$$F(2) = F(1) + p\Delta F(1) + \frac{p(p-1)}{2!}\Delta^2 F(1) + \frac{p(p-1)(p-2)}{3!}\Delta^3 F(1)$$

where

$$p = \frac{x - x_0}{h} = \frac{2-1}{3} = \frac{1}{3}$$

Also, from the table we note that $\Delta F(1) = -171186$, $\Delta^2 F(1) = 17158$ and $\Delta^3 F(1) = 2023$. Substituting these values, the above equation gives.

$$F(2) = 500426 - \frac{171186}{3} - \frac{17158}{9} + \frac{5(2023)}{81} = 441582.4325$$

Therefore, we finally have

$$f(2) = F(1) - F(2) = 500426 - 441582.4325 = 58843.5675$$

## 6.10 HERMITE INTERPOLATION

In Hermite interpolation, we use the expansion involving not only the function values but also its first derivative. To state the problem: given a set of data points $(x_i, y_i, y'_i)$, $i = 0, 1, 2, \ldots, n$, we have to determine a polynomial $P(x)$ of degree $(2n + 1)$. Thus, keeping in mind the Lagrange interpolation formula, we seek $P(x)$ in the form

$$P(x) = \sum_{i=0}^{n} U_i(x) y_i + \sum_{i=0}^{n} V_i(x) y'_i \tag{6.95}$$

where $U_i(x)$ and $V_i(x)$ are polynomials of degree $(2n + 1)$ that satisfy the relations

$$\left.\begin{aligned} & U_i(x_j) = \delta_{ij} \\ & \left.\frac{\partial U_i}{\partial x}\right|_{x = x_j} = 0 \\ \text{and} \quad & V_i(x_j) = 0 \\ & \left.\frac{\partial V_i}{\partial x}\right|_{x = x_j} = \delta_{ij} \end{aligned}\right\} \tag{6.96}$$

Here, $\delta_{ij}$ is a kronecker delta, whose value is unity if $i = j$, otherwise zero. Polynomials satisfying the above conditions are called Hermite polynomials. Now, we define

$$U_i = \left\{1 - 2(x - x_i)\left.\frac{dL_i}{dx}\right|_{x = x_i}\right\}[L_i(x)]^2$$

and

$$V_i = (x - x_i)[L_i(x)]^2 \tag{6.97}$$

which of course meets the requirements as defined in Eq. (6.96), where $L_i(x)$ is a Lagrange polynomial satisfying

$$L_i(x_j) = \delta_{ij}$$

Substituting $x = x_i$ in Eq. (6.97), we find that

$$U_i(x_i) = [L_i(x_i)]^2 = 1$$

and

$$V_i(x_i) = 0$$

Now, differentiating Eqs. (6.97), we have

$$U_i'(x) = [1 - 2L_i'(x_i)(x - x_i)]2\,L_i(x)\,L_i'(x)$$

$$-2L_i'(x_i)[L_i(x)]^2$$

and

$V_i'(x) = (x - x_i)\,2L_i(x)\,L_i'(x) + L_i(x)^2$ observe that $U_i'(x_j) = 0$, $V_i'(x_j) = 0$ for $i \neq j$. Since $L_i(x_i) = 1$, we get

$$U_i'(x_i) = 2L_i'(x_i) - 2L_i'(x_i) = 0$$

and

$$V_i'(x_i) = [L_i(x_i)]^2 = 1$$

Hence, the Hermite Interpolation formula is given as

$$P(x) = \sum_{i=0}^{n} [1 - 2L_i'(x_i)(x - x_i)][L_i(x)]^2\, y_i$$

$$+ (x - x_i)[L_i(x)]^2\, y_i' \tag{6.98}$$

For illustration, we consider the following example.

**Example 6.25** Estimate the value of $y(1.05)$ using Hermite interpolation formula from the following data:

| $x$ | $y$ | $y'$ |
|---|---|---|
| 1.00 | 1.00000 | 0.5000 |
| 1.10 | 1.04881 | 0.47673 |

***Solution:*** At first we compute

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{1.05 - 1.10}{1.00 - 1.10} = 0.5$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{1.05 - 1.00}{1.10 - 1.00} = 0.5$$

$$L_0'(x) = \frac{1}{x_0 - x_1} = -\frac{1}{0.10}$$

$$L_1'(x) = \frac{1}{x_1 - x_0} = \frac{1}{0.1}$$

Substituting these expressions in Hermite formula,

$$P(x) = \sum_{i=0}^{n} [1 - 2L_i'(x_i)(x - x_i)][L_i(x)]^2 y_i$$

$$+ (x - x_i)[L_i(x)]^2 y_i'$$

we find

$$y(1.05) = \left[1 - 2\left(-\frac{1}{0.1}\right)(0.05)\right]\left(\frac{1}{2}\right)^2 (1) + (0.05)\left(\frac{1}{2}\right)^2 (0.5)$$

$$+\left[1 - 2\left(\frac{1}{0.1}\right)(-0.05)\right]\left(\frac{1}{2}\right)^2 (1.04881)$$

$$+ (-0.05)\left(\frac{1}{2}\right)^2 (0.47673)$$

$$= 1.0247$$

## EXERCISES

**6.1** Express $\Delta^2 y_1$ and $\Delta^4 y_0$ in terms of the values of the function $y$.

**6.2** Compute the missing values of $y_n$ and $\Delta y_n$ in the following table

| $y_n$ | $\Delta y_n$ | $\Delta^2 y_n$ |
|---|---|---|
| – | | |
| – | – | |
| – | – | 1 |
| – | – | 4 |
| 6 | 5 | 13 |
| – | – | 18 |
| – | – | 24 |
| – | – | |

**6.3** Show that $E\nabla = \Delta = \delta E^{1/2}$.

**6.4** Prove that (i) $\delta = 2 \sinh (hD/2)$ and, (ii) $\mu = 2 \cosh (hD/2)$.