $f(0.4)$, giving the following table.

$x_0 = 0$	$P_0 = 1$				
$x_1 = 0.25$	$P_1 = 2$	$P_{0,1} = 2.6$			
$x_2 = 0.5$	P_2	$P_{1,2}$	$P_{0,1,2}$		
$x_3 = 0.75$	$P_3 = 8$	$P_{2,3} = 2.4$	$P_{1,2,3} = 2.96$	$P_{0,1,2,3} = 3.016$	

Determine $P_2 = f(0.5)$.

6. Neville's method is used to approximate $f(0.5)$, giving the following table.

$x_0 = 0$	$P_0 = 0$			
$x_1 = 0.4$	$P_1 = 2.8$	$P_{0,1} = 3.5$		
$x_2 = 0.7$	P_2	$P_{1,2}$	$P_{0,1,2} = \frac{27}{7}$	

Determine $P_2 = f(0.7)$.

7. Suppose $x_j = j$, for $j = 0, 1, 2, 3$ and it is known that

$$P_{0,1}(x) = 2x + 1, \quad P_{0,2}(x) = x + 1, \quad \text{and} \quad P_{1,2,3}(2.5) = 3.$$

Find $P_{0,1,2,3}(2.5)$.

8. Suppose $x_j = j$, for $j = 0, 1, 2, 3$ and it is known that

$$P_{0,1}(x) = x + 1, \quad P_{1,2}(x) = 3x - 1, \quad \text{and} \quad P_{1,2,3}(1.5) = 4.$$

Find $P_{0,1,2,3}(1.5)$.

9. Neville's Algorithm is used to approximate $f(0)$ using $f(-2)$, $f(-1)$, $f(1)$, and $f(2)$. Suppose $f(-1)$ was understated by 2 and $f(1)$ was overstated by 3. Determine the error in the original calculation of the value of the interpolating polynomial to approximate $f(0)$.
10. Neville's Algorithm is used to approximate $f(0)$ using $f(-2)$, $f(-1)$, $f(1)$, and $f(2)$. Suppose $f(-1)$ was overstated by 2 and $f(1)$ was understated by 3. Determine the error in the original calculation of the value of the interpolating polynomial to approximate $f(0)$.
11. Construct a sequence of interpolating values y_n to $f(1 + \sqrt{10})$, where $f(x) = (1 + x^2)^{-1}$ for $-5 \leq x \leq 5$, as follows: For each $n = 1, 2, \dots, 10$, let $h = 10/n$ and $y_n = P_n(1 + \sqrt{10})$, where $P_n(x)$ is the interpolating polynomial for $f(x)$ at the nodes $x_0^{(n)}, x_1^{(n)}, \dots, x_n^{(n)}$ and $x_j^{(n)} = -5 + jh$, for each $j = 0, 1, 2, \dots, n$. Does the sequence $\{y_n\}$ appear to converge to $f(1 + \sqrt{10})$?

Inverse Interpolation Suppose $f \in C^1[a, b]$, $f'(x) \neq 0$ on $[a, b]$ and f has one zero p in $[a, b]$. Let x_0, \dots, x_n , be $n + 1$ distinct numbers in $[a, b]$ with $f(x_k) = y_k$, for each $k = 0, 1, \dots, n$. To approximate p construct the interpolating polynomial of degree n on the nodes y_0, \dots, y_n for f^{-1} . Since $y_k = f(x_k)$ and $0 = f(p)$, it follows that $f^{-1}(y_k) = x_k$ and $p = f^{-1}(0)$. Using iterated interpolation to approximate $f^{-1}(0)$ is called *iterated inverse interpolation*.

12. Use iterated inverse interpolation to find an approximation to the solution of $x - e^{-x} = 0$, using the data

x	0.3	0.4	0.5	0.6
e^{-x}	0.740818	0.670320	0.606531	0.548812

13. Construct an algorithm that can be used for inverse interpolation.

3.3 Divided Differences

Iterated interpolation was used in the previous section to generate successively higher-degree polynomial approximations at a specific point. Divided-difference methods introduced in this section are used to successively generate the polynomials themselves.

Suppose that $P_n(x)$ is the n th Lagrange polynomial that agrees with the function f at the distinct numbers x_0, x_1, \dots, x_n . Although this polynomial is unique, there are alternate algebraic representations that are useful in certain situations. The divided differences of f with respect to x_0, x_1, \dots, x_n are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1}), \quad (3.5)$$

for appropriate constants a_0, a_1, \dots, a_n . To determine the first of these constants, a_0 , note that if $P_n(x)$ is written in the form of Eq. (3.5), then evaluating $P_n(x)$ at x_0 leaves only the constant term a_0 ; that is,

$$a_0 = P_n(x_0) = f(x_0).$$

Similarly, when $P(x)$ is evaluated at x_1 , the only nonzero terms in the evaluation of $P_n(x_1)$ are the constant and linear terms,

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1);$$

so

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (3.6)$$

We now introduce the divided-difference notation, which is related to Aitken's Δ^2 notation used in Section 2.5. The *zereth divided difference* of the function f with respect to x_i , denoted $f[x_i]$, is simply the value of f at x_i :

$$f[x_i] = f(x_i). \quad (3.7)$$

The remaining divided differences are defined recursively; the *first divided difference* of f with respect to x_i and x_{i+1} is denoted $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}. \quad (3.8)$$

The *second divided difference*, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

Similarly, after the $(k - 1)$ st divided differences,

$$f[x_i, x_{i+1}, x_{i+2}, \dots, x_{i+k-1}] \quad \text{and} \quad f[x_{i+1}, x_{i+2}, \dots, x_{i+k-1}, x_{i+k}],$$

have been determined, the **k th divided difference** relative to $x_i, x_{i+1}, x_{i+2}, \dots, x_{i+k}$ is

$$f[x_i, x_{i+1}, \dots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}. \quad (3.9)$$

The process ends with the single *n th divided difference*,

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

Because of Eq. (3.6) we can write $a_1 = f[x_0, x_1]$, just as a_0 can be expressed as $a_0 = f(x_0) = f[x_0]$. Hence the interpolating polynomial in Eq. (3.5) is

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}).$$

As in so many areas, Isaac Newton is prominent in the study of difference equations. He developed interpolation formulas as early as 1675, using his Δ notation in tables of differences. He took a very general approach to the difference formulas, so explicit examples that he produced, including Lagrange's formulas, are often known by other names.

As might be expected from the evaluation of a_0 and a_1 , the required constants are

$$a_k = f[x_0, x_1, x_2, \dots, x_k],$$

for each $k = 0, 1, \dots, n$. So $P_n(x)$ can be rewritten in a form called Newton's Divided-Difference:

$$P_n(x) = f[x_0] + \sum_{k=1}^n f[x_0, x_1, \dots, x_k](x - x_0) \cdots (x - x_{k-1}). \quad (3.10)$$

The value of $f[x_0, x_1, \dots, x_k]$ is independent of the order of the numbers x_0, x_1, \dots, x_k , as shown in Exercise 21.

The generation of the divided differences is outlined in Table 3.9. Two fourth and one fifth difference can also be determined from these data.

Table 3.9

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
		$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
		$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			



Newton's Divided-Difference Formula

To obtain the divided-difference coefficients of the interpolatory polynomial P on the $(n+1)$ distinct numbers x_0, x_1, \dots, x_n for the function f :

INPUT numbers x_0, x_1, \dots, x_n ; values $f(x_0), f(x_1), \dots, f(x_n)$ as $F_{0,0}, F_{1,0}, \dots, F_{n,0}$.

OUTPUT the numbers $F_{0,0}, F_{1,1}, \dots, F_{n,n}$ where

$$P_n(x) = F_{0,0} + \sum_{i=1}^n F_{i,i} \prod_{j=0}^{i-1} (x - x_j). \quad (F_{i,i} \text{ is } f[x_0, x_1, \dots, x_i].)$$

Step 1 For $i = 1, 2, \dots, n$

For $j = 1, 2, \dots, i$

$$\text{set } F_{i,j} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}. \quad (F_{i,j} = f[x_{i-j}, \dots, x_i].)$$

Step 2 OUTPUT $(F_{0,0}, F_{1,1}, \dots, F_{n,n})$;
STOP.

The form of the output in Algorithm 3.2 can be modified to produce all the divided differences, as shown in Example 1.

Example 1**Table 3.10**

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Complete the divided difference table for the data used in Example 1 of Section 3.2, and reproduced in Table 3.10, and construct the interpolating polynomial that uses all this data.

Solution The first divided difference involving x_0 and x_1 is

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{0.6200860 - 0.7651977}{1.3 - 1.0} = -0.4837057.$$

The remaining first divided differences are found in a similar manner and are shown in the fourth column in Table 3.11.

Table 3.11

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \dots, x_i]$	$f[x_{i-4}, \dots, x_i]$
0	1.0	0.7651977				
1	1.3	0.6200860	-0.4837057			
2	1.6	0.4554022	-0.5489460	-0.1087339		
3	1.9	0.2818186	-0.5786120	-0.0494433	0.0658784	
4	2.2	0.1103623	-0.5715210	0.0118183	0.0680685	0.0018251

The second divided difference involving x_0 , x_1 , and x_2 is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{-0.5489460 - (-0.4837057)}{1.6 - 1.0} = -0.1087339.$$

The remaining second divided differences are shown in the 5th column of Table 3.11. The third divided difference involving x_0 , x_1 , x_2 , and x_3 and the fourth divided difference involving all the data points are, respectively,

$$\begin{aligned} f[x_0, x_1, x_2, x_3] &= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-0.0494433 - (-0.1087339)}{1.9 - 1.0} \\ &= 0.0658784, \end{aligned}$$

and

$$\begin{aligned} f[x_0, x_1, x_2, x_3, x_4] &= \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0} = \frac{0.0680685 - 0.0658784}{2.2 - 1.0} \\ &= 0.0018251. \end{aligned}$$

All the entries are given in Table 3.11.

The coefficients of the Newton forward divided-difference form of the interpolating polynomial are along the diagonal in the table. This polynomial is

$$\begin{aligned} P_4(x) &= 0.7651977 - 0.4837057(x - 1.0) - 0.1087339(x - 1.0)(x - 1.3) \\ &\quad + 0.0658784(x - 1.0)(x - 1.3)(x - 1.6) \\ &\quad + 0.0018251(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9). \end{aligned}$$

Notice that the value $P_4(1.5) = 0.5118200$ agrees with the result in Table 3.6 for Example 2 of Section 3.2, as it must because the polynomials are the same. ■

We can use Maple with the *NumericalAnalysis* package to create the Newton Divided-Difference table. First load the package and define the x and $f(x) = y$ values that will be used to generate the first four rows of Table 3.11.

```
xy := [[1.0, 0.7651977], [1.3, 0.6200860], [1.6, 0.4554022], [1.9, 0.2818186]]
```

The command to create the divided-difference table is

```
p3 := PolynomialInterpolation(xy, independentvar = 'x', method = newton)
```

A matrix containing the divided-difference table as its nonzero entries is created with the *DividedDifferenceTable*(p3)

We can add another row to the table with the command

```
p4 := AddPoint(p3, [2.2, 0.1103623])
```

which produces the divided-difference table with entries corresponding to those in Table 3.11.

The Newton form of the interpolation polynomial is created with

```
Interpolant(p4)
```

which produces the polynomial in the form of $P_4(x)$ in Example 1, except that in place of the first two terms of $P_4(x)$:

$$0.7651977 - 0.4837057(x - 1.0)$$

Maple gives this as $1.248903367 - 0.4837056667x$.

The Mean Value Theorem 1.8 applied to Eq. (3.8) when $i = 0$,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

implies that when f' exists, $f[x_0, x_1] = f'(\xi)$ for some number ξ between x_0 and x_1 . The following theorem generalizes this result.

Theorem 3.6 Suppose that $f \in C^n[a, b]$ and x_0, x_1, \dots, x_n are distinct numbers in $[a, b]$. Then a number ξ exists in (a, b) with

$$f[x_0, x_1, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}. \quad \blacksquare$$

Proof Let

$$g(x) = f(x) - P_n(x).$$

Since $f(x_i) = P_n(x_i)$ for each $i = 0, 1, \dots, n$, the function g has $n + 1$ distinct zeros in $[a, b]$. Generalized Rolle's Theorem 1.10 implies that a number ξ in (a, b) exists with $g^{(n)}(\xi) = 0$, so

$$0 = f^{(n)}(\xi) - P_n^{(n)}(\xi).$$

Since $P_n(x)$ is a polynomial of degree n whose leading coefficient is $f[x_0, x_1, \dots, x_n]$,

$$P_n^{(n)}(x) = n!f[x_0, x_1, \dots, x_n],$$

for all values of x . As a consequence,

$$f[x_0, x_1, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}. \quad \blacksquare \quad \blacksquare \quad \blacksquare$$

Newton's divided-difference formula can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing. In this case, we introduce the notation $h = x_{i+1} - x_i$, for each $i = 0, 1, \dots, n-1$ and let $x = x_0 + sh$. Then the difference $x - x_i$ is $x - x_i = (s - i)h$. So Eq. (3.10) becomes

$$\begin{aligned} P_n(x) &= P_n(x_0 + sh) = f[x_0] + shf[x_0, x_1] + s(s-1)h^2f[x_0, x_1, x_2] \\ &\quad + \dots + s(s-1)\dots(s-n+1)h^n f[x_0, x_1, \dots, x_n] \\ &= f[x_0] + \sum_{k=1}^n s(s-1)\dots(s-k+1)h^k f[x_0, x_1, \dots, x_k]. \end{aligned}$$

Using binomial-coefficient notation,

$$\binom{s}{k} = \frac{s(s-1)\dots(s-k+1)}{k!},$$

we can express $P_n(x)$ compactly as

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^n \binom{s}{k} k! h^k f[x_0, x_1, \dots, x_k]. \quad (3.11)$$

Forward Differences

The **Newton forward-difference formula**, is constructed by making use of the forward difference notation Δ introduced in Aitken's Δ^2 method. With this notation,

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}(f(x_1) - f(x_0)) = \frac{1}{h}\Delta f(x_0) \\ f[x_0, x_1, x_2] &= \frac{1}{2h} \left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h} \right] = \frac{1}{2h^2}\Delta^2 f(x_0), \end{aligned}$$

and, in general,

$$f[x_0, x_1, \dots, x_k] = \frac{1}{k!h^k}\Delta^k f(x_0).$$

Since $f[x_0] = f(x_0)$, Eq. (3.11) has the following form.

Newton Forward-Difference Formula

$$P_n(x) = f(x_0) + \sum_{k=1}^n \binom{s}{k} \Delta^k f(x_0) \quad (3.12)$$

Backward Differences

If the interpolating nodes are reordered from last to first as x_n, x_{n-1}, \dots, x_0 , we can write the interpolatory formula as

$$\begin{aligned} P_n(x) &= f[x_n] + f[x_n, x_{n-1}](x - x_n) + f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1}) \\ &\quad + \dots + f[x_n, \dots, x_0](x - x_n)(x - x_{n-1})\dots(x - x_1). \end{aligned}$$

If, in addition, the nodes are equally spaced with $x = x_n + sh$ and $x = x_i + (s + n - i)h$, then

$$\begin{aligned} P_n(x) &= P_n(x_n + sh) \\ &= f[x_n] + shf[x_n, x_{n-1}] + s(s + 1)h^2 f[x_n, x_{n-1}, x_{n-2}] + \cdots \\ &\quad + s(s + 1) \cdots (s + n - 1)h^n f[x_n, \dots, x_0]. \end{aligned}$$

This is used to derive a commonly applied formula known as the **Newton backward-difference formula**. To discuss this formula, we need the following definition.

Definition 3.7 Given the sequence $\{p_n\}_{n=0}^\infty$, define the backward difference ∇p_n (read *nabla p_n*) by

$$\nabla p_n = p_n - p_{n-1}, \quad \text{for } n \geq 1.$$

Higher powers are defined recursively by

$$\nabla^k p_n = \nabla(\nabla^{k-1} p_n), \quad \text{for } k \geq 2. \quad \blacksquare$$

Definition 3.7 implies that

$$f[x_n, x_{n-1}] = \frac{1}{h} \nabla f(x_n), \quad f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h^2} \nabla^2 f(x_n),$$

and, in general,

$$f[x_n, x_{n-1}, \dots, x_{n-k}] = \frac{1}{k!h^k} \nabla^k f(x_n).$$

Consequently,

$$P_n(x) = f[x_n] + s \nabla f(x_n) + \frac{s(s + 1)}{2} \nabla^2 f(x_n) + \cdots + \frac{s(s + 1) \cdots (s + n - 1)}{n!} \nabla^n f(x_n).$$

If we extend the binomial coefficient notation to include all real values of s by letting

$$\binom{-s}{k} = \frac{-s(-s - 1) \cdots (-s - k + 1)}{k!} = (-1)^k \frac{s(s + 1) \cdots (s + k - 1)}{k!},$$

then

$$P_n(x) = f[x_n] + (-1)^1 \binom{-s}{1} \nabla f(x_n) + (-1)^2 \binom{-s}{2} \nabla^2 f(x_n) + \cdots + (-1)^n \binom{-s}{n} \nabla^n f(x_n).$$

This gives the following result.

Newton Backward–Difference Formula

$$P_n(x) = f[x_n] + \sum_{k=1}^n (-1)^k \binom{-s}{k} \nabla^k f(x_n) \tag{3.13}$$

Illustration

The divided-difference Table 3.12 corresponds to the data in Example 1.

Table 3.12

		First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	<u>0.7651977</u>				
		<u>-0.4837057</u>			
1.3	0.6200860		<u>-0.1087339</u>		
		-0.5489460		<u>0.0658784</u>	
1.6	0.4554022		-0.0494433		<u>0.0018251</u>
		-0.5786120		<u>0.0680685</u>	
1.9	0.2818186		<u>0.0118183</u>		
		<u>-0.5715210</u>			
2.2	<u>0.1103623</u>				

Only one interpolating polynomial of degree at most 4 uses these five data points, but we will organize the data points to obtain the best interpolation approximations of degrees 1, 2, and 3. This will give us a sense of accuracy of the fourth-degree approximation for the given value of x .

If an approximation to $f(1.1)$ is required, the reasonable choice for the nodes would be $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$, and $x_4 = 2.2$ since this choice makes the earliest possible use of the data points closest to $x = 1.1$, and also makes use of the fourth divided difference. This implies that $h = 0.3$ and $s = \frac{1}{3}$, so the Newton forward divided-difference formula is used with the divided differences that have a *solid* underline (—) in Table 3.12:

$$\begin{aligned}
 P_4(1.1) &= P_4\left(1.0 + \frac{1}{3}(0.3)\right) \\
 &= 0.7651977 + \frac{1}{3}(0.3)(-0.4837057) + \frac{1}{3}\left(-\frac{2}{3}\right)(0.3)^2(-0.1087339) \\
 &\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(0.3)^3(0.0658784) \\
 &\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(0.3)^4(0.0018251) \\
 &= 0.7196460.
 \end{aligned}$$

To approximate a value when x is close to the end of the tabulated values, say, $x = 2.0$, we would again like to make the earliest use of the data points closest to x . This requires using the Newton backward divided-difference formula with $s = -\frac{2}{3}$ and the divided differences in Table 3.12 that have a *wavy* underline (~~~~). Notice that the fourth divided difference is used in both formulas.

$$\begin{aligned}
 P_4(2.0) &= P_4\left(2.2 - \frac{2}{3}(0.3)\right) \\
 &= 0.1103623 - \frac{2}{3}(0.3)(-0.5715210) - \frac{2}{3}\left(\frac{1}{3}\right)(0.3)^2(0.0118183) \\
 &\quad - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3(0.0680685) - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0.3)^4(0.0018251) \\
 &= 0.2238754.
 \end{aligned}$$

□

Centered Differences

The Newton forward- and backward-difference formulas are not appropriate for approximating $f(x)$ when x lies near the center of the table because neither will permit the highest-order difference to have x_0 close to x . A number of divided-difference formulas are available for this case, each of which has situations when it can be used to maximum advantage. These methods are known as **centered-difference formulas**. We will consider only one centered-difference formula, Stirling’s method.

For the centered-difference formulas, we choose x_0 near the point being approximated and label the nodes directly below x_0 as x_1, x_2, \dots and those directly above as x_{-1}, x_{-2}, \dots . With this convention, **Stirling’s formula** is given by

$$\begin{aligned}
 P_n(x) = P_{2m+1}(x) = & f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2 h^2 f[x_{-1}, x_0, x_1] \quad (3.14) \\
 & + \frac{s(s^2 - 1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2] \\
 & + \dots + s^2(s^2 - 1)(s^2 - 4) \dots (s^2 - (m - 1)^2)h^{2m} f[x_{-m}, \dots, x_m] \\
 & + \frac{s(s^2 - 1) \dots (s^2 - m^2)h^{2m+1}}{2} (f[x_{-m-1}, \dots, x_m] + f[x_{-m}, \dots, x_{m+1}]),
 \end{aligned}$$

if $n = 2m + 1$ is odd. If $n = 2m$ is even, we use the same formula but delete the last line. The entries used for this formula are underlined in Table 3.13.

James Stirling (1692–1770) published this and numerous other formulas in *Methodus Differentialis* in 1720. Techniques for accelerating the convergence of various series are included in this work.

Table 3.13

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
x_{-2}	$f[x_{-2}]$				
		$f[x_{-2}, x_{-1}]$			
x_{-1}	$f[x_{-1}]$		$f[x_{-2}, x_{-1}, x_0]$		
		<u>$f[x_{-1}, x_0]$</u>		<u>$f[x_{-2}, x_{-1}, x_0, x_1]$</u>	
x_0	<u>$f[x_0]$</u>		<u>$f[x_{-1}, x_0, x_1]$</u>		<u>$f[x_{-2}, x_{-1}, x_0, x_1, x_2]$</u>
		<u>$f[x_0, x_1]$</u>		<u>$f[x_{-1}, x_0, x_1, x_2]$</u>	
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	$f[x_2]$				

Example 2 Consider the table of data given in the previous examples. Use Stirling’s formula to approximate $f(1.5)$ with $x_0 = 1.6$.

Solution To apply Stirling’s formula we use the *underlined* entries in the difference Table 3.14.

Table 3.14

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		<u>−0.5489460</u>		<u>0.0658784</u>	
1.6	<u>0.4554022</u>		<u>−0.0494433</u>		<u>0.0018251</u>
		<u>−0.5786120</u>		<u>0.0680685</u>	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

The formula, with $h = 0.3$, $x_0 = 1.6$, and $s = -\frac{1}{3}$, becomes

$$\begin{aligned} f(1.5) &\approx P_4 \left(1.6 + \left(-\frac{1}{3} \right) (0.3) \right) \\ &= 0.4554022 + \left(-\frac{1}{3} \right) \left(\frac{0.3}{2} \right) ((-0.5489460) + (-0.5786120)) \\ &\quad + \left(-\frac{1}{3} \right)^2 (0.3)^2 (-0.0494433) \\ &\quad + \frac{1}{2} \left(-\frac{1}{3} \right) \left(\left(-\frac{1}{3} \right)^2 - 1 \right) (0.3)^3 (0.0658784 + 0.0680685) \\ &\quad + \left(-\frac{1}{3} \right)^2 \left(\left(-\frac{1}{3} \right)^2 - 1 \right) (0.3)^4 (0.0018251) = 0.5118200. \quad \blacksquare \end{aligned}$$

Most texts on numerical analysis written before the wide-spread use of computers have extensive treatments of divided-difference methods. If a more comprehensive treatment of this subject is needed, the book by Hildebrand [Hild] is a particularly good reference.

EXERCISE SET 3.3

- Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
 - $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
 - $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$
- Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
 - $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
 - $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$
- Use Newton the forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
 - $f(-\frac{1}{3})$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
 - $f(0.25)$ if $f(0.1) = -0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$
- Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
 - $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
 - $f(0.18)$ if $f(0.1) = -0.29004986$, $f(0.2) = -0.56079734$, $f(0.3) = -0.81401972$, $f(0.4) = -1.0526302$
- Use the Newton backward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
 - $f(-1/3)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
 - $f(0.25)$ if $f(0.1) = -0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$

6. Use the Newton backward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
- $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
 - $f(0.25)$ if $f(-1) = 0.86199480$, $f(-0.5) = 0.95802009$, $f(0) = 1.0986123$, $f(0.5) = 1.2943767$

7. a. Use Algorithm 3.2 to construct the interpolating polynomial of degree three for the unequally spaced points given in the following table:

x	$f(x)$
-0.1	5.30000
0.0	2.00000
0.2	3.19000
0.3	1.00000

- b. Add $f(0.35) = 0.97260$ to the table, and construct the interpolating polynomial of degree four.
8. a. Use Algorithm 3.2 to construct the interpolating polynomial of degree four for the unequally spaced points given in the following table:

x	$f(x)$
0.0	-6.00000
0.1	-5.89483
0.3	-5.65014
0.6	-5.17788
1.0	-4.28172

- b. Add $f(1.1) = -3.99583$ to the table, and construct the interpolating polynomial of degree five.
9. a. Approximate $f(0.05)$ using the following data and the Newton forward-difference formula:

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	1.00000	1.22140	1.49182	1.82212	2.22554

- Use the Newton backward-difference formula to approximate $f(0.65)$.
 - Use Stirling's formula to approximate $f(0.43)$.
10. Show that the polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

11. a. Show that the cubic polynomials

$$P(x) = 3 - 2(x + 1) + 0(x + 1)(x) + (x + 1)(x)(x - 1)$$

and

$$Q(x) = -1 + 4(x + 2) - 3(x + 2)(x + 1) + (x + 2)(x + 1)(x)$$

both interpolate the data

x	-2	-1	0	1	2
$f(x)$	-1	3	1	-1	3

- Why does part (a) not violate the uniqueness property of interpolating polynomials?
12. A fourth-degree polynomial $P(x)$ satisfies $\Delta^4 P(0) = 24$, $\Delta^3 P(0) = 6$, and $\Delta^2 P(0) = 0$, where $\Delta P(x) = P(x + 1) - P(x)$. Compute $\Delta^2 P(10)$.

13. The following data are given for a polynomial $P(x)$ of unknown degree.

x	0	1	2
$P(x)$	2	-1	4

Determine the coefficient of x^2 in $P(x)$ if all third-order forward differences are 1.

14. The following data are given for a polynomial $P(x)$ of unknown degree.

x	0	1	2	3
$P(x)$	4	9	15	18

Determine the coefficient of x^3 in $P(x)$ if all fourth-order forward differences are 1.

15. The Newton forward-difference formula is used to approximate $f(0.3)$ given the following data.

x	0.0	0.2	0.4	0.6
$f(x)$	15.0	21.0	30.0	51.0

Suppose it is discovered that $f(0.4)$ was understated by 10 and $f(0.6)$ was overstated by 5. By what amount should the approximation to $f(0.3)$ be changed?

16. For a function f , the Newton divided-difference formula gives the interpolating polynomial

$$P_3(x) = 1 + 4x + 4x(x - 0.25) + \frac{16}{3}x(x - 0.25)(x - 0.5),$$

on the nodes $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$ and $x_3 = 0.75$. Find $f(0.75)$.

17. For a function f , the forward-divided differences are given by

$x_0 = 0.0$	$f[x_0]$		
		$f[x_0, x_1]$	
$x_1 = 0.4$	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{50}{7}$
		$f[x_1, x_2] = 10$	
$x_2 = 0.7$	$f[x_2] = 6$		

Determine the missing entries in the table.

18. a. The introduction to this chapter included a table listing the population of the United States from 1950 to 2000. Use appropriate divided differences to approximate the population in the years 1940, 1975, and 2020.
 b. The population in 1940 was approximately 132,165,000. How accurate do you think your 1975 and 2020 figures are?
19. Given

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}),$$

use $P_n(x_2)$ to show that $a_2 = f[x_0, x_1, x_2]$.

20. Show that

$$f[x_0, x_1, \dots, x_n, x] = \frac{f^{(n+1)}(\xi(x))}{(n+1)!},$$

for some $\xi(x)$. [Hint: From Eq. (3.3),

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0) \cdots (x - x_n).$$