$$= \frac{x^3 - 13x^2 + 54x - 72}{-30}(-3) + \frac{x^3 - 11x^2 + 34x - 24}{6}(0)$$

$$+ \frac{x^3 - 10x^2 + 27x - 18}{-6}(30) + \frac{x^3 - 8x^2 + 19x - 12}{30}(132)$$

On simplification, we get

$$y(x) = \frac{1}{10}\left(-5x^3 + 135x^2 - 460x + 300\right) = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$$

which is the required Lagrange's interpolation polynomial. Now, $y(5) = 75$.

**Example 6.15** Given the following data, evaluate $f(3)$ using Lagrange's interpolating polynomial.

| $x$ | 1 | 2 | 5 |
|---|---|---|---|
| $f(x)$ | 1 | 4 | 10 |

***Solution*** Using Lagrange's interpolation formula given by Eq. (6.37), we have

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Therefore,

$$f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1) + \frac{(3-1)(3-5)}{(2-1)(2-5)}(4) + \frac{(3-1)(3-2)}{(5-1)(5-2)}(10) = 6.4999$$

## 6.6 DIVIDED DIFFERENCES

When the function values are given at non-equispaced points, we have already developed the Lagrange's interpolation formula for interpolation in Section 6.5. Now, we shall introduce the concept of divided differences and then develop Newton's divided difference interpolation formula, whose accuracy is same as that of Lagrange's formula, but has the advantage of being computationally economical in the sense that it involves less number of arithmetic operations.

Let us assume that the function $y = f(x)$ is known for several values of $x$, $(x_i, y_i)$, for $i = 0\,(1)\,n$. The divided differences of orders 0, 1, 2, ..., $n$ are defined recursively as follows:

$$y[x_0] = y(x_0) = y_0$$

is the 0th order divided difference. The first order divided difference is defined as

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, the higher order divided differences are defined in terms of lower order divided differences by the relations (Hildebrand, 1982) of the form

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

while

$$y[x_0, x_1, \cdots, x_n] = \frac{y[x_1, x_2, \cdots, x_n] - y[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \tag{6.46}$$

The standard format of the divided differences are displayed in Table 6.4.

**Table 6.4 Divided Differences**

| $x$ | $y(x)$ | 1st order | 2nd order | 3rd order | 4th order |
|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | |
| | | $y[x_1, x_0]$ | | | |
| $x_1$ | $y_1$ | | $y[x_0, x_1, x_2]$ | | |
| | | $y[x_2, x_1]$ | | $y[x_0, x_1, x_2, x_3]$ | |
| $x_2$ | $y_2$ | | $y[x_1, x_2, x_3]$ | | $y[x_0, x_1, x_2, x_3, x_4]$ |
| | | $y[x_3, x_2]$ | | $y[x_1, x_2, x_3, x_4]$ | |
| $x_3$ | $y_3$ | | $y[x_2, x_3, x_4]$ | | |
| | | $y[x_4, x_3]$ | | | |
| $x_4$ | $y_4$ | | | | |

We can easily verify that the divided difference is a symmetric function of its arguments. That is,

$$y[x_1, x_0] = y[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

Now,

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{1}{x_2 - x_0}\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

Therefore,

$$y[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

which is a symmetric form, hence suggests the general result as

$$y[x_0, \cdots, x_k] = \frac{y_0}{(x_0 - x_1)\cdots(x_0 - x_k)} + \frac{y_1}{(x_1 - x_0)\cdots(x_1 - x_k)} + \cdots$$

$$+ \frac{y_k}{(x_k - x_0)\cdots(x_k - x_{k-1})}$$

$$= \sum_{i=0}^{k} \frac{y_i}{\prod_{\substack{i=0 \\ i \neq j}}^{k} (x_i - x_j)} \tag{6.47}$$

In Eq. (6.47), it can be noted that zero factor $(x_i - x_i)$ is omitted in the denominator of each term of the sum.

### 6.6.1 Newton's Divided Difference Interpolation Formula

Let $y = f(x)$ be a function which takes values $y_0, y_1, \ldots, y_n$ corresponding to $x = x_i$, $i = 0, 1, \ldots, n$. We choose an interpolating polynomial, interpolating at $x = x_i$, $i = 0, 1, \ldots, n$ in the following convenient form

$$y = f(x) = a_0 + a_1 (x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n (x - x_0)(x - x_1) \cdots (x - x_{n-1}) \quad (6.48)$$

Here, the coefficients $a_k$ are so chosen as to satisfy Eq. (6.48) by the $(n + 1)$ pairs $(x_i, y_i)$. Thus, we have

$$\left.\begin{aligned} y(x_0) &= f(x_0) = y_0 = a_0 \\ y(x_1) &= f(x_1) = y_1 = a_0 + a_1(x_1 - x_0) \\ y(x_2) &= f(x_2) = y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ &\vdots \\ y_n &= a_0 + a_1(x_n - x_0) + a_2(x_n - x_0)(x_n - x_1) + \cdots \\ &\quad + a_n(x_n - x_0) \cdots (x_n - x_{n-1}) \end{aligned}\right\} \quad (6.49)$$

The coefficients $a_0, a_1, \ldots, a_n$ can be easily obtained from the system of Eqs. (6.49), as they form a lower triangular matrix. The first equation of (6.49) gives

$$a_0 = y(x_0) = y_0 \quad (6.50)$$

The second equation of (6.49) and Eq. (6.50) gives

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = y[x_0, x_1] \quad (6.51)$$

The third equation of (6.49) after using $a_0$ and $a_1$ as given in Eqs. (6.50) and (6.51) yields

$$a_2 = \frac{y_2 - y_0 - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

which can be rewritten as

$$a_2 = \frac{\left[y_2 - y_1 + \dfrac{y_1 - y_0}{x_1 - x_0}(x_1 - x_0)\right] - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

That is,

$$a_2 = \frac{y_2 - y_1 + y[x_0, x_1](x_1 - x_2)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

Thus, in terms of second order divided differences, we have

$$a_2 = y[x_0, x_1, x_2] \quad (6.52)$$

Similarly, we can show that

$$a_n = y[x_0, x_1, \ldots, x_n] \quad (6.53)$$

Hence, Newton's divided difference interpolation formula can be written as

$$y = f(x) = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1})\, y[x_0, x_1, \dots, x_n] \tag{6.54}$$

Newton's divided differences can also be expressed in terms of forward, backward and central differences. They can be easily derived as follows: Assuming equispaced values of abscissa, we have

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\dfrac{\Delta y_1}{h} - \dfrac{\Delta y_0}{h}}{2h} = \frac{\Delta^2 y_0}{2!h^2}$$

By induction, we can in general arrive at the result

$$y[x_0, x_1, \cdots, x_n] = \frac{\Delta^n y_0}{n!h^n} \tag{6.55}$$

Similarly

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\nabla y_1}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\dfrac{\nabla y_2}{h} - \dfrac{\nabla y_1}{h}}{2h} = \frac{\nabla^2 y_2}{2!h^2}$$

In general, we have

$$y[x_0, x_1, \cdots, x_n] = \frac{\nabla^n y_n}{n!h^n} \tag{6.56}$$

Also, in terms of central differences, we have

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\delta y_{1/2}}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\dfrac{\delta y_{3/2}}{h} - \dfrac{\delta y_{1/2}}{h}}{2h} = \frac{\delta^2 y_1}{2!h^2}$$

In general, the following pattern is arrived:

$$\left.\begin{aligned} y[x_0, x_1, \dots, x_{2m}] &= \frac{\delta^{2m} y_m}{(2m)!\,h^{2m}} \\ \text{or} \quad & \\ y[x_0, x_1, \dots, x_{2m+1}] &= \frac{\delta^{2m+1} y_{m+(1/2)}}{(2m+1)!\,h^{2m+1}} \end{aligned}\right\} \tag{6.57}$$

We present below few examples for illustration.

**Example 6.16** Find the interpolating polynomial by (i) Lagrange's formula, and (ii) Newton's divided difference formula for the following data, and hence show that they represent the same interpolating polynomial.

| $x$ | 0 | 1 | 2 | 4 |
|---|---|---|---|---|
| $y$ | 1 | 1 | 2 | 5 |

***Solution*** The divided difference table for the given data is constructed as follows:

| $x$ | $y$ | 1st divided difference | 2nd divided difference | 3rd divided difference |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | 1 | 0 | | |
| 2 | 2 | 1 | 1/2 | −1/12 |
| 4 | 5 | 3/2 | 1/6 | |

(i) Lagrange's interpolation formula (6.37) gives

$$y = f(x) = \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(1)$$

$$+ \frac{(x-0)(x-1)(x-4)}{(2)(2-1)(2-4)}(2) + \frac{(x-0)(x-1)(x-2)}{4(4-1)(4-2)}(5)$$

$$= \frac{-(x^3 - 7x^2 + 14x - 8)}{8} + \frac{x^3 - 6x^2 + 8x}{3} - \frac{x^3 - 5x^2 + 4x}{2}$$

$$+ \frac{5(x^3 - 3x^2 + 2x)}{24}$$

$$= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1 \qquad (1)$$

(ii) Newton's divided difference formula gives

$$y = f(x) = 1 + (x-0)(0) + (x-0)(x-1)\left(\frac{1}{2}\right) + (x-0)(x-1)(x-2)\left(-\frac{1}{12}\right)$$

$$= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1 \qquad (2)$$

From Eqs. (1) and (2) we observe that the interpolating polynomial by both Lagrange's and Newton's divided difference formulae is one and the same. Also Newton's formula involves less number of arithmetic operations than that of Lagrange's.

**Example 6.17** Using Newton's divided difference formula, find the quadratic equation for the following data. Hence find $y(2)$.

| $x$ | 0 | 1 | 4 |
|---|---|---|---|
| $y$ | 2 | 1 | 4 |

***Solution*** The divided difference table for the given data is constructed as follows:

| $x$ | $y$ | 1st divided difference | 2nd divided difference |
|---|---|---|---|
| 0 | 2 | | |
| | | −1 | |
| 1 | 1 | | 1/2 |
| | | 1 | |
| 4 | 4 | | |

Now, using Newton's divided difference formula, we have

$$y = 2 + (x - 0)(-1) + (x - 0)(x - 1)\left(\frac{1}{2}\right) = \frac{1}{2}(x^2 - 3x + 4)$$

Hence, $y(2) = 1$.

**Example 6.18** A function $y = f(x)$ is given at the sample points $x = x_0, x_1$ and $x_2$. Show that the Newton's divided difference interpolation formula and the corresponding Lagrange's interpolation formula are identical.

***Solution*** For the function $y = f(x)$, we have the data $(x_i, y_i)$, $i = 0, 1, 2$. The interpolation polynomial using Newton's divided difference formula is given as

$$y = f(x) = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] \quad (1)$$

Using the definition of divided differences and Eq. (6.47), we can rewrite Eq. (1) in the form

$$y = y_0 + (x - x_0)\frac{(y_1 - y_0)}{(x_1 - x_0)} + (x - x_0)(x - x_1)\left[\frac{y_0}{(x_0 - x_1)(x_0 - x_2)}\right.$$

$$\left. + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_{}}\right]$$

$$= \left[1 - \frac{(x_0 - x)}{(x_0 - x_1)} + \frac{(x - x_0)(x - x_1)}{(x_0 - x_1)(x_0 - x_2)}\right]$$

$$+ \left[\frac{(x - x_0)}{(x_1 - x_0)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)}\right] y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\, y_2$$