$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

That is, $f(x) = x^3 - 2x^2 + 7x - 3$, is the required cubic polynomial.

6.4 NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

If one wishes to interpolate the value of the function $y = f(x)$ near the end of table of values, and to extrapolate value of the function a short distance forward from y_n, Newton's backward interpolation formula is used, which can be derived as follows:

Let $y = f(x)$ be a function which takes on values $f(x_n), f(x_n - h), f(x_n - 2h), \ldots, f(x_0)$ corresponding to equispaced values $x_n, x_n - h, x_n - 2h, \ldots, x_0$. Suppose, we wish to evaluate the function $f(x)$ at $(x_n + ph)$, where p is any real number, then we have the shift operator E, such that

$$f(x_n + ph) = E^p f(x_n) = (E^{-1})^{-p} f(x_n) = (1 - \nabla)^{-p} f(x_n)$$

Binomial expansion yields,

$$f(x_n + ph) = \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n + \text{Error}\right] f(x_n)$$

That is,

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n f(x_n) + \text{Error} \quad (6.34)$$

This formula is known as ***Newton's backward interpolation formula***. This formula is also known as ***Newtons–Gregory backward difference interpolation formula***. If we retain $(r + 1)$ terms in Eq. (6.34), we obtain a polynomial of degree r agreeing with $f(x)$ at $x_n, x_{n-1}, \ldots, x_{n-r}$. Alternatively, this formula can also be written as

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error} \quad (6.35)$$

where

$$p = \frac{x - x_n}{h}$$

Here follows a couple of examples for illustration.

Example 6.12 For the following table of values, estimate $f(7.5)$.

x	1	2	3	4	5	6	7	8
$y = f(x)$	1	8	27	64	125	216	343	512

Solution The value to be interpolated is at the end of the table. Hence, it is appropriate to use Newton's backward interpolation formula. We shall first construct the backward difference table for the given data:

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
2	8	7			
			12		
3	27	19		6	
			18		0
4	64	37		6	
			24		0
5	125	61		6	
			30		0
6	216	91		6	
			36		0
7	343	127		6	
			42		
8	512	169			

Since the fourth and higher order differences are zero, the required Newton's backward interpolation formula is

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

In this problem,

$$p = \frac{x - x_n}{h} = \frac{7.5 - 8.0}{1} = -0.5$$

and

$$\nabla y_n = 169, \quad \nabla^2 y_n = 42, \quad \nabla^3 y_n = 6$$

Therefore,

$$y_{7.5} = 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2}(42) + \frac{(-0.5)(0.5)(1.5)}{6}(6)$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.875$$

Example 6.13 The sales in a particular department store for the last five years is given in the following table:

Year	1974	1976	1978	1980	1982
Sales (in lakhs)	40	43	48	52	57

Estimate the sales for the year 1979.

Solution At the outset, we shall construct Newton's backward difference table for the given data as

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
		3			
1976	43		2		
		5		−3	
1978	48		−1		5
		4		2	
1980	52		1		
		5			
1982	57				

In this example,

$$p = \frac{1979 - 1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1, \quad \nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

Newton's interpolation formula gives

$$y_{1979} = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1)$$

$$+ \frac{(-1.5)(-0.5)(0.5)}{6}(2) + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5)$$

$$= 57 - 7.5 + 0.375 + 0.125 + 0.1172$$

Therefore,

$$y_{1979} = 50.1172$$

6.5 LAGRANGE'S INTERPOLATION FORMULA

Newton's interpolation formulae developed in the earlier sections can be used only when the values of the independent variable x are equally spaced. Also the differences of y must ultimately become small. If the values of the independent variable are not given at equidistant intervals, then we have the basic formula associated with the name of *Lagrange* which is derived as follows:

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$. Since there are $(n + 1)$ values of y corresponding to $(n + 1)$ values of x, we can represent the function $f(x)$ by a polynomial of degree n. Suppose we write this polynomial in the form

$$f(x) = A_0 x^n + A_1 x^{n-1} + \ldots + A_n$$

or, more conveniently, in the form

$$y = f(x) = a_0 (x - x_1)(x - x_2) \cdots (x - x_n) + a_1 (x - x_0)(x - x_2) \cdots (x - x_n)$$
$$+ a_2 (x - x_0)(x - x_1) \cdots (x - x_n) + \ldots + a_n (x - x_0)(x - x_1) \cdots (x - x_{n-1}) \qquad (6.36)$$

Here, the coefficients a_k are so chosen as to satisfy Eq. (6.36) by the $(n + 1)$ pairs

(x_i, y_i). Thus, Eq. (6.36) yields

$$y_0 = f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

Therefore,

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

Similarly, we obtain

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

$$a_i = \frac{y_i}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

and

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

Now, substituting the values of $a_0, a_1, \ldots, a_n$ into Eq. (6.36), we get

$$y = f(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y_0$$

$$+ \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} y_1 + \cdots$$

$$+ \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} y_i + \cdots$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})} y_n \tag{6.37}$$

Equation (6.37) is Lagrange's formula for interpolation. This formula can be used whether the values $x_0, x_1, x_2, \ldots, x_n$ are equally spaced or not. Alternatively, Eq. (6.37) can also be written in compact form as

$$y = f(x) = L_0(x) y_0 + L_1(x) y_1 + \cdots + L_i(x) y_i + \cdots + L_n(x) y_n$$

$$= \sum_{k=0}^{n} L_k(x) y_k$$

$$= \sum_{k=0}^{n} L_k(x) f(x_k) \tag{6.38}$$

where,

$$L_i(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} \tag{6.39}$$

we can easily observe that, $L_i(x_i) = 1$ and $L_i(x_j) = 0$, $i \neq j$. Thus introducing

Kronecker delta notation

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Further, if we introduce the notation

$$\Pi(x) = \prod_{i=0}^{n} (x - x_i) = (x - x_0)(x - x_1) \cdots (x - x_n) \tag{6.40}$$

that is, $\Pi(x)$ is a product of $(n + 1)$ factors. Clearly, its derivative $\Pi'(x)$ contains a sum of $(n + 1)$ terms in each of which one of the factors of $\Pi(x)$ will be absent. We also define,

$$P_k(x) = \prod_{i \neq k} (x - x_i) \tag{6.41}$$

which is same as $\Pi(x)$ except that the factor $(x - x_k)$ is absent. Then

$$\Pi'(x) = P_0(x) + P_1(x) + \cdots + P_n(x) \tag{6.42}$$

But, when $x = x_k$, all terms in the above sum vanishes except $P_k(x_k)$. Hence,

$$\Pi'(x_k) = P_k(x_k) = (x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n) \tag{6.43}$$

Therefore, using Eqs. (6.40)–(6.43), Eq. (6.39) can be rewritten as

$$L_k(x) = \frac{P_k(x)}{P_k(x_k)} = \frac{P_k(x)}{\Pi'(x_k)} = \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} \tag{6.44}$$

Finally, the Lagrange's interpolation polynomial of degree n can be written as

$$y(x) = f(x) = \sum_{k=0}^{n} \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} f(x_k) = \sum_{k=0}^{n} L_k(x) f(x_k) = \sum_{k=0}^{n} L_k(x) y_k \tag{6.45}$$

Lagrange's interpolation is illustrated through the following examples.

Example 6.14 Find Lagrange's interpolation polynomial fitting the points $y(1) = -3$, $y(3) = 0$, $y(4) = 30$, $y(6) = 132$. Hence find $y(5)$.

Solution The given data can be arranged as follows:

x	1	3	4	6
$y = f(x)$	−3	0	30	132

using Lagrange's interpolation formula (6.37), we have

$$y(x) = f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$
$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$= \frac{x^3 - 13x^2 + 54x - 72}{-30}(-3) + \frac{x^3 - 11x^2 + 34x - 24}{6}(0)$$

$$+ \frac{x^3 - 10x^2 + 27x - 18}{-6}(30) + \frac{x^3 - 8x^2 + 19x - 12}{30}(132)$$

On simplification, we get

$$y(x) = \frac{1}{10}\left(-5x^3 + 135x^2 - 460x + 300\right) = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$$

which is the required Lagrange's interpolation polynomial. Now, $y(5) = 75$.

Example 6.15 Given the following data, evaluate $f(3)$ using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

Solution Using Lagrange's interpolation formula given by Eq. (6.37), we have

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Therefore,

$$f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1) + \frac{(3-1)(3-5)}{(2-1)(2-5)}(4) + \frac{(3-1)(3-2)}{(5-1)(5-2)}(10) = 6.4999$$

6.6 DIVIDED DIFFERENCES

When the function values are given at non-equispaced points, we have already developed the Lagrange's interpolation formula for interpolation in Section 6.5. Now, we shall introduce the concept of divided differences and then develop Newton's divided difference interpolation formula, whose accuracy is same as that of Lagrange's formula, but has the advantage of being computationally economical in the sense that it involves less number of arithmetic operations.

Let us assume that the function $y = f(x)$ is known for several values of x, (x_i, y_i), for $i = 0\,(1)\,n$. The divided differences of orders 0, 1, 2, ..., n are defined recursively as follows:

$$y[x_0] = y(x_0) = y_0$$

is the 0th order divided difference. The first order divided difference is defined as

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$